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SHARP INEQUALITY FOR MULTILINEAR COMMUTATOR OF MULTIPLIER OPERATOR

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Abstract. In this paper, we prove the sharp inequality for multilinear commutator related to the multiplier operators, in which functions belong to BMO space. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutator.

1. Introduction

It is well known that the commutator $[b, T]$ is defined as follows: $[b, T]f = bTf - T(bf)$. If T is a Calderón-Zygmund singular integral operator, Coifman, Rochberg and Weiss[1] stated that $[b, T]$ was bounded on $L^p(R^n)(1 < p < \infty)$. In [12], You proved that $[b, T]$ is bounded in $L^p(R^n)$ when T is a multiplier operator and $b \in \dot{\wedge}_\beta(R^n)$. In [13, 14], Zhang studied the $(L^p, \dot{F}_p^{\beta,\infty})$-boundedness of commutators of multipliers. In this paper, following them, we will introduce the multilinear commutator associated to the multiplier operator and study the sharp inequality of the multilinear commutator. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutator.

2. Notations and some lemmas

First, let us introduce some notations. In this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function b, let $b_Q = |Q|^{-1} \int_Q b(x)dx$, and the sharp function of b is defined by

$$b^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy.$$

It is well-known that (see [2])

$$b^\#(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbf{C}} \frac{1}{|Q|} \int_Q |b(y) - c| dy.$$

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We say that b belongs to $BMO(R^n)$, if $b^{\#}$ belongs to $L^\infty(R^n)$ and define

$$\|b\|_{BMO} = \|b^{\#}\|_{L^\infty}.$$

For $b_j \in BMO(R^n)(j = 1, \cdots, m)$, set $\|\overrightarrow{b}\|_{BMO} = \prod_{j=1}^m \|b_j\|_{BMO}$.

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\delta = \{\delta(1), \cdots, \delta(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\delta \in C_j^m$, set $\delta^c = \{1, \cdots, m\} \setminus \delta$. For $\overrightarrow{b} = (b_1, \cdots, b_m)$ and $\delta = \{\delta(1), \cdots, \delta(j)\} \in C_j^m$, set $\overrightarrow{b}_\delta = (b_{\delta(1)}, \cdots, b_{\delta(j)})$, $b_\delta = b_{\delta(1)} \dots b_{\delta(j)}$ and

$$\|\overrightarrow{b}_\delta\|_{BMO} = \|b_{\delta(1)}\|_{BMO} \cdots \|b_{\delta(j)}\|_{BMO}.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

we write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$.

We denote the Muckenhoupt weights by A_1 (see [11]), that is

$$A_1 = \{w : M(w)(x) \le Cw(x), a.e.\}.$$

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index of non-negative integers $\alpha_j (j = 1, 2, \cdots, n)$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Denote by D^α the partial differential operators of order α as follows:

$$D^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

A bounded measurable function k defined on $R^n \setminus 0$ is called a multiplier. The multiplier operator T_k associated with k is defined by

$$(T_k f)(x) = k(x) \hat{f}(x), \text{ for } f \in S(R^n),$$

where $\hat{f}$ denotes the Fourier transform of f and $S(R^n)$ is the Schwartz test function class.

Now, we recall the definition of the class $M(s, l)$. Denote by $|x| \sim t$ the fact that the value of x lies in the annulus $\{x \in R^n : at < |x| < bt\}$, where $0 < a \le 1 < b < \infty$ are values specified in each instance.

Definition 2.1. ([11]). Let $l \ge 0$ be a real number and $1 \le s \le 2$. we say that the multiplier k satisfies the condition $M(s, l)$, if

$$\left(\int_{|\xi| \sim R} |D^\alpha k(\xi)|^s d\xi \right)^{\frac{1}{s}} < CR^{n/s - |\alpha|}$$

for all $R > 0$ and multi-indices α with $|\alpha| \le l$ when l is a positive integer, and, in addition, if

$$\left(\int_{|\xi| \sim R} |D^\alpha k(\xi) - D^\alpha k(\xi - z)|^s d\xi \right)^{\frac{1}{s}} \le C \left(\frac{|z|}{R} \right)^\gamma R^{\frac{n}{s} - |\alpha|}$$

for all $|z| < R/2$ and all multi-indices α with $|\alpha| = [l]$, the integer part of l,i.e., $[l]$ is the greatest integer less than or equal to l, and $l = [l] + \gamma$ when l is not an integer.

