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QUASI-CONVEX DUALITY FOR A MIXED 0-1 VARIABLE PROBLEM AND APPLICATIONS IN PRODUCTION PLANNING WITH SET UP COSTS

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Abstract. One of the intractable nonlinear structures comes from the 0-1 variables that formulate, for instance, the discontinuity of set up costs. In this article we consider a mixed 0-1 variable problem that occurs in production planning. By quasi-convex duality we can solve efficiently the problem by linear programs.

1. Introduction

A special class of nonlinear mathematical programming problems, that is quasi-concave maximization, has attracted an increasing attention from practitioners in both applications and theoretical studies (cf.[1]-[6]). In certain applications, the discontinuity of set up costs is hardly represented within the framework of convexity, but it can be well treated by using the extension in quasi-convexity. In theoretical studies, quasi-concave maximization problems have been equipped with optimality conditions, in which the supdifferential calculus is carried out by the quasi-supdifferentials (cf.[5, 6]). The zero gap duality scheme has also been developed for quasi-concave maximization problems, and on the basis of this scheme we can obtain appropriate solution methods. In Section 2 a mixed 0-1 variable problem is formulated for an application in production planning. In Section 3 the mixed 0-1 variable problem is reduced to a quasi-concave maximization problem. In Section 4 the reduced problem is converted by duality into a quasi-affine minimization problem under linear constraints. In Section 5 we present an optimality criterion and a solution method that decomposes the problem into linear programs. Finally, several concluding remarks are drawn in Section 6.

2. Problem setting

In the production planning problem of our interest we have to produce a product t from $(n-1)$ resources $x_1, x_2, \ldots, x_{n-1}$, and a monetary budget x_n $(n > 1)$

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by a production process in which m technologies ($m \geq 1$) are performed consecutively from the first technology to the m-th technology. In the i-th technology ($i \in \{1, 2, \ldots, m\}$) the set up cost is s_i ($s_i > 0$), and in order to produce one unit of the product we consume a^i_j ($a^i_j \geq 0$) units of the j-th resource ($j \in \{1, 2, \ldots, n-1\}$) and a^i_n monetary units. However, the i-th technology has a capacity t^*_i ($t^*_i > 0$) of the product, i.e., the i-th technology can produce at most the value t^*_i of the product. A special case, in which $a^i_j = a_j$ for any $i \in \{1, 2, \ldots, m\}$ and any $j \in \{1, 2, \ldots, n\}$, refers to a single technology production process where t^*_i is the capacity of the i-th production level and s_i is the capacity improvement cost.

Denote by t_i ($t_i \geq 0$) the value of the product produced by the i-th technology. Then, the vector $(t_1, t_2, \ldots, t_m)^T$ is acceptable if

$$0 \leq t_i \leq t^*_i \quad i = 1, 2, \ldots, m, \tag{1}$$

$$\delta(t_i)(t^*_{i-1} - t_{i-1}) = 0 \quad i = 2, 3, \ldots, m, \tag{2}$$

where

$$\delta(t_i) = \begin{cases} 0 & \text{if} \quad t_i = 0, \\ 1 & \text{if} \quad t_i > 0. \end{cases}$$

The condition (1) tells that t_i satisfies the capacity constraint, and the condition (2) tells that the m technologies are performed consecutively, i.e., the i-th technology is performed ($t_i > 0$) only if the $(i-1)$-th technology was performed in full capacity.

An acceptable product vector $(t_1, t_2, \ldots, t_m)^T \in R^m_+$ is called feasible to a vector $x = (x_1, x_2, \ldots, x_n)^T \in R^n_+$ of the resources and the monetary budget if

$$x_j \geq \sum_{i=1}^{m} a^i_j t_i \quad j = 1, 2, \ldots, n-1, \tag{3}$$

$$x_n \geq \sum_{i=1}^{m} a^i_n t_i + \sum_{i=1}^{m} \delta(t_i) s_i. \tag{4}$$

The condition (3) means that the products $t_1, t_2, \ldots, t_m$ consume the value $\sum_{i=1}^m a^i_j t_i$ of the j-th resource ($j \in \{1, 2, \ldots, n-1\}$) that does not exceed the given value x_j of the j-th resource, and the condition (4) means that the products $t_1, t_2, \ldots, t_m$ consume the monetary value

$$\sum_{i=1}^{m} a^i_n t_i + \sum_{i=1}^{m} \delta(t_i) s_i$$

that does not exceed the given monetary value x_n.

