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GLOBAL ATTRACTORS FOR A CLASS OF DEGENERATE PARABOLIC EQUATIONS

CUNG THE ANH AND PHAN QUOC HUNG

Abstract. The aim of this paper is to prove the existence of global solutions and global attractors for a class of semilinear degenerate parabolic equations in an arbitrary domain.

1. Introduction

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for dissipative dynamical system is to consider its global attractor. This is an invariant set that attracts all the trajectories of the system. The existence of the global attractor has been derived for a large class of PDEs (see [6, 12] and references therein). One of the most studied gradient partial differential equations is the reaction-diffusion equation, which models several physical phenomena like heat conduction, population dynamics, etc. There is an extensive literature concerning the existence and asymptotic behavior of solutions of reaction-diffusion equations and systems, both in bounded and in unbounded domains (see e.g. [2, 3, 6-9, 12, 13, 15, 16]). However, to the best of our knowledge, little seems to be known for the asymptotic behavior of solutions of degenerate equations.

In this paper we study the following semilinear degenerate parabolic equation with variable, nonnegative coefficients, defined on an arbitrary domain (bounded or unbounded) $\Omega \subset \mathbb{R}^N, N \geqslant 2$,

$$\begin{aligned} u_t - \operatorname{div}(\sigma(x)\nabla u) + f(u) + g(x) = 0, x \in \Omega, t > 0 \\ u(x,0) = u_0, x \in \Omega \\ u(x,t) = 0,\ x \in \partial\Omega, t > 0. \end{aligned} \tag{1.1}$$

Problem (1.1) can be derived as a simple model for neutron diffusion (feedback control of nuclear reactor) (see [5]). In this case u and σ stand for the neutron flux and neutron diffusion respectively.

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The degeneracy of problem (1.1) is considered in the sense that the measurable nonnegative diffusion coefficient $\sigma(x)$ is allowed to have at most a finite number of (essential) zeros at some points. Motivated by [4], where a degenerate elliptic problem is studied, we assume that the function $\sigma : \Omega \to \mathbb{R}$ satisfies the following assumptions

$(\mathcal{H}_\alpha)$ $\sigma \in L^1_{\text{loc}}(\Omega)$ and for some $\alpha \in (0,2), \liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$, when the domain Ω is bounded.

$(\mathcal{H}^\infty_\beta)$ σ satisfies condition $(\mathcal{H}_\alpha)$ and $\liminf_{|x|\to\infty} |x|^{-\beta}\sigma(x) > 0$ for some $\beta > 2$, when the domain Ω is unbounded.

The physical motivation of the assumption $(\mathcal{H}_\alpha)$ is related to the modelling of reaction diffusion processes in composite materials, occupying the bounded domain Ω, which at some points behave as *perfect insulator*. Following [5, p. 79], when at some points the medium is perfectly insulating, it is natural to assume that $\sigma(x)$ vanishes at these points. On the other hand, when condition $(\mathcal{H}^\infty_\beta)$ is satisfied, it is easy to see that the diffusion coefficient has to be unbounded. Physically, this situation corresponds to a nonhomogeneous medium, occupying the unbounded domain Ω, which behaves as a perfect conductor in $\Omega \setminus B_R(0)$ (see [5, p. 79]), and as a perfect insulator in a finite number of points in $B_R(0)$. Note that in various diffusion processes, the equation involves diffusion $\sigma(x) \sim |x|^\alpha, \alpha \in (0,2)$, in the case of a bounded domain, and $\sigma(x) \sim |x|^\alpha + |x|^\beta, \alpha \in (0,2), \beta > 2$, in the case of an unbounded domain.

Denote $A = -\text{div}(\sigma(x)\nabla)$, the positive and self-adjoint operator with domain of definition

$$D(A) = \{u \in \mathcal{D}^1_0(\Omega,\sigma) : Au \in L^2(\Omega)\}$$

(see Subsection 2.1) and define the corresponding Nemytski map f by

$$f(u)(x) = f(u(x)),\ u \in \mathcal{D}^1_0(\Omega,\sigma).$$

Then, the problem (1.1) can be formulated as an abstract evolutionary equation

$$\frac{du}{dt} + Au + f(u) + g = 0,\ u(0) = u_0. \tag{1.2}$$

The main purpose of this paper is to study the existence of a global attractor for the dynamic system generated by (1.1). We restrict ourselves to the case $N \geqslant 2$ for the coherence of presentation, since the case $N = 1$ is similar to the higher dimensional case with respect to the definition and properties of the appropriate functional setting.

Let us describe the organization and methods used in this paper. For the clarity, we first consider the case of a bounded domain in Section 3 and Section 4, for the weight function $\sigma(x)$ satisfying condition $(\mathcal{H}_\alpha)$. More precisely, in Section 3 we consider problem (1.1), where $g \in L^2(\Omega)$, $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions

$$|f(u) - f(v)| \leqslant C_1|u-v|(1 + |u|^\rho + |v|^\rho),\ 0 \leqslant \rho \leqslant \frac{2-\alpha}{N-2+\alpha}, \tag{1.3}$$

$$f(u)u \geqslant -\mu u^2 - C_2, \tag{1.4}$$

$$F(u) \geqslant -\frac{1}{2}\mu u^2 - C_2, \tag{1.5}$$

where $C_2 \geqslant 0$, F is the primitive $F(y) = \int\limits_0^y f(s)ds$ of f, $\mu < \lambda_1$, λ_1 is the first eigenvalue of the operator A in Ω with homogeneous Dirichlet condition (see Sec. 2.1). The main aim of Section 3 is to prove the existence of a global attractor in the space $\mathcal{D}_0^1(\Omega, \sigma)$. Firstly, under the assumption (1.3), one can check that the Nemytski f is a locally Lipschitzian map from $\mathcal{D}_0^1(\Omega, \sigma)$ to $L^2(\Omega)$. This combining with A being a sectorial operator guarantees the existence and uniqueness of a local solution. Then, by using the condition (1.5) and the remarkable fact that the equation admits a natural Lyapunov functional

$$\Phi(u) = \frac{1}{2}\|u\|^2_{\mathcal{D}_0^1} + \int_\Omega (F(u) + gu)dx, \tag{1.6}$$

we are able to prove that the solution exists globally in time. Besides, we also show that orbits of bounded sets are bounded. Finally, by proving the asymptotically compact property of the semigroup $S(t)$ generated by (1.1) and using the dissipativeness condition (1.4) for proving the boundedness of the set E of equilibrium points, we obtain the existence of a global attractor $\mathcal{A}$ in $\mathcal{D}_0^1(\Omega, \sigma)$.