Denote $D(R^n) = \{\phi \in S(R^n) : \text{supp}(\phi) \text{ is compact}\}$ and $\hat{D}_0(R^n) = \{\phi \in S(R^n) : \hat{\phi} \in D(R^n)$ and $\hat{\phi}$ vanishes in a neighborhood of the origin$\}$. The following boundedness property of T_k on $L^p(R^n)$ is proved in Strömberg and Torkinsky ([4]).

Lemma 2.2. ([11]) *Let $k \in M(s, l), 1 \le s \le 2$, and $l > \frac{n}{s}$; then the associated mapping T_k, defined a priori for $f \in \hat{D}_0(R^n)$, $T_k f(x) = (f * K)(x)$, extends to a bounded mapping from $L^p(R^n)$ into itself for $1 < p < \infty$, $K(x) = \check{k}(x)$.*

Definition 2.3. ([11]) For a real number $\tilde{l} \ge 0$ and $1 \le \tilde{s} < \infty$, we say that K verifies the condition $\tilde{M}(\tilde{s}, \tilde{l})$, and we often write $K \in \tilde{M}(\tilde{s}, \tilde{l})$, if

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}} K(x)|^{\tilde{s}} dx\right)^{\frac{1}{\tilde{s}}} \le C R^{n/\tilde{s} - n - |\tilde{\alpha}|}, \qquad R > 0,$$

for all multi-indices $|\tilde{\alpha}| \le \tilde{l}$ and, in addition, if

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x - z)|^{\tilde{s}} dx\right)^{\frac{1}{\tilde{s}}} \le C \left(\frac{|z|}{R}\right)^{v} R^{\frac{n}{\tilde{s}} - n - u}, \text{ if } 0 < v < 1,$$

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x - z)|^{\tilde{s}} dx\right)^{\frac{1}{\tilde{s}}} \le C \left(\frac{|z|}{R}\right)\left(\log \frac{R}{|z|}\right) R^{\frac{n}{\tilde{s}} - n - u}, \text{ if } v = 1,$$

for all $|z| < \frac{R}{2}, R > 0$, and all multi-indices $\tilde{\alpha}$ with$|\tilde{\alpha}| = u$, where u denotes the largest integer strictly less than $\tilde{l}$ with $\tilde{l} = u + v$.

Lemma 2.4. ([11]) *Suppose $k \in M(s, l)$, $1 \le s \le 2$. Given $1 \le \tilde{s} < \infty$, let $r \ge 1$ be such that $\frac{1}{r} = \max\{\frac{1}{s}, 1 - \frac{1}{\tilde{s}}\}$. Then $K \in \tilde{M}(\tilde{s}, \tilde{l})$, where $\tilde{l} = l - \frac{n}{r}$.*

Lemma 2.5. *Let $1 \le s \le 2$, suppose that l is a positive real number with $l > n/r$, $1/r = \max\{1/s, 1 - 1/\tilde{s}\}$, and $k \in M(s, l)$. Then there is a positive constant a, such that*

$$(\int_{B_k} |K(x - z) - K(x_Q - z)|^{\tilde{s}} dz)^{1/\tilde{s}} \le C 2^{-ka} (2^k h)^{-n/\tilde{s}'}.$$

Proof. We split our proof into two cases:

Case 1. $1 \le s \le 2$ and $0 < l - n/s \le 1$. We choose a real number $1 < \tilde{s} < \infty$ such that $s \le \tilde{s}$, and set $\tilde{l} = l - \frac{n}{s} > 0$. Since $k \in M(s, l)$, then by Lemma 2.4 there is $K \in \tilde{M}(\tilde{s}, \tilde{l})$.

When $\tilde{l} = l - \frac{n}{s} < 1$, note that l is a positive real number and $l > \frac{n}{s}$. Applying the condition $K \in \tilde{M}(\tilde{s}, \tilde{l})$ for $v = l - \frac{n}{s}$ and $u = 0$, one has

$$\left(\int_{B_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \leq C2^{-k(l-\frac{n}{s})}(2^k h)^{-\frac{n}{\tilde{s}'}},$$

let $a = l - \frac{n}{s}$,

$$\left(\int_{B_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \leq C2^{-ka}(2^k h)^{-\frac{n}{\tilde{s}'}}.$$