The vector $x = (x_1, x_2, \ldots, x_n)^T \in R^n_+$ of the resources and the monetary budget is feasible if it satisfies the following constraint

$$x \in X, \tag{5}$$

where X is a bounded polyhedral set with the nonempty interior in R^n_+ satisfying the free disposal condition:

$$x \in X \implies y \in X \quad \forall y: \; 0 \le y \le x.$$

An illustration of the constraint (5) can be found in a multiple integrated production with a given budgetary constraint that can be interpreted as follows. There are ℓ plants that integratedly produce the resources $x_1, x_2, \dots, x_{n-1}$. In the k-th plant ($k \in \{1, 2, \dots, \ell\}$) one monetary unit integratedly yields b^k_j units of the j-th resource ($j = 1, 2, \dots, n-1$), and consequently μ_k monetary units integratedly yields $\mu_k b^k_j$ units of the j-th resource ($j = 1, 2, \dots, n-1$). For a given M budgetary units the constraint (5) is represented as follows

$$x_j \le \sum_{k=1}^{\ell} \mu_k b^k_j \quad j = 1, 2, \dots, n-1, \tag{6}$$

$$x_n + \sum_{k=1}^{\ell} \mu_k \le M, \tag{7}$$

$$x_j \ge 0 \quad j = 1, 2, \dots, n. \tag{8}$$

In this illustrative example the feasible set X is a bounded polyhedral subset in R^n_+ defined via (6)-(8). Now our problem is to maximize the product $t = t_1 + t_2 + \dots + t_m$, subject to the constraints (1)-(5) :

$$\begin{array}{ll} \max & t_1 + t_2 + \dots + t_m, \\ \text{s.t.} & 0 \le t_i \le t^*_i \quad i = 1, 2, \dots, m, \\ & \delta(t_i)(t^*_{i-1} - t_{i-1}) = 0 \quad i = 2, 3, \dots, m, \\ & x_j \ge \displaystyle\sum_{i=1}^{m} a^i_j t_i \quad j = 1, 2, \dots, n-1, \\ & x_n \ge \displaystyle\sum_{i=1}^{m} a^i_n t_i + \sum_{i=1}^{m} \delta(t_i) s_i, \\ & (x_1, x_2, \dots, x_n)^T \in X. \end{array} \tag{9}$$

Since $\delta(t_i)$ is a 0-1 variable and the constraint (2) is of the complementary type, the above problem is a nonlinear program.

3. Reduction to a quasi-concave maximization problem

For any acceptable vector $(t_1, t_2, \dots, t_m)^T$ we define

$$t = \sum_{i=1}^{m} t_i. \tag{10}$$

Obviously,

$$0 \le t \le \sum_{i=1}^{m} t^*_i.$$

The acceptable vector $(t_1, t_2, \dots, t_m)^T$ can be uniquely defined through the equation (10) for a given $t \in [0, t^*]$ where $t^* = \sum_{i=1}^m t_i^*$, i.e., if an acceptable vector $(t'_1, t'_2, \dots, t'_m)^T$ satisfies the condition (10) then $t'_i = t_i$ for any $i = 1, 2, \dots, m$. Indeed, if there is an index $i \in \{1, 2, \dots, m\}$ such that $t_i \neq t'_i$ then without loss of generality we can assume that $t_i < t'_i$. So, from the conditions (1) and (2) it follows that on one hand

$$t_r \;=\; 0 \;\leq\; t'_r \quad \forall r \in \{i+1, i+2, \dots, m\},$$

and on the other hand

$$t'_r \;=\; t^*_r \;\geq\; t_r \quad \forall r \in \{1, 2, \dots, i-1\}.$$

Consequently, we arrive at a contradiction:

$$t \;=\; \sum_{i=1}^m t_i \;<\; \sum_{i=1}^m t'_i \;=\; t. \tag{11}$$

From now on for given $t \in [0, t^*]$ we denote by $(t_1, t_2, \dots, t_m)^T$ the unique acceptable vector satisfying the condition (10). It is simple to see that if $t' \in [0, t^*]$ and $t' \geq t$ then $t'_i \geq t_i$ for all $i = 1, 2, \dots, m$. Let us define

$$\begin{aligned} f_j(t) \;&=\; \sum_{i=1}^m a^i_j t_i \quad j = 1, 2, \dots, n-1, \\ f_n(t) \;&=\; \sum_{i=1}^m a^i_n t_i + \sum_{i=1}^m \delta(t_i) s_i. \end{aligned}$$

Since $t'_i \geq t_i$ $i = 1, 2, \dots, m$ for $t' \geq t$, the functions f_j, $j = 1, 2, \dots, n$, are nondecreasing on $[0, t^*]$ and

$$f_j(0) \;=\; 0 \quad j = 1, 2, \dots, n.$$

Setting

$$f^*_j \;=\; f_j(t^*) \quad j = 1, 2, \dots, n$$

we can see that the function f_j is defined on $[0, t^*]$ with the value in $[0, f^*_j]$ for any $j = 1, 2, \dots, n$. Without loss of generality we can assume that $f^*_j > 0$ for any $j = 1, 2 \dots, n$. Indeed, $f^*_j = 0$ means that the j-th resource is not consumed in the production process and therefore it can be excluded from the model of our consideration. Note that the functions f_j, $j = 1, 2, \dots, n-1$, are continuous while the function f_n is lower semi-continuous but may not be upper-semicontinuous on $[0, t^*]$. With the above notations the problem (9) can be reformulated as follows

$$\begin{array}{ll} \max & t, \\ \text{s.t.} & 0 \;\leq\; t \;\leq\; t^*, \\ & x_j \;\geq\; f_j(t) \quad j = 1, 2, \dots, n, \\ & x \;\in\; X. \end{array} \tag{12}$$

For any $j = 1, 2, \dots, n$ we define the function $f_j^{-1} : \; [0, f^*_j] \mapsto [0, t^*]$ as follows

$$f_j^{-1}(\theta) \;=\; \max\{t : \; f_j(t) \leq \theta\}. \tag{13}$$

It is obvious that f_j^{-1} is nondecreasing on $[0, f_J^*]$ for any $j = 1, 2, \dots, n$.

Proposition 3.1. *For any $j = 1, 2, \dots, n$, any $t \in [0, t^*]$ and any $\theta \in [0, f_j^*]$ we have*

$$f_j(t) \ \leq \ \theta \quad \Longleftrightarrow \quad t \ \leq \ f_j^{-1}(\theta).$$

Proof. First suppose $t \leq f_j^{-1}(\theta)$. By the definition (13) this implies

$$t \ \leq \ \max\{\xi : \ f_j(\xi) \leq \theta\},$$

and consequently there is ξ such that

$$f_j(\xi) \ \leq \ \theta \quad \text{and} \quad t \ \leq \ \xi.$$

Since $t \leq \xi$, we have $f_j(t) \leq f_j(\xi)$. This together with $f_j(\xi) \leq \theta$ implies $f_j(t) \leq \theta$. Conversely suppose $f_j(t) \leq \theta$. This implies $f_j^{-1} f_j(t) \leq f_j^{-1}(\theta)$. So, by the definition (13) we have

$$f_j^{-1}(f_j(t)) = \max\{\xi : \ f_j(\xi) \leq f_j(t)\} \geq t,$$

completing the proof. ☐

As a consequence of Proposition 3.1 we have Proposition 3.2 as follows.