Since the critical exponent of the embedding $\mathcal{D}_0^1(\Omega, \sigma) \hookrightarrow L^p(\Omega)$ is $2_\alpha^* = \frac{2N}{N-2+\alpha}$, condition $\rho \leqslant \frac{2-\alpha}{N-2+\alpha}$ in (1.3) is necessary to ensure that the Nemytski f is a map from $D_0^1(\Omega, \sigma)$ into $L^2(\Omega)$ and the Lyapunov function $\Phi(u(t))$ is well defined. These are basic tools for the approach method with respect to gradient systems. In Section 4, however, in the case $f(u)$ has the following form

$$f(u) = k|u|^\rho u + h(u),\ 0 \leqslant \rho \leqslant \frac{4 - 2\alpha}{N - 2 + \alpha} = 2_\alpha^* - 2, \tag{1.7}$$

where $k > 0$ and $h \in C(\mathbb{R})$ satisfies $|h(u)| \leqslant C(1 + |u|^\gamma)$, $\gamma \in [0, \rho + 1)$, and some certain conditions (see Sec. 4), we are able to prove the existence of a global attractor in the space $L^2(\Omega)$. Here we prove the existence of a global weak solution in $L^2(\Omega)$ via the Galerkin method (see [10]). As a result, we obtain the existence of an absorbing set of the semigroup $S(t)$ in $L^2(\Omega)$. Then we prove the asymptotic compactness of $S(t)$ in $L^2(\Omega)$, and we therefore obtain the existence of a global attractor of $S(t)$ in $L^2(\Omega)$. Finally, in Section 5 we give some similar results on the existence of global attractors in the case of an unbounded domain, for the weight function $\sigma(x)$ satisfying the condition $(\mathcal{H}_\beta^\infty)$.

2. Preliminary results

2.1. **Function spaces.** We recall some of the basic results on functional spaces defined in [4]. Let $N \geqslant 2, \alpha \in (0, 2)$, and

$$2_\alpha^* = \begin{cases} \frac{4}{\alpha} \in (2, +\infty) \text{ if } \alpha \in (0,2), N = 2, \\ \frac{2N}{N-2+\alpha} \in (2, \frac{2N}{N-2}) \text{ if } \alpha \in (0,2), N \geqslant 3. \end{cases}$$

The exponent 2^*_α has the role of the critical exponent in the classical Sobolev embedding. We have the following generalized version of Poincaré inequality ([4, Corollary 2.6]).

Lemma 2.1. *Let Ω be a bounded (unbounded) domain of $\mathbb{R}^N$, $N \geqslant 2$, and assume that condition $(\mathcal{H}_\alpha)$ ($(\mathcal{H}^\infty_\beta)$) is satisfied. Then there exists a constant $c > 0$, such that*

$$\int_\Omega |u|^2 dx \leqslant c \int_\Omega \sigma(x)|\nabla u|^2 dx, \text{ for every } u \in C_0^\infty(\Omega). \tag{2.1}$$

We emphasize that conditions $(\mathcal{H}_\alpha)$, $(\mathcal{H}^\infty_\beta)$ are optimal in the following sense: For $\alpha > 2$ there exist functions such that (2.1) is not satisfied (see [4]). Note also that in the case of an unbounded domain, (2.1) does not hold in general, if $\beta \leqslant 2$ in $(\mathcal{H}^\infty_\beta)$. We refer also to the examples of [1].

The natural energy space for the problem (1.1) involves the space $\mathcal{D}^1_0(\Omega,\sigma)$, defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}^1_0} := \Big(\int_\Omega \sigma(x)|\nabla u|^2 dx\Big)^{1/2}.$$

The space $\mathcal{D}^1_0(\Omega,\sigma)$ is a Hilbert space with respect to the scalar product

$$(u,v)_\sigma := \int_\Omega \sigma(x)\nabla u \nabla v dx.$$

The two following lemmas refer to the continuous and compact inclusion of $\mathcal{D}^1_0(\Omega,\sigma)$ ([4, Propositions 3.3-3.5]).

Lemma 2.2. *Assume that Ω is a bounded domain in $\mathbb{R}^N, N \geqslant 2$, and σ satisfies $(\mathcal{H}_\alpha)$. Then the following embeddings hold*

i) $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^{2^*_\alpha}(\Omega)$ *continuously,*
ii) $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^p(\Omega)$ *compactly if* $p \in [1, 2^*_\alpha)$.

Lemma 2.3. *Assume that Ω is an unbounded domain in $\mathbb{R}^N, N \geqslant 2$, and σ satisfies $(\mathcal{H}^\infty_\beta)$. Then the following embeddings hold*

i) $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^p(\Omega)$ *continuously for every* $p \in [2^*_\beta, 2^*_\alpha]$,
ii) $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^p(\Omega)$ *compactly if* $p \in (2^*_\beta, 2^*_\alpha)$.