When $\tilde{l} = l - \frac{n}{s} = 1$, we choose $0 < \xi < 1$, such that $t^{1-\xi}\log(1/t) \leq C$ for $0 < t < 1/2$. Notice that $K \in \tilde{M}(\tilde{s}, \tilde{l})$, by Definition 2.3 for $u = 0, v = 1$,

$$\begin{aligned}
&\left(\int_{B_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \\
&\leq C\left(\frac{|y - x_Q|}{2^k h}\right)^{\xi}\left(\frac{|y - x_Q|}{2^k h}\right)^{1-\xi}\left(\log\frac{2^k h}{|y - x_Q|}\right)(2^k h)^{n/\tilde{s} - n} \\
&\leq C2^{-k\xi}(2^k h)^{-n/\tilde{s}'},
\end{aligned}$$

let $a = \xi$

$$\left(\int_{B_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \leq C2^{-ka}\left(2^k h\right)^{-n/\tilde{s}'}.$$

Case 2. $1 \leq s \leq 2$ and $l - n/s > 1$. Set $d = [l - n/s]$, if $l - n/s > 1$ is not an integer, and $d = l - n/s - 1$ if $l - n/s > 1$ is an integer. Choose $l_1 = l - d$; then $0 < l_1 - n/s \leq 1$ and $0 < l_1 < l$. So, from $k \in M(s, l)$ we know $k \in M(s, l_1)$. Set $\tilde{l} = l_1 - n/s$; by Lemma 2.4, $K \in \tilde{M}(\tilde{s}, \tilde{l})$. Repeating the proof of Case 1, except for replacing l by l_1, we can obtain the same result under the assumption $l - n/s > 1$. We omit the details here.

Certainly when $0 < \tilde{s}' < s$, which is the same as the above. □

3. Main results

Definition 3.1. Let $\overrightarrow{b} = (b_1, b_2, \cdots, b_m)$, by Lemma 2.2, $T_k f(x) = (K * f)(x)$, for $K(x) = \check{k}(x)$. We define the multilinear commutator of the multiplier operator as follows

$$T^{\overrightarrow{b}}(f)(x) = \left[\overrightarrow{b}, T_k\right] f(x) = \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(y)) K(x-y) f(y) dy,$$

and let $T(f)(x) = T_k f(x) = \int_{R^n} K(x-y) f(y) dy$.

Theorem 3.2. *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$\left(T^{\vec{b}}(f)\right)^{\#}(x) \le C \left\|\vec{b}\right\|_{BMO} M_r(f)(x) + \sum_{j=1}^{m} \sum_{\delta \in C_j^m} \left\|\vec{b}_\delta\right\|_{BMO} M_r(T^{\vec{b}_{\delta^c}}(f))(x).$$

Theorem 3.3. *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then $T^{\vec{b}}$ is bounded on $L^p(w)$ for $w \in A_1$ and $1 < p < \infty$.*

4. Proofs of theorems

Proof of Theorem 3.2. It suffices to prove that for $f \in C_0^\infty$ and some constant C_0, the following inequality holds:

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T^{\vec{b}}(f)(x) - C_0| dx \\
&\le C\|\vec{b}\|_{BMO}\left(M_r(f)(\tilde{x}) + \sum_{j=1}^{m}\sum_{\sigma \in C_j^m} M_r(T^{\vec{b}_{\sigma^c}}(f))(\tilde{x})\right).
\end{aligned}$$

Fix a cube $Q = Q(x_0, h)$, and $\tilde{x} \in Q$.
Case $m = 1$**.** For $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$, let

$$\begin{aligned}
T^{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})\, F(f)(x) - F\left((b_1 - (b_1)_{2Q})f_1\right)(x) \\
- F\left((b_1 - (b_1)_{2Q})f_2\right)(x).
\end{aligned}$$