Proposition 3.2. *The function f_j^{-1} is upper semi-continuous on $[0, f_j^*]$.*

Proof. By Proposition 3.1 we can see that (t, θ) belongs to the epigraph of f_j if and only if it belongs to the hypograph of f_j^{-1}. Since f_j is lower semi-continuous, its epigraph is closed. Therefore, the hypograph of f_j^{-1} is also closed. Thus, the function f_j^{-1} is upper semi-continuous. ☐

Proposition 3.3. *If $a_j^i > 0$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, then f_j^{-1} is continuous on $[0, f_j^*]$ for any $j = 1, 2, \dots, n$.*

Proof. If $a_j^i > 0$, $i = 1, 2, \dots, m$ $\ j = 1, 2, \dots, n-1$, then it is obvious that f_j, $j = 1, 2, \dots, n-1$, are increasing and continuous, and therefore f_j^{-1}, $j = 1, 2, \dots, n-1$, are their usual inverses, respectively. Hence, f_j^{-1}, $j = 1, 2, \dots, n-1$, are continuous. If $a_n^i > 0$, $i = 1, 2, \dots, m$, then f_n is increasing. By virtue of Proposition 3.2 it remains to prove that f_n^{-1} is lower semi-continuous. Indeed, suppose that $f_n^{-1}(\theta) > t$ for some $\theta \in [0, f_n^*]$ and some $t \in [0, t^*]$. Then

$$\max\{\xi : \ f_n(\xi) \leq \theta\} \ > \ t.$$

So, there is $\xi' \in [0, t^*]$ such that $f_n(\xi') \leq \theta$ and $\xi' > t$. Let $\xi'' \in (t, \xi')$ and $\theta' = f_n(\xi'')$. Then $\theta' = f_n(\xi'') < f_n(\xi') \leq \theta$ and therefore

$$f_n^{-1}(\theta') \ = \ \max\{\xi : \ f_n(\xi) \leq \theta'\} \ \geq \ \xi'' \ > \ t.$$

So, the subset $\{\theta \in [0, f_n^*] : \ f_n^{-1}(\theta) > t\}$ is open in $[0, f_n^*]$ for any $t \in [0, t^*]$, i.e., f_n^{-1} is lower semi-continuous. ☐

By virtue of Proposition 3.1 we can transform the problem (12) into the following problem

$$\begin{array}{ll} \max & t, \\ \text{s.t.} & t \le f_j^{-1}(x_j) \quad j = 1, 2, \ldots, n, \\ & 0 \le x_j \le f_j^* \quad j = 1, 2, \ldots, n, \\ & x \in X, \end{array}$$

or equivalently,

$$\begin{array}{ll} \max & F(x), \\ \text{s.t.} & x \in \overline{X}, \end{array} \tag{14}$$

where

$$\begin{aligned} F(x) &= \min\{f_j^{-1}(x_j) \;\; j = 1, 2, \ldots, n\}, \\ \overline{X} &= \{x \ge 0 : \; x_j \le f_j^* \;\; j = 1, 2, \ldots, n, \; x \in X\}. \end{aligned}$$

Since F is a minimum of nondecreasing upper semi-comtinuous functions, F is a quasi-concave upper semi-continuous function. Therefore, the problem (14) is a quasi-concave maximization problem.

4. Duality

Set

$$x^* = (f_1^*, f_2^*, \ldots, f_n^*)^T.$$

If $x^* \in X$ then x^* is an optimal solution of the problem (14). Indeed, since $x^* \ge x$ for any $x \in \overline{X}$, from the nondecreasing property of F we have

$$F(x^*) \ge F(x) \quad \forall x \in \overline{X},$$

i.e., x^* is optimal to (14). For the nontriviality we assume from now on that

$$x^* \notin X. \tag{15}$$

Define

$$P = \{p \ge 0 : \; p^T x \le 1 \;\; \forall x \in \overline{X}\}.$$

The boundedness and the nonempty interior of P follows from the nonempty interior and the boundedness of $\overline{X}$, repectively. It is simple to see that P is a bounded, closed polyhedral subset with the nonempty interior and satisfying

$$p \in P \implies q \in P \quad \forall q : \; 0 \le q \le p.$$

In the terminology of Convex Analysis P is called the polar of $\overline{X}$, and it has been known that $\overline{X}$ is also the polar of P :