Assuming condition $(\mathcal{H}_\alpha)$ or $(\mathcal{H}^\infty_\beta)$, the operator $A = -\mathrm{div}(\sigma(x)\nabla)$ is positive and self-adjoint, with domain of definition

$$D(A) = \{u \in \mathcal{D}^1_0(\Omega,\sigma) : Au \in L^2(\Omega)\}.$$

The space $D(A)$ is a Hilbert space endowed with the usual graph scalar product. Moreover, there exists a complete system of eigensolutions (e_j, λ_j),

$$-\mathrm{div}(\sigma(x)\nabla e_j) = \lambda_j e_j, j = 1, 2, \dots$$

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots, \quad \lambda_j \to \infty, \text{ as } j \to \infty.$$

The fraction powers are defined as follows: For every $s > 0$, A^s is a bounded self-adjoint operator in $L^2(\Omega)$, with domain $D(A^s)$ to be a dense subset in $L^2(\Omega)$. The operator A^s is strictly positive and injective. Also, $D(A^s)$ endowed with the scalar product $(u,v)_{D(A^s)} = (A^s u, A^s v)_{L^2}$, becomes a Hilbert space. We write as usual, $X^s = D(A^s)$ and we have the following identifications $D(A^{-1/2}) = \mathcal{D}_0^{-1}(\Omega,\sigma) =$ the dual of $\mathcal{D}_0^1(\Omega,\sigma)$, $D(A^0) = L^2(\Omega)$ and $D(A^{1/2}) = \mathcal{D}_0^1(\Omega,\sigma)$. Moreover, the injection $X^{s_1} \subset X^{s_2}, s_1, s_2 \in \mathbb{R}, s_1 > s_2$, is compact and dense.

2.2. **Existence of global attractors.** For convenience of the readers, we summarize some definitions and results of theory of infinite dimensional dynamical dissipative systems in [6, 12] which we will use later.

Let X be a metric space (not necessarily complete) with metric d. If $C \subset X$ and $b \in X$ we set $\rho(b,C) := \inf_{c\in C} d(b,c)$, and if $B \subset X, C \subset X$ we set $\operatorname{dist}(B,C) := \sup_{b\in B} \rho(b,C)$. Let $S(t)$ be a *continuous semigroup* on the metric space X.

A set $A \subset X$ is *invariant* if $S(t)A = A$, for any $t \geqslant 0$.

The positive orbit of $x \in X$ is the set $\gamma^+(x) = \{S(t)x | t \geqslant 0\}$. If $B \subset X$, the *positive orbit* of B is the set

$$\gamma^+(B) = \cup_{t\geqslant 0} S(t)B = \cup_{z\in B}\gamma^+(z).$$

More generally, for $\tau \geqslant 0$, we define the orbit after the time τ of B by

$$\gamma_\tau^+(B) = \gamma^+(S(\tau)B).$$

The subset $A \subset X$ *attracts* a set B if $\operatorname{dist}(S(t)B, A) \to 0$ as $t \to \infty$.

The subset A is a *global attractor* if A is closed, bounded, invariant, and attracts all bounded sets.

The semigroup $S(t)$ is *asymptotically compact* if, for any bounded subset B of X such that $\gamma_\tau^+(B)$ is bounded for some $\tau \geqslant 0$, every set of the form $\{S(t_n)z_n\}$, with $z_n \in B$ and $t_n \geqslant \tau, t_n \to +\infty$ as $n \to \infty$, is relatively compact.

The semigroup $S(t)$ is *point* (*bounded*) *dissipative* on X if there exists a bounded set $B_0 \subset X$, which attracts each point (bounded set, respectively) of X.

If the semigroup is bounded dissipative, there exists a bounded set $B_1 \subset X$ with the property that, for any bounded set $B \subset X$, there exists $\tau = \tau(B) \geqslant 0$ such that $\gamma_\tau^+(B) \subset B_1$. Such a set is called an *absorbing set* for $S(t)$.

We now state the fundamental theorem of existence of a compact global attractor (see [6, Theorem 3.4.6] or [12, Theorem 2.26]).

Theorem 2.1. *If $S(t), t \geqslant 0$, is asymptotically compact, point dissipative, and orbits of bounded sets are bounded, then there exists a compact global attractor $\mathcal{A}$. Moreover, if X is a Banach space, then $\mathcal{A}$ is connected.*

A continuous semigroup $S(t)$ is a *continuous gradient system* if there exists a function $\Phi \in C^0(X,\mathbb{R})$ such that $\Phi(S(t)u) \leqslant \Phi(u), \forall t \geqslant 0, \forall u \in X$, and

$\Phi(S(t)u) = \Phi(u), \forall t \geqslant 0$ implies that u is an equilibrium point, i.e. $S(t)u = u\ \forall t \geqslant 0$. The function Φ is called a strict Lyapunov functional.

Let E be the set of equilibrium points for the semigroup $S(t)$. We give the definition of the unstable set of E by

$$W^u(E) = \{y \in X : S(-t)y \text{ is defined for } t \geqslant 0 \text{ and } S(-t)y \to E \text{ as } t \to \infty\}.$$

From Proposition 2.19 and Theorem 4.6 in [12], we have

Theorem 2.2. *Let $S(t), t \geqslant 0$, be an asymptotically compact gradient system, which has the property that, for any bounded set $B \subset X$, there exists $\tau \geqslant 0$ such that $\gamma_\tau^+(B)$ is bounded. If the set of equilibrium points E is bounded, then $S(t)$ has a compact global attractor $\mathcal{A}$ and $\mathcal{A} = W^u(E)$. Moreover, if X is a Banach space, then $\mathcal{A}$ is connected.*

2.3. **Sectorial evolutionary equations.** Assume that A is a sectorial operator on X and there is an $\alpha \in [0, 1)$ such that $f : X^\alpha \to X$ is locally Lipschitz continuous. Consider the equation

$$\frac{du}{dt} + Au = f(u), \ \ t > 0, \ \ u(0) = u_0. \tag{2.2}$$

A solution of (2.2) on $[0, \tau)$ is a continuous function $u : [0, \tau) \to X^\alpha, u(0) = u_0$, such that $f(u(.)) : [0, \tau) \to X$ is a continuous function, $u(t) \in D(A)$ and u satisfies (2.2) on $(0, \tau)$. One can show that the solutions of (2.2) coincide with those solutions of the integral equation

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(u(s))ds, 0 \leqslant t < \tau, \tag{2.3}$$

for which $u : [0, \tau) \to X^\alpha$ is continuous and $f(u(.)) : [0, \tau) \to X$ is continuous.

We have the following result ([6, Theorem 4.2.1]).