Then

$$\begin{aligned}
&|T^{b_1}(f)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)| \\
&\le |(b_1(x) - (b_1)_{2Q})T(f)(x)| + |T((b_1 - (b_1)_{2Q})f_1)(x)| \\
&\quad + |T((b_1 - (b_1)_{2Q})f_2)(x) - T((b_1 - (b_1)_{2Q})f_2)(x_0)| \\
&= A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, by Hölder inequality with exponent $1/r + 1/r' = 1$ and Lemma 2.2, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q A(x) dx \\
&= \frac{1}{|Q|}\int_Q |b_1(x) - (b_1)_{2Q}||T(f)(x)| dx \\
&\le C\left(\frac{1}{|2Q|}\int_{2Q} |b_1(x) - (b_1)_{2Q}|^{r'} dx\right)^{1/r'} \left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r} \\
&\le C\|b_1\|_{BMO} M_r(T(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, choose p such that $1 < p < r$. By Hölder inequality and Lemma 2.2, we know that T_k is bounded on $L^p(R^n)$, so we can obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q B(x)dx \\
&= \frac{1}{|Q|}\int_Q |T((b_1-(b_1)_{2Q})f_1)(x)|dx \\
&\le \left(\frac{1}{|Q|}\int_{R^n} |T((b_1-(b_1)_{2Q})f\chi_{2Q})(x)|^p dx\right)^{1/p} \\
&\le C\left(\frac{1}{|Q|}\int_{R^n} (|b_1(x)-(b_1)_{2Q}||f(x)\chi_{2Q}(x)|)^p dx\right)^{1/p} \\
&\le C\left(\frac{1}{|Q|}\int_{2Q} |f(x)|^r dx\right)^{1/r}\left(\frac{1}{|Q|}\int_{2Q} |b_1(x)-(b_1)_{2Q}|^{rp/(r-p)}dx\right)^{(r-p)/rp} \\
&\le C\|b_1\|_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, when $x \in Q = Q(x_0, h)$, we write $B_k = \{z \in R^n : 2^k h < |x-z| \le 2^{k+1}h\}$. By Hölder inequality with exponent $1/p + 1/p' = 1$ and $1/t + 1/t' = 1$,

$$\begin{aligned}
C(x) =& |T((b_1-(b_1)_{2Q})f_2)(x) - T((b_1-(b_1)_{2Q})f_2)(x_0)| \\
&\le \int_{R^n} |(K(x-z)-K(x_0-z))(b_1(z)-(b_1)_{2Q})f_2(z)|dz \\
&\le C\int_{(2Q)^c} |(K(x-z)-K(x_0-z))(b_1(z)-(b_1)_{2Q})f(z)|dz \\
&\le C\sum_{k=1}^{\infty}\int_{B_k} |(K(x-z)-K(x_0-z))f(z)||b_1(z)-(b_1)_{2Q}|dz \\
&\le C\sum_{k=1}^{\infty}\left(\int_{B_k} |(K(x-z)-K(x_0-z))f(z)|^p dz\right)^{1/p} \times \\
&\quad \times \left(\int_{B_k} |b_1(z)-(b_1)_{2Q}|^{p'}dz\right)^{1/p'} \\
&\le C\sum_{k=1}^{\infty}\left(\int_{B_k} |K(x-z)-K(x_0-z)|^{pt}dz\right)^{1/pt}\left(\int_{B_k} |f(z)|^{pt'}dz\right)^{1/pt'} \times \\
&\quad \times \left(\int_{B_k} |b_1(z)-(b_1)_{2Q}|^{p'}dz\right)^{1/p'}.
\end{aligned}$$

According to Lemma 2.5, as $1 < pt < \infty$, here we denote $\tilde{s} = pt$,

$$\begin{aligned}
C(x) \le& \sum_{k=1}^{\infty} C2^{-ka}(2^k h)^{-n/(pt)'} \times \\
&\times \left(\int_{B_k} |f(z)|^{pt'}dz\right)^{1/pt'}\left(\int_{B_k} |b_1(z)-(b_1)_{2Q}|^{p'}dz\right)^{1/p'}
\end{aligned}$$

$$\le \sum_{k=1}^{\infty} C2^{-ka} \left(\frac{1}{|2^{k+1}Q|}\right)^{\frac{1}{(pt)'}} \left(\int_{B_k} |f(z)|^{pt'} dz\right)^{1/pt'} \times$$
$$\times \left(\int_{B_k} |b_1(z) - (b_1)_{2Q}|^{p'} dz\right)^{1/p'}$$
$$\le \sum_{k=1}^{\infty} C2^{-ka} \left(\frac{1}{|2^{k+1}Q|}\right)^{\frac{1}{p'}+\frac{1}{pt'}} \left(\int_{2^{k+1}Q} |f(z)|^{pt'} dz\right)^{1/pt'} \times$$
$$\times \left(\int_{2^{k+1}Q} |b_1(z) - (b_1)_{2Q}|^{p'} dz\right)^{1/p'}$$
$$\le C\|b_1\|_{BMO} M_{pt'}(f)(\tilde{x}) \sum_{k=1}^{\infty} k2^{-ka}$$
$$\le C\|b_1\|_{BMO} M_{pt'}(f)(\tilde{x}).$$