$$\overline{X} = \{x \ge 0 : \; p^T x \le 1 \;\; \forall p \in P\}.$$

For $p \in R_+^n$ we define

$$g(p) = \max\{F(x) : \; p^T x \le 1, \; 0 \le x \le x^*\}. \tag{16}$$

Then

$$\begin{aligned} g(p) &= \max\{t:\ t \le F(x),\ p^T x \le 1,\ 0 \le x \le x^*\} \\ &= \max\{t:\ p^T x \le 1,\ t \le f_j^{-1}(x_j),\ 0 \le x_j \le f_j^* \ \ j = 1, 2, \dots, n\} \\ &= \max\{t:\ p^T x \le 1,\ f_j(t) \le x_j,\ 0 \le x_j \le f_j^* \ \ j = 1, 2, \dots, n\}. \end{aligned}$$

Therefore

$$g(p) \ = \ \max\left\{ t:\ \sum_{j=1}^{n} p_j f_j(t) \le 1,\ t \in [0, t^*] \right\}. \tag{17}$$

Theorem 4.1. *The function g is quasi-affine on R_+^n.*

Proof. Let $p^1 \in R_+^n$, $p^2 \in R_+^n$ and $\gamma \in (0,1)$. The function g is quasi-convex, because by virtue of (16) we have

$$\begin{aligned} g(\gamma p^1 + (1-\gamma)p^2) &= \max\{F(x):\ (\gamma p^1 + (1-\gamma)p^2)^T x \le 1,\ 0 \le x \le x^*\} \\ &\le \max\{\max\{F(x):\ {p^1}^T x \le 1,\ 0 \le x \le x^*\}, \\ &\qquad \max\{F(x):\ {p^2}^T x \le 1,\ 0 \le x \le x^*\}\} \\ &= \max\{g(p^1), g(p^2)\}. \end{aligned}$$

It remains to prove the quasi-concavity of g. Let $t_1 = g(p^1)$, $t_2 = g(p^2)$. Then by virtue of (17) we have

$$\begin{aligned} \sum_{j=1}^{n} p_j^1 f_j(t_1) &\le 1, \\ \sum_{j=1}^{n} p_j^2 f_j(t_2) &\le 1. \end{aligned} \tag{18}$$

Let $t_0 = \min\{t_1, t_2\}$. Then from (18) it follows that

$$\begin{aligned} \sum_{j=1}^{n} p_j^1 f_j(t_0) &\le 1, \\ \sum_{j=1}^{n} p_j^2 f_j(t_0) &\le 1. \end{aligned}$$

So,

$$\sum_{j=1}^{n} (\gamma p_j^1 + (1-\gamma)p_j^2) f_j(t_0) \ \le \ 1.$$

This together with (17) implies

$$g(\gamma p^1 + (1-\gamma)p^2) \ \ge \ t_0 \ = \ \min\{g(p^1), g(p^2)\},$$

proving the quasi-concavity of g. □

Theorem 4.2. *For any $p \in R^n_+$ and $q \in R^n_+$ we have the following equivalence*

$$g(p) \; > \; g(q) \quad \Longleftrightarrow \quad \sum_{j=1}^{n} q_j f_j(g(p)) \; > \; 1. \tag{19}$$

Proof. Suppose

$$\sum_{j=1}^{n} q_j f_j(g(p)) \; \leq \; 1.$$

By virtue of (17) we have $g(q) \geq g(p)$. Conversely suppose

$$\sum_{j=1}^{n} q_j f_j(g(p)) \; > \; 1.$$

Since f_j, $j = 1, 2, \ldots, n$, are lower semi-continuous, there is $\alpha < g(p)$ such that

$$\sum_{j=1}^{n} q_j f_j(\alpha) \; > \; 1.$$

By virtue of (17) we have $g(q) \leq \alpha < g(p)$. □

Now a dual problem of the primal problem (14) can be stated as follows

$$\begin{array}{ll} \min & g(p), \\ \text{s.t.} & p \in P. \end{array} \tag{20}$$

The duality between the primal problem and the dual problem is presented in the following theorem.