Theorem 2.3. *Under the above hypotheses on A, f, there is a unique classical solution $u \in C^0([0, t_{\max}); X^\alpha) \cap C^1((0, t_{\max}); X) \cap C^0((0, t_{\max})); D(A))$ of (2.2) on a maximal interval of existence $[0, t_{\max}(u_0))$. If $t_{\max}(u_0) < \infty$, then there is a sequence $t_n \to t_{\max}^-(u_0)$ such that $\|u(t_n)\|_{X^\alpha} \to \infty$. If, in addition, f is a C^r-function in u, then the solution $u(t, u_0)$ is a C^r-function in (t, u_0) in the domain of definition of the function.*

3. Global attractors in $\mathcal{D}_0^1(\Omega, \sigma)$

From now on, for the sake of brevity we denote by $\|.\|$ the norm in $L^2(\Omega)$.

Theorem 3.1. *Assume that f satisfies the conditions (1.3), (1.5). Then for any $u_0 \in \mathcal{D}_0^1(\Omega, \sigma)$ given, the problem (1.1) has a unique global solution $u \in C([0, \infty); \mathcal{D}_0^1(\Omega, \sigma)) \cap C^1((0, \infty); L^2(\Omega)) \cap C((0, \infty); D(A))$. Moreover, for the solution u, $\Phi(u(t)) \in C^1((0, \infty))$ with*

$$\frac{d}{dt}\Phi(u(t)) = -\|u_t(t)\|^2, \ \ t \in (0, \infty).$$

Proof. Firstly, we prove that f is a map from $\mathcal{D}_0^1(\Omega,\sigma)$ to $L^2(\Omega)$. By (1.3) we have

$$|f(s)| \leqslant C(1+|s|^{\rho+1}).$$

Let $u \in \mathcal{D}_0^1(\Omega,\sigma)$, we have

$$\int_\Omega |f(u(x))|^2 dx \leqslant C\Big(1+\int_\Omega |u(x)|^{2\rho+2}dx\Big).$$

Since $2\rho+2 \leqslant 2_\alpha^*$, $\mathcal{D}_0^1(\Omega,\sigma)$ is continuously embedded in $L^{2\rho+2}(\Omega)$ and

$$\|f(u)\|^2 \leqslant C(1+\|u\|_{\mathcal{D}_0^1}^{2\rho+2}).$$

Therefore $f : \mathcal{D}_0^1(\Omega,\sigma) \to L^2(\Omega)$ and f maps bounded sets of $\mathcal{D}_0^1(\Omega,\sigma)$ to bounded sets of $L^2(\Omega)$.

Secondly, we prove that $f : \mathcal{D}_0^1(\Omega,\sigma) \to L^2(\Omega)$ is Lipschitz continuous on every bounded set of $\mathcal{D}_0^1(\Omega,\sigma)$. Let $u, v \in \mathcal{D}_0^1(\Omega,\sigma)$, $\|u\|_{\mathcal{D}_0^1} \leqslant r$, $\|u\|_{\mathcal{D}_0^1} \leqslant r$, we have

$$\int_\Omega |f(u(x))-f(v(x))|^2 dx \leqslant 2C\int_\Omega |u(x)-v(x)|^2(1+|u(x)|^{2\rho}+|v(x)|^{2\rho})dx.$$

Using Hölder inequality with $p=\dfrac{2_\alpha^*}{2}, q=\dfrac{2_\alpha^*}{2_\alpha^*-2}$, we get

$$\|f(u)-f(v)\|^2 \leqslant C\|u-v\|_{L^{2_\alpha^*}}^2(C_1+\|u\|_{L^{2\rho q}}^{2\rho}+\|v\|_{L^{2\rho q}}^{2\rho}).$$

Since $2\rho q \leqslant 2_\alpha^*$, $\mathcal{D}_0^1(\Omega,\sigma)$ is continuously embedded in $L^{2\rho q}(\Omega)$,

$$\|f(u)-f(v)\|^2 \leqslant C\|u-v\|_{\mathcal{D}_0^1}^2(C_1+\|u\|_{\mathcal{D}_0^1}^{2\rho}+\|v\|_{\mathcal{D}_0^1}^{2\rho}).$$

Therefore

$$\|f(u)-f(v)\| \leqslant M(r)\|u-v\|_{\mathcal{D}_0^1},\ \text{if } \|u\|_{\mathcal{D}_0^1} \leqslant r, \|v\|_{\mathcal{D}_0^1} \leqslant r.$$

Now, applying Theorem 2.3, the problem (1.1) has a unique local solution $u \in C([0,t_{max});\mathcal{D}_0^1(\Omega,\sigma)) \cap C^1((0,t_{max});L^2(\Omega)) \cap C((0,t_{max});D(A))$.

We now prove that the solution exists globally. Putting

$$\Phi(u) = \frac{1}{2}\|u\|_{\mathcal{D}_0^1}^2 + \int_\Omega (F(u)+gu)dx,$$

where $F(y)=\int\limits_0^y f(s)ds$. It is easy to check that

$$\frac{d}{dt}\Phi(u(t)) = -\|u_t(t)\|^2,\ t\in(0,t_{\max}).$$

Using hypothesis (1.5) and Cauchy inequality we get

$$\Phi(u(t)) \geqslant \frac{1}{2}\|u(t)\|_{\mathcal{D}_0^1}^2 - \frac{\mu}{2}\|u(t)\|^2 - C(\Omega) - \varepsilon\|u(t)\|^2 - \frac{1}{4\varepsilon}\|g\|^2.$$

By choosing ε small enough such that $\mu+2\varepsilon<\lambda_1$ we obtain

$$\Phi(u(0)) \geqslant \Phi(u(t)) \geqslant \frac{1}{2}(1-\frac{\mu+2\varepsilon}{\lambda_1})\|u\|_{\mathcal{D}_0^1}^2 - C.$$