Set $pt' = r$, then

$$\frac{1}{|Q|}\int_Q C(x)dx \le C\|b_1\|_{BMO} M_r(f)(\tilde{x}).$$

So we get

$$\left(T^{\vec{b}}(f)\right)^{\#}(\tilde{x}) \le C\|b_1\|_{BMO}(M_r(f)(\tilde{x}) + M_r(T(f))(\tilde{x}).$$

Case $m \ge 2$. We have known that, for $\vec{b} = (b_1, \cdots, b_m)$,

$$T^{\vec{b}}(f)(x) = [\vec{b}, T_k]f(x)$$
$$= \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(y))K(x-y)f(y)dy$$
$$= \int_{R^n} ((b_1(x) - (b_1)_{2Q}) - (b_1(y) - (b_1)_{2Q})) \cdots ((b_m(x) - (b_m)_{2Q})$$
$$- (b_m(y) - (b_m)_{2Q}))K(x-y)f(y)dy$$
$$= \sum_{j=0}^{m} \sum_{\delta \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\delta \int_{R^n} (b(y) - (b)_{2Q})_{\delta^c} K(x-y)f(y)dy$$
$$= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(y) - (b_m)_{2Q})T(f)(x)$$
$$+ (-1)^m T((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q})f)(x)$$
$$+ \sum_{j=1}^{m-1} \sum_{\delta \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\delta \int_{R^n} (b(y) - (b)_{2Q})_{\delta^c} K(x-y)f(y)dy$$
$$= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q})T(f)(x)$$
$$+ (-1)^m T((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q})f)(x)$$

$$+\sum_{j=1}^{m-1}\sum_{\delta\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\delta T((b-(b)_{2Q})_{\delta^c}f)(x),$$

thus

$$\begin{aligned}
&|T^{\overrightarrow{b}}(f)(x)-T(\prod_{j=1}^{m}((b_j-(b_j)_{2Q})f_2)(x_0)|\\
&\leq |(b_1(x)-(b_1)_{2Q}\cdots(b_m(x)-(b_m)_{2Q}))T(f)(x)|\\
&\quad+\sum_{j=1}^{m-1}\sum_{\delta\in C_j^m}|(b(x)-(b)_{2Q})_\delta T((b-(b)_{2Q})_{\delta^c}f)(x)|\\
&\quad+|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)|\\
&\quad+|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x)\\
&\quad-T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)|\\
&=I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}$$

For $I_1(x)$, by Hölder inequality with exponent $1/p_1+\cdots+1/p_m+1/r=1$ and Lemma 2.2, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_1(x)dx\\
&\leq \frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||T(f)(x)|dx\\
&\leq \left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|^{p_1}dx\right)^{1/p_1}\cdots\left(\frac{1}{|Q|}\int_Q |b_m(x)-(b_m)_{2Q}|^{p_m}dx\right)^{1/p_m}\times\\
&\qquad\times\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r}\\
&\leq C\|\overrightarrow{b}\|_{BMO}M_r(T(f))(\tilde{x}).
\end{aligned}$$

For $I_2(x)$, by Minkowski and Hölder inequalities with exponent $1/p_1+\cdots+1/p_j=1$, $1/r+1/r'=1$, and Lemma 2.2, we get

$$\begin{aligned}
\frac{1}{|Q|}\int_Q I_2(x)dx &\leq \sum_{j=1}^{m-1}\sum_{\delta\in C_j^m}\frac{1}{|Q|}\int_Q|(b(x)-(b)_{2Q})_\delta T((b-(b)_{2Q})_{\delta^c}f)(x)|dx\\
&\leq C\sum_{j=1}^{m-1}\sum_{\delta\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\delta|^{r'}dx\right)^{1/r'}\\
&\quad\times\left(\frac{1}{|Q|}\int_Q|T((b-(b)_{2Q})_{\delta^c}f)(x)|^r dx\right)^{1/r}
\end{aligned}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\delta\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_{\delta(j)}|^{r'p_1}dx\right)^{1/r'p_1}\cdots$$
$$\times\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_{\delta(j)}|^{r'p_j}dx\right)^{1/r'p_j}$$
$$\times\left(\frac{1}{|Q|}\int_{Q}|T((b-(b)_{2Q})_{\delta^c}f)(x)|^{r}dx\right)^{1/r}$$
$$\leq C\sum_{j=1}^{m-1}\sum_{\delta\in C_j^m}\|\overrightarrow{b}_{\delta}\|_{BMO}M_r(T^{\overrightarrow{b}_{\delta^c}}(f))(\tilde{x}).$$