Theorem 4.3. *We have the following assertions.*

(i) *Both the primal problem and the dual problem are solvable, and their optimal values are equal;*

(ii) *If $\overline{p}$ is an optimal solution of the dual problem then $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$ is an optimal solution of the primal problem, where $\overline{x}_j = f_j(g(\overline{p}))$, $j = 1, 2, \ldots, n$.*

Proof. Let $x \in \overline{X}$ and $p \in P$. Then we have $p^T x \leq 1$. From (16) it follows that

$$g(p) = \max\{F(x') : \; p^T x' \leq 1, \; 0 \leq x' \leq x^*\} \geq \quad F(x).$$

Thus,

$$g(p) \; \geq \; F(x) \quad \forall x \in \overline{X} \;\; \forall p \in P. \tag{21}$$

Since g is quasi-affine and P is a polytope, the dual problem (20) achieves an optimal solution at a vertex of P. In particular, the dual problem is solvable. Let $\overline{p}$ be an optimal solution of the dual problem. Set

$$\begin{aligned} \overline{x}_j &= f_j(g(\overline{p})) \quad j = 1, 2, \ldots, n, \\ \overline{x} &= (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T. \end{aligned}$$

If there is $p \in P$ such that $p^T\overline{x} > 1$ then

$$\sum_{j=1}^{n} p_j f_j(g(\overline{p})) \; > \; 1$$

and consequently by virtue of Theorem 4.2 we have $g(\overline{p}) > g(p)$, contradictory with the optimality of $\overline{p}$. So, $p^T\overline{x} \leq 1$ for any $p \in P$. This implies $\overline{x} \in X$. Moreover,

$$\begin{aligned} F(\overline{x}) &= \min\{f_j^{-1}(\overline{x}_j) \;\; j = 1, 2, \dots, m\} \\ &= \min\{f_j^{-1} f_j(g(\overline{p})) \;\; j = 1, 2, \dots, n\} \\ &= \min\{\max\{t : \; f_j(t) \leq f_j(g(\overline{p}))\} \;\; j = 1, 2, \dots, n\} \\ &\geq g(\overline{p}). \end{aligned}$$

This together with (21) implies that $\overline{x}$ is optimal to the primal problem, and the optimal value $F(\overline{x})$ of the primal problem and the optimal value $g(\overline{p})$ of the dual problem are equal. □

5. Optimality criterion and solution method

The following theorem gives us an optimality criterion in the dual program.

Theorem 5.1. *Let $\overline{p} \in P$. A sufficient and necessary condition for the optimality of $\overline{p}$ in the dual problem is as follows*

$$\max\left\{\sum_{j=1}^{n} p_j f_j(g(\overline{p})) : \; p \in P\right\} \; \leq \; 1. \tag{22}$$

If the above condition does not hold then q will be a feasible solution of the dual problem that is better than $\overline{p}$: $g(q) < g(\overline{p})$, where q is an optimal solution of the following linear program

$$\max \sum_{j=1}^{n} p_j f_j(g(\overline{p})), \; s.t. \; p \; \in \; P. \tag{23}$$

Proof. Suppose that the condition (22) holds at $\overline{p} \in P$. Then

$$\sum_{j=1}^{n} p_j f_j(g(\overline{p})) \; \leq \; 1 \quad \forall p \in P.$$

By Theorem 4.2 this implies

$$g(\overline{p}) \; \leq \; g(p) \quad \forall p \in P,$$

i.e., $\overline{p}$ is optimal to the dual problem. If the condition (22) does not hold at $\overline{p} \in P$ and q is an optimal solution of the linear program (23), then

$$\sum_{j=1}^{n} q_j f_j(g(\overline{p})) \; > \; 1.$$

Again by Theorem 4.2 this implies $g(q) < g(\overline{p})$, and consequently $\overline{p}$ is not optimal to the dual problem. □

Theorem 5.2. *The subset*

$$\left\{ p \in P : \sum_{j=1}^{n} p_j f_j^* > 1 \right\}$$

is nonempty. Moreover, for any vector $\overline{p}$ in this subset we have

$$g(\overline{p}) \ < \ t^*. \tag{24}$$

Proof. Since $x^* = (f_1^*, f_2^*, \ldots, f_n^*)^T \notin \overline{X}$ and P is the polar of $\overline{X}$, the subset

$$\left\{ p \in P : \sum_{j=1}^{n} p_j f_j^* > 1 \right\}$$

is nonempty. Let $\overline{p}$ belong to this subset. Then

$$1 \ < \ \sum_{j=1}^{n} \overline{p}_j f_j^* \ = \ \sum_{j=1}^{n} \overline{p}_j f_j(t^*).$$