Hence

$$\|u(t)\|_{\mathcal{D}_0^1} \leqslant M, \ \forall t \in [0, t_{max}).$$

This implies that $t_{\max} = +\infty$. Indeed, let $t_{\max} < +\infty$ and $\limsup_{t \to t_{\max}^-} \|u(t)\|_{\mathcal{D}_0^1} < +\infty$. Then there exist a sequence $(t_n)_{n \geqslant 1}$ and a constant K such that $t_n \to t_{\max}^-$, as $n \to +\infty$ and $\|u(t_n)\|_{\mathcal{D}_0^1} < K, \ n = 1, 2, \ldots$. As we have already shown above, for each $n \in \mathbb{N}$ there exists a unique solution of the problem (1.1) with initial data $u(t_n)$ on $[t_n, t_n + T^*]$, where $T^* > 0$ depending on K and independent of $n \in \mathbb{N}$. Thus, we can get $t_{\max} < t_n + T^*$, for $n \in \mathbb{N}$ large enough. This contradicts the maximality of $t_{\max}$ and the proof of Theorem 3.1 is complete. □

In order to prove the asymptotic compactness of the semigroup $S(t)$, we first note that $A = -\mathrm{div}(\sigma(x)\nabla)$ is a sectorial operator in the space $X = L^2(\Omega)$ with the fractional power spaces X^α. From the fact that $X^{1/2} = \mathcal{D}_0^1(\Omega, \sigma), \ X^0 = L^2(\Omega)$ (see Sec. 2.1) and the properties of the sectorial operator (see e.g. [7]) we have the analytic semigroup e^{-tA} generated by the operator A satisfying the following estimates

$$\|e^{-At}u\| \leqslant Me^{at}\|u\|, \text{ for all } u \in L^2(\Omega) \text{ and all } t > 0, \tag{3.1}$$

$$\|e^{-At}u\|_{\mathcal{D}_0^1} \leqslant Me^{at}t^{-1/2}\|u\|, \text{ for all } u \in L^2(\Omega) \text{ and all } t > 0, \tag{3.2}$$

where M and a are two positive constants. Furthermore, we need the following lemma (see [7, Chapter 7]).

Lemma 3.1. *Assume that $\varphi(t)$ is a continuous nonnegative function on the interval $(0, T)$ such that*

$$\varphi(t) \leqslant c_0 t^{-\gamma_0} + c_1 \int_0^t (t-s)^{-\gamma_1}\varphi(s)ds, \ t \in (0, T),$$

where $c_0, c_1 \geqslant 0$ and $0 \leqslant \gamma_0, \gamma_1 < 1$. Then there exists a constant $K = K(\gamma_1, c_1, T)$ such that

$$\varphi(t) \leqslant \frac{c_0}{1-\gamma_0} t^{-\gamma_0} K(\gamma_1, c_1, T), \ t \in (0, T).$$

Theorem 3.2. *Under the conditions* (1.3) − (1.5)*, the semigroup $S(t)$ generated by* (1.1) *has a compact connected global attractor $\mathcal{A} = W^u(E)$ in $\mathcal{D}_0^1(\Omega, \sigma)$.*

Proof. Firstly, from the proof of Theorem 3.1 we see that $\gamma^+(B)$ is bounded for any bounded subset B of $\mathcal{D}_0^1(\Omega, \sigma)$ and the function Φ defined by (1.6) is a strict Lyapunov functional.

Notice that the set of equilibrium points

$$E = \{z | Az + f(z) + g = 0\}.$$

Let $z \in E$, we have

$$0 = \|z\|^2_{\mathcal{D}_0^1} + \int_\Omega (f(z)z + gz)dx.$$

Using hypothesis (1.4) and Cauchy inequality we obtain that

$$\|z\|_{\mathcal{D}_0^1} \leqslant M, \text{ for all } z \in E,$$

i.e. E is bounded in $\mathcal{D}_0^1(\Omega,\sigma)$. Thus, in order to prove the existence of the global attractor, we only need to prove that $S(t)$ is asymptotically compact in $\mathcal{D}_0^1(\Omega,\sigma)$.

Let $(u_n)_{n\geqslant 1}$ be a bounded sequence in $\mathcal{D}_0^1(\Omega,\sigma)$ and $t_n \to +\infty$. Fix $T>0$, since $\{u_n\}$ is bounded and orbits of bounded sets are bounded, $\{S(t_n-T)u_n\}$ is bounded in $\mathcal{D}_0^1(\Omega,\sigma)$. Because $\mathcal{D}_0^1(\Omega,\sigma)$ is compactly embedded in $L^2(\Omega)$, there are subsequence $\{S(t_{n_k}-T)u_{n_k}\}$ and $v \in \mathcal{D}_0^1(\Omega,\sigma)$ such that $v_k = S(t_{n_k}-T)u_{n_k} \rightharpoonup v$ weakly in $\mathcal{D}_0^1(\Omega,\sigma)$ and $v_k \to v$ strongly in $L^2(\Omega)$ as $k\to\infty$. We will prove that $S(t_{n_k})u_{n_k} = S(T)v_k$ converges strongly to $S(T)v$ in $\mathcal{D}_0^1(\Omega,\sigma)$, and thus $S(t)$ is asymptotically compact.

Denote $v_k(t) = S(t)v_k, v(t) = S(t)v$, we have

$$v_k(t) = e^{-At}v_k + \int_0^t e^{-A(t-s)}(-f(v_k(s)) - g)ds,$$

$$v(t) = e^{-At}v + \int_0^t e^{-A(t-s)}(-f(v(s)) - g)ds.$$

It follows from (3.1) and (3.2) that for $t\in(0,T]$, we have

$$\|v_k(t)-v(t)\|_{\mathcal{D}_0^1} \leqslant Me^{aT}t^{-1/2}\|v_k - v\| + MC\int_0^t (t-s)^{-1/2}\|v_k(s)-v(s)\|_{\mathcal{D}_0^1}ds.$$

By the singular Gronwall inequality (see Lemma 3.1), there is a constant C_1 such that, for $t\in(0,T]$,

$$\|v_k(t)-v(t)\|_{\mathcal{D}_0^1} \leqslant C_1 Me^{aT}t^{-1/2}\|v_k - v\|,$$

in particular,

$$\|v_k(T)-v(T)\|_{\mathcal{D}_0^1} \leqslant C_2 T^{-1/2}\|v_k - v\|.$$

Since $v_k \to v$ in $L^2(\Omega)$, $v_k(T)\to v(T)$ in $\mathcal{D}_0^1(\Omega,\sigma)$ as $k\to+\infty$. This implies that $S(t)$ is asymptotically compact. Applying Theorem 2.2, we obtain the conclusion of the theorem. □