For $I_3(x)$, choose $1<p<r, 1<q_j<\infty, j=1,\cdots,m$ such that $1/q_1+\cdots+1/q_m+p/r=1$. By Lemma 2.2 and Hölder inequality, we get

$$\frac{1}{|Q|}\int_Q I_3(x)dx$$
$$=\frac{1}{|Q|}\int_Q|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)|dx$$
$$\leq\left(\frac{1}{|Q|}\int_{R^n}|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f\chi_{2Q})(x)|^p dx\right)^{1/p}$$
$$\leq\left(\frac{1}{|Q|}\int_{R^n}|(b_1(x)-(b_1))_{2Q}\cdots(b_m(x)-(b_m)_{2Q})f(x)\chi_{2Q}(x)|^p dx\right)^{1/p}$$
$$\leq C\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^r dx\right)^{1/r}\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{pq_1}\right)^{1/pq_1}\cdots$$
$$\times\left(\frac{1}{|2Q|}\int_{2Q}|b_m(x)-(b_m)_{2Q}|^{pq_m}\right)^{1/pq_m}$$
$$\leq C\|\overrightarrow{b}\|_{BMO}M_r(f)(\tilde{x}).$$

For $I_4(x)$, when $x\in Q=Q(x_0,h)$, we write $B_k=\{z\in R^n: 2^k h<|x-z|\leq 2^{k+1}h\}$. For $1<p, 1<t$, by Hölder inequality with exponent $1/p+1/p'=1, 1/t+1/t'=1$ and Lemma 2.2, we get

$$I_4=|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x)$$
$$\quad -T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)|$$
$$=\int_{R^n}|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})f(z)\chi_{(2Q)^c}(z)(K(x-z)-K(x_0-z))|dz$$
$$=\int_{(2Q)^c}|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})||(K(x-z)-K(x_0-z))f(z)|dz$$

$$\leq \sum_{k=1}^{\infty}\int_{B_k}|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})||(K(x-z)-K(x_0-z))f(z)|dz$$
$$\leq \sum_{k=1}^{\infty}\left(\int_{B_k}|K(x-z)-K(x_0-z)|^p|f(z)|^p dz\right)^{1/p}\times$$
$$\times\left(\int_{B_k}|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})|^{p'}dz\right)^{1/p'}$$
$$\leq \sum_{k=1}^{\infty}\left(\int_{B_k}|K(x-z)-K(x_0-z)|^{pt}dz\right)^{1/pt}\left(\int_{B_k}|f(z)|^{pt'}dz\right)^{1/pt'}\times$$
$$\times\left(\int_{B_k}|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})|^{p'}dz\right)^{1/p'}.$$

According to Lemma 2.5, we get

$$\left(\int_{B_k}|K(x-z)-K(x_Q-z)|^{\tilde{s}}dz\right)^{1/\tilde{s}}\leq C2^{-ka}(2^k h)^{-n/\tilde{s}'},$$

as $1<pt<\infty$, here we denote $\tilde{s}=pt$, by Hölder inequality with exponent $1/r_1+\cdots+1/r_m=1$. According to the above inequality, we get

$$I_4(x)\leq C\sum_{k=1}^{\infty}2^{-ka}(2^k h)^{-n/(pt)'}\left(\int_{B_k}|f(z)|^{pt'}dz\right)^{1/pt'}\times$$
$$\times\left(\int_{B_k}|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})|^{p'}dz\right)^{1/p'}$$
$$\leq C\sum_{k=1}^{\infty}2^{-ka}\left(\frac{1}{|2^{k+1}Q|}\right)^{(\frac{1}{p'}+\frac{1}{pt'})}\left(\int_{B_k}|f(z)|^{pt'}dz\right)^{1/pt'}\times$$
$$\times\left(\int_{B_k}|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})|^{p'}dz\right)^{1/p'}$$
$$\leq C\sum_{k=1}^{\infty}2^{-ka}M_{pt'}(f)(\tilde{x})\left(\frac{1}{|2^{k+1}Q|}\right)^{1/p'}\times$$
$$\times\left(\int_{2^{k+1}Q}|b_1(z)-(b_1)_{2Q}|^{p'r_1}dz\right)^{1/p'r_1}\times$$
$$\times\cdots\times\left(\int_{2^{k+1}Q}|b_m(z)-(b_m)_{2Q}|^{p'r_m}dz\right)^{1/p'r_m}$$