Therefore

$$g(\overline{p}) = \max \left\{ t : \sum_{j=1}^{n} \overline{p}_j f_j(t) \le 1, \ 0 \le t \le t^* \right\} < t^*,$$

proving (24). □

On the basis of the above theorems we obtain the following algorithm to solve both the primal problem and the dual problem.

Algorithm

Initialization. Let p^0 be an optimal basic solution of the following linear program

$$\max \sum_{j=1}^{n} p_j f_j^*, \ \text{s.t. } p \ \in \ P.$$

Set $k = 0$.

Iteration k $(k = 0, 1, 2, \ldots)$

- Step 1. Compute $g(p^k)$:

$$g(p^k) \ = \ \max \left\{ t : \sum_{j=1}^{n} p_j^k f_j(t) \le 1, \ t \in [0, t^*] \right\}.$$

- Step 2. Solve the following linear program

$$\max \sum_{j=1}^{n} p_j f_j(g(p^k)), \ \text{s.t. } p \ \in \ P, \tag{25}$$

  obtaining its optimal basic solution q^k.

- Step 3. If

$$\sum_{j=1}^{n} q_j^k f_j(g(p^k)) \leq 1, \tag{26}$$

then stop: p^k is an optimal solution of the dual problem and $x^k = (x_1^k, x_2^k, \ldots, x_n^k)^T$, where $x_j^k = f_j(g(p^k))$, $j = 1, 2, \ldots, n$, is an optimal solution of the primal problem, otherwise set $p^{k+1} = q^k$ and go to iteration $k+1$.

For the convergence of the algorithm we have the following theorems.

Theorem 5.3. *If the algorithm does not stop at iteration k ($k \geq 0$) then*

$$g(p^{k+1}) < g(p^k). \tag{27}$$

Proof. Suppose that the algorithm does not stop at iteration k. Then

$$\sum_{j=1}^{n} q_j^k f_j(g(p^k)) > 1.$$

By Theorem 4.2 this implies $g(q^k) < g(p^k)$. So, we have (27). □

Theorem 5.4. *The algorithm stops after finitely many iterations, yielding optimal solutions of the primal and dual problems, respectively.*

Proof. If the algorithm does not stop after finitely many iterations, then it generates an infinite sequence $\{p^k \ \ k = 1, 2, 3, \ldots\}$, where p^k is a basic solution and $g(p^{k+1}) < g(p^k)$ for any k. So, $p^{k'} \neq p^k$ for any $k' \neq k$. Thus, we arrive at a contradiction with the fact that the number of basic solutions is finite. Suppose now that the algorithm stops at iteration k. Then, since q^k solves (25), we have

$$1 \geq \sum_{j=1}^{n} q_j^k f_j(g(p^k)) = \max\left\{\sum_{j=1}^{n} p_j f_j(g(p^k)) : \ p \in P\right\}.$$

By Theorem 5.1 this implies that p^k solves the dual problem, and therefore by Theorem 4.3 x^k solves the primal problem. □

6. Concluding remarks

In this article we are concerned with a problem of maximizing the production under the conditions of limited resources, a monetary budget and a consecutive multiple technology performance. The set up costs involved in the production create the 0-1 variables indicating the technology complex. Using the acceptability of the productions we reduce the mixed 0-1 variable production problem to a quasi-concave maximization problem. By duality the reduced problem can be converted into a quasi-affine minimization under linear constraints in the dual space. On the basis of strong duality we can obtain an optimal solution of the primal problem from an optimal solution of the dual problem. The optimality test in the dual problem can be realized at the cost of a linear program. Thus, we are able to construct an algorithm to decompose the dual problem into a finite sequence of linear programs. The algorithm is efficient because all the linear programs have the same set of linear constraints.

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