4. Global attractors in $L^2(\Omega)$

If the exponent ρ in (1.3) is larger than $\frac{2-\alpha}{N-2+\alpha}$, we now no longer have that $f : \mathcal{D}_0^1(\Omega,\sigma) \to L^2(\Omega)$, so the methods of Section 3 do not apply. In this section, we prove the existence of a global attractor in $L^2(\Omega)$ in the case f has the following form

$$f(u) = k|u|^\rho u + h(u),\ 0\leqslant \rho \leqslant \frac{4-2\alpha}{N-2+\alpha}, \tag{4.1}$$

where $k > 0$ and $h \in C(\mathbb{R})$ with $|h(u)| \leqslant C(1+|u|^\gamma)$ for $\gamma \in [0, \rho+1)$. Moreover, we assume that

$$(f(u)-f(v))(u-v) \geqslant -C(u-v)^2,\ C>0. \tag{4.2}$$

From Hölder inequality $|h(u)u| \leqslant C(\varepsilon) + \varepsilon|u|^{\rho+2}$, by choosing ε small enough we see that f satisfies the dissipativeness condition

$$f(u)u \geqslant -C. \tag{4.3}$$

In Section 3, the local in time solvability was discussed via the semigroup method. This method does not work in this case. However, we may use the Galerkin method to prove the existence of a global weak solution. Firstly, we introduce the definition of weak solutions of problem (1.1).

Definition 4.1. For a given $u_0 \in L^2(\Omega)$ and $T > 0$, a weak solution on $(0,T)$ for the problem (1.1) is a function

$$u \in L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma)) \cap L^{\rho+2}(0,T;L^{\rho+2}(\Omega))$$

satisfying for every $\eta \in C_0^\infty((0,T)\times\Omega)$ the weak formula

$$\begin{aligned} \big\langle \partial_t u, \eta \big\rangle + \int_\Omega \sigma(x)\nabla u \nabla \eta dx + \big\langle f(u)+g, \eta\big\rangle &= 0, \\ u(x,0) &= u_0. \end{aligned} \tag{4.4}$$

Here the symbol $\langle , \rangle$ denotes the pairing between $\mathcal{D}_0^{-1}(\Omega,\sigma)$ and $\mathcal{D}_0^1(\Omega,\sigma)$.

The weak form (4.4) is also equivalent to the following initial value problem

$$\begin{aligned} u_t + Au + f(u) + g &= 0, \\ u(0) &= u_0. \end{aligned} \tag{4.5}$$

Theorem 4.1. *Let $u_0 \in L^2(\Omega)$ and the conditions (4.1) − (4.2) be fulfilled. Then the problem (1.1) has a unique global in time weak solution $u(t)$ satisfying*

$$u \in C([0,\infty);L^2(\Omega)) \cap L^2_{loc}(0,\infty;\mathcal{D}_0^1(\Omega,\sigma)) \cap L^p_{loc}(0,\infty;L^p(\Omega)),$$

with $u(0) = u_0$ and $p = \rho + 2$.

Proof. Firstly, we prove the uniqueness of solutions. If u and v are two solutions of (1.1) with the same intial data $u_0 \in L^2(\Omega)$, then $\omega = u-v$ satisfies the equation

$$\begin{aligned} \omega_t + A\omega + f(u) - f(v) &= 0, \\ \omega(0) &= 0. \end{aligned} \tag{4.6}$$

Multiplying (4.6) by ω and intergrating over Ω we obtain

$$\frac{d}{dt}\|\omega(t)\|^2 + 2\|\omega(t)\|^2_{\mathcal{D}_0^1} + 2\int_\Omega (f(u)-f(v))(u-v)dx = 0.$$

Using (4.2) we get

$$\frac{d}{dt}\|\omega(t)\|^2 \leqslant C\|\omega(t)\|^2.$$

By Gronwall inequality we obtain $\|\omega(t)\| = 0, \forall t \geqslant 0$, i.e. $u = v$.

Let $\{e_i\}$ be the complete set of orthonormal eigenvectors of the operator $A = -\mathrm{div}(\sigma(x)\nabla)$ with eigenvalues $\{\lambda_i\}$, where $\{\lambda_i\}$ is increasing. For each integer $m \geqslant 1$, let $H_m = \mathrm{span}\{e_1, \dots, e_m\}$. Choose a sequence $\{u_{0m}\}$ in H_m such that

$$u_{0m} \to u_0 \text{ in } L^2(\Omega) \text{ as } m \to \infty.$$

We look for an approximate solution of the form $u_m(t) = \sum_{n=1}^{m} b_{im}(t)e_i$ in H_m, where coefficients $b_{im}(t)$ satisfy

$$\int_\Omega \frac{d}{dt}u_m(t)e_i dx + \int_\Omega \sigma(x)\nabla u_m(t)\nabla e_i dx + \int_\Omega (f(u_m(t)) + g)e_i dx = 0, \tag{4.7}$$

where $i = 1, \dots, m$

$$u_m(0) = u_{0m}. \tag{4.8}$$

According to the theory of ODEs, there is a unique solution $b_{im}(t)$ satisfying (4.7) - (4.8) on $[0, \tau_m)$. Therefore, we obtain a sequence of approximate solutions $u_m(t)$.