$$\leq C\sum_{k=1}^{\infty}k^m 2^{-ka}\|\overrightarrow{b}\|_{BMO}M_{pt'}(f)(\tilde{x})$$
$$\leq C\|\overrightarrow{b}\|_{BMO}M_{pt'}(f)(\tilde{x}),$$

set $pt' = r$, then

$$\frac{1}{|Q|}\int_Q I_4(x)dx \leq C\|\overrightarrow{b}\|_{BMO}M_r(f)(\tilde{x}).$$

Finally we can get Theorem 3.2:

$$\left(T^{\overrightarrow{b}}(f)\right)^{\#}(x) \leq C\|\overrightarrow{b}\|_{BMO}M_r(f)(x) + \sum_{j=1}^{m}\sum_{\delta\in C_j^m}\|\overrightarrow{b}_\delta\|_{BMO}M_r(T^{\overrightarrow{b}_{\delta^c}}(f))(x).$$

This completes the proof of the theorem. □

Proof of Theorem 3.3. We choose $1 < r < p$ as in Theorem 3.2. We first consider the case $m = 1$ and have

$$\begin{aligned}||T^{b_1}(f)||_{L^p(w)} &\leq \|M(T^{b_1})(f)\|_{L^p(w)} \leq C\|(T^{b_1}(f))^{\#}\|_{L^p(w)}\\ &\leq C\|M_r(T(f))\|_{L^p(w)} + C\|M_r(f)\|_{L^p(w)}\\ &\leq C\|T(f)\|_{L^p(w)} + C\|M_r(f)\|_{L^p(w)}\\ &\leq C\|f\|_{L^p(w)} + C\|f\|_{L^p(w)}\\ &\leq C\|f\|_{L^p(w)}.\end{aligned}$$

When $m \geq 2$, we may get the conclusion of the theorem by induction. This completes the proof. □

References

[1] R. R. Coifman, R. Rochberg and G. Weiss, Fractorization theorems for Hardy spaces in several variables, *Ann. of Math.* **103** (1976), 611–635.
[2] J. Garcia-Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics, *North-Holland Math.* **16**, Amsterdam, 1985.
[3] D. S. Kurtz and R. L. Wheeden, Results on weighted norm inequalities for multiplies, *Trans. Amer. Math. Soc.* **255** (1979), 343–362.
[4] D. S. Kurtz, Sharp function estimates for fractional integrals and related operators, *J. Austral. Math. Soc.* **49** (A) (1990), 129–137.
[5] S. Z. Lu and D. C. Yang, Multiplier theorem for Herz type Hardy spaces, *Proc. Amer. Math. Soc.* **126** (1998), 3337–3346.
[6] B. Muckenhoupt, R. L. Wheeden and W. S. Young, Sufficient conditions for L^p multipliers with general weights, *Trans. Amer. Math. Soc.* **300** (2) (1987), 463–502.
[7] C. Pérez, Endpoint estimates for commutators of singular integral operators, *J. Funct. Anal.* **128** (1995), 163–185.
[8] M. Paluszyński, Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss, *Indiana Univ. Math. J.* **44** (1) (1995), 1–17.
[9] E. M. Stein, *Singular Integrals and Differentiability Properties of Function*, Princeton Univ. Press, Princeton NJ, 1970.
[10] E. M. Stein, *Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[11] J. O. Strömberg and A.Torkinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. Vol.1381, Springer Verlag, Berlin, 1989.
[12] Z. You, Results of commutators obtained norm inequalities, *Adv. in Math.* **17** (1) (1988), 79–84 (in Chinese).
[13] P. Zhang and J. C. Chen, The $(L^p, \dot{F}_p^{\beta,\infty})$-boundedness of commutators of multiplies, *Acta Math. Sinica* **21** (4) (2005), 765–772.
[14] P. Zhang and J. C. Chen, Boundedness properties for commutators of multiplies, *Acta Math. Sinica* **49** (6) (2006), 1387–1396 (Chinese Series).

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