In (4.7), replacing e_i by $u_m(t)$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + \|u_m(t)\|^2_{\mathcal{D}_0^1} + k\int_\Omega |u_m(t)|^p dx \\ + \int_\Omega (h(u_m(t))u_m(t) + gu_m(t))dx = 0.\end{aligned}$$

By Cauchy inequality we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + \|u_m(t)\|^2_{\mathcal{D}_0^1} + k\int_\Omega |u_m(t)|^p dx \\ &\leqslant \int_\Omega [-h(u_m(t))u_m(t) + u_m^2(t)]dx + \|g\|^2.\end{aligned}$$

It follows from (4.1) and Hölder inequality that

$$(-h(u_m(t))u_m(t) + u_m^2(t)) \leqslant C(\varepsilon) + \varepsilon|u_m(t)|^p.$$

Thus, by choosing $\varepsilon = k/2$ we get

$$\frac{d}{dt}\|u_m(t)\|^2 + 2\|u_m(t)\|^2_{\mathcal{D}_0^1} + k\int_\Omega |u_m(t)|^p dx \leqslant R. \tag{4.9}$$

Integrating the above inequality over $[0, t] \subset [0, \tau_m)$ we get

$$\|u_m(t)\|^2 + 2\int_0^t \|u_m(t)\|^2_{\mathcal{D}_0^1} dt + k\int_0^t\int_\Omega |u_m(t)|^p dxdt \leqslant \|u_{0m}\|^2 + Ct.$$

This implies that the solution $u_m(t)$ can be extended to $[0, T)$ for $T > 0$ and we have the priori estimates

$$\|u_m\|_{L^\infty(0,T;L^2(\Omega))} \leqslant R,\ \|u_m\|_{L^2(0,T;\mathcal{D}_0^1)} \leqslant R,\ \|u_m\|_{L^p(0,T;L^p(\Omega))} \leqslant R. \tag{4.10}$$

It follows from (4.10) that

$$\|f(u_m)\|_{L^q(0,T;L^q(\Omega))} \leqslant R, \text{ where } q = \frac{p}{p-1}.$$

Therefore

$$\|\partial_t u_m\|_Y \leqslant R, Y = L^2(0,T;\mathcal{D}_0^{-1}(\Omega,\sigma)) + L^q(0,T;L^q(\Omega)).$$

$\mathcal{D}_0^1(\Omega,\sigma)$ is continuously embedded in $L^p(\Omega)$, so $L^q(\Omega)$ is continuously embedded in $\mathcal{D}_0^{-1}(\Omega,\sigma)$. Thus, $L^q(0,T;L^q(\Omega)) \hookrightarrow L^q(0,T;\mathcal{D}_0^{-1}(\Omega,\sigma))$ is continuous. Therefore we have

$$\|\partial_t u_m\|_{L^q(0,T,\mathcal{D}_0^{-1})} \leqslant R. \tag{4.11}$$

Now we may apply the results of [14] to obtain that $\{u_m\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$. Therefore we may extract a subsequence, still denoted by u_m, such that

$$\begin{aligned}
&u_m \stackrel{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega))\\
&u_m \longrightarrow u \text{ in } L^2(0,T;L^2(\Omega))\\
&u_m \rightharpoonup u \text{ in } L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma)) \cap L^p(0,T;L^p(\Omega))\\
&\partial_t u_m \rightharpoonup \partial_t u \text{ in } L^q(0,T;\mathcal{D}_0^{-1}(\Omega,\sigma)).
\end{aligned} \tag{4.12}$$

By passing to the limit in the weak form, we obtain the solution u of the problem. Because $u \in L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma))$ and $u_t \in L^q(0,T;\mathcal{D}_0^{-1}(\Omega,\sigma))$, $u \in C([0,T];\mathcal{D}_0^{-1}(\Omega,\sigma))$. In addition, since $u \in L^\infty(0,T;L^2(\Omega))$, we have

$$u \in L^\infty(0,T;L^2(\Omega)) \cap C([0,T];\mathcal{D}_0^{-1}(\Omega,\sigma)) = C_w([0,T];L^2(\Omega)]$$

(see [11, Lemma 8.1, p. 275]). This and the continuity of the L^2-norm, imply that $u \in C([0,T];L^2(\Omega))$.

We now prove the global existence of the solution. Analogously to (4.9) we have

$$\frac{d}{dt}\|u(t)\|^2 + 2\|u(t)\|^2_{\mathcal{D}_0^1} + k\int_\Omega |u(t)|^p dx \leqslant R.$$

Therefore

$$\frac{d}{dt}\|u(t)\|^2 + 2\lambda_1\|u(t)\|^2 \leqslant R.$$

By Gronwall inequality we get

$$\|u(t)\|^2 \leqslant e^{-2\lambda_1 t}\|u_0\|^2 + \frac{R}{2\lambda_1}(1 - e^{-2\lambda_1 t}). \tag{4.13}$$

This implies that the solution u exists globally. □

Let $S(t)$ be the semigroup in $L^2(\Omega)$ generated by (1.1). From (4.13), by choosing $R_1 > \frac{R}{2\lambda_1}$, the ball $\mathcal{B}_0 = B_{L^2}(0,R_1)$ is an absorbing set of $S(t)$ in $L^2(\Omega)$. This implies that the semigroup $S(t)$ is dissipative and eventually bounded. Therefore, in order to prove the existence of a global attractor of $S(t)$ in $L^2(\Omega)$, we only have to prove that $S(t)$ is asymptotically compact in $L^2(\Omega)$. Firstly, we need the following lemma.

Lemma 4.1. *The semigroup $S(t)$ associated to the problem* (1.1) *is weakly continuous on $L^2(\Omega)$.*

Proof. Assume that u_{0m} is a sequence in $L^2(\Omega)$ such that $u_{0m} \rightharpoonup u_0$ in $L^2(\Omega)$. Let $u_m(t) = S(t)u_{0m}$ and $\widetilde{u}(t) = S(t)u_0$. By the similar argument in (4.10) and (4.12), we may extract a subsequence (still denoted by) u_m such that

$$
\begin{aligned}
&u_m \rightharpoonup u \text{ in } L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma)) \cap L^p(0,T;L^p(\Omega))\\
&u_m \to u \text{ in } L^2(0,T;L^2(\Omega))\\
&\partial_t u_m \rightharpoonup \partial_t u \text{ in } L^q(0,T;\mathcal{D}_0^{-1}(\Omega,\sigma))\\
&A(u_m) \rightharpoonup z \text{ in } L^2(0,T;\mathcal{D}_0^{-1}(\Omega,\sigma))\\
&f(u_m) \rightharpoonup \omega \text{ in } L^q(0,T;L^q(\Omega)),
\end{aligned}
\tag{4.14}
$$

and the weak limit u satisfies the equation

$$
\partial_t u = -z - \omega - g. \tag{4.15}
$$

Putting

$$
I_m = \int_0^t s\langle Au_m + f(u_m) - A\psi - f(\psi), u_m - \psi\rangle ds
$$

for any $\psi \in L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma)) \cap L^p(0,T;L^p(\Omega))$, it follows from the strong monotonicity of A and (4.2) that

$$
I_m \geqslant \int_0^t s\langle f(u_m) - f(\psi), u_m - \psi\rangle \geqslant -C\int_0^t s\|u_m - \psi\|^2 ds.
$$

Since $\partial_t u_m = -Au_m - f(u_m) - g$ then

$$
\begin{aligned}
I_m &= \int_0^t s\langle Au_m + f(u_m), u_m\rangle ds - \int_0^t s\langle Au_m + f(u_m), \psi\rangle ds\\
&\quad - \int_0^t s\langle A\psi + f(\psi), u_m - \psi\rangle ds\\
&= -\frac{1}{2}t\|u_m(t)\|^2 + \frac{1}{2}\int_0^t \|u_m(s)\|^2 ds - \int_0^t s\langle g, u_m\rangle ds\\
&\quad - \int_0^t s\langle Au_m + f(u_m), \psi\rangle ds - \int_0^t s\langle A\psi + f(\psi), u_m - \psi\rangle ds.
\end{aligned}
$$

Using (4.14) to pass to the limit, we obtain

$$
\begin{aligned}
&-\frac{1}{2}t\|u(t)\|^2 + \frac{1}{2}\int_0^t \|u(s)\|^2 ds - \int_0^t s\langle g, u\rangle ds\\
&\quad - \int_0^t s\langle z + \omega, \psi\rangle ds - \int_0^t s\langle A\psi + f(\psi), u - \psi\rangle ds \geqslant -C\int_0^t s\|u - \psi\|^2 ds.
\end{aligned}
$$

Hence

$$\begin{aligned}
&\frac{1}{2}t\|u(t)\|^2 - \frac{1}{2}\int_0^t \|u(s)\|^2 ds \\
&\leqslant -\int_0^t s\langle g, u\rangle ds - \int_0^t s\langle z+\omega, \psi\rangle ds - \int_0^t s\langle A\psi + f(\psi), u-\psi\rangle ds \\
&\quad + C\int_0^t s\|u-\psi\|^2 ds. \qquad (4.16)
\end{aligned}$$

It follows from (4.15) that

$$-\int_0^t s\langle z+\omega+g, u\rangle ds = \int_0^t s\langle \partial_s u, u\rangle ds = \frac{1}{2}t\|u(t)\|^2 - \frac{1}{2}\int_0^t \|u(s)\|^2 ds. \qquad (4.17)$$

Combining (4.16) and (4.17) we get

$$\int_0^t s\langle z+\omega - A\psi - f(\psi), u-\psi\rangle ds \geqslant -C\int_0^t s\|u-\psi\|^2 ds.$$

Choosing $\psi = u - \theta v$, for some $\theta \in (0,1]$ and

$$v \in L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma)) \cap L^p(0,T;L^p(\Omega)),$$

we have

$$\int_0^t s\langle z+\omega - A(u-\theta v) - f(u-\theta v), v\rangle ds \geqslant -C\theta\int_0^t s\|v\|^2 ds.$$

Using Lebesgue dominated convergence theorem, let $\theta \to 0$ we obtain

$$\int_0^t s\langle z+\omega - A(u) - f(u), v\rangle ds \geqslant 0. \qquad (4.18)$$

Since (4.18) holds for any $v \in L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma)) \cap L^p(0,T;L^p(\Omega))$, we have

$$z+\omega = A(u) + f(u).$$

Hence and from (4.15) it follows that

$$\partial_t u = -A(u) - f(u) - g.$$

Therefore u is a solution of the problem (1.1) with $u(0) = u_0$. By the uniqueness of solutions we have $u = \widetilde{u}$.

For any subsequence $S(t)u_{m_k}$ of the sequence $S(t)u_m$, which converges weakly in $L^2(\Omega)$, we have $S(t)u_{m_k} \rightharpoonup S(t)u$ in $L^2(\Omega)$ as $k \to \infty$. Therefore, by Proposition 21.23 in [17, p. 258], the whole sequence $S(t)u_m \rightharpoonup S(t)u$ in $L^2(\Omega)$. This completes the proof. □

Lemma 4.2. *The semigroup $S(t)$ associated to the problem* (1.1) *is asymptotically compact on $L^2(\Omega)$.*

Proof. Let (u_m) be a sequence such that $\|u_m\|^2 \leqslant M$, and $t_n \to +\infty$. Since $S(t)$ is dissipative, there exists $t(M)$ such that $\{S(t)u_m\} \subset \mathcal{B}_0 = B_{L^2}(0,R_1)$, for every

$t \geqslant t(M)$. Let $T > 0$ we can extract a subsequence (still denoted by) u_m such that

$$S(t_m)u_m \rightharpoonup \psi \text{ for some } \psi \in \mathcal{B}_0 \text{ in } L^2(\Omega),$$
$$S(t_m - T)u_m \rightharpoonup \psi^* \text{ for some } \psi^* \in \mathcal{B}_0 \text{ in } L^2(\Omega).$$

It follows from the weak continuity of $S(t)$ that

$$S(T)\psi^* = \psi, \tag{4.19}$$

and we get

$$\liminf_{m\to\infty} \|S(t_m)u_m\| \geqslant \|\psi\|. \tag{4.20}$$

For any solution $u(t) = S(t)u_0$ we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u\|^2_{\mathcal{D}_0^1} + \gamma\|u\|^2 = -I_1(u) - I_2(u), \tag{4.21}$$

where $I_1(u) := \int_\Omega (f(u) + C)udx$, $\;I_2(u) := \int_\Omega (g - \gamma u - C)udx$, $\gamma > 0$, C is the constant in (4.3) ($f(u)u \geqslant -C$). By the variation of constants formula and (4.21) we get

$$\begin{aligned}\|S(t)u_0\|^2 = \|u_0\|^2 e^{-2\gamma t} &- 2\int_0^t e^{-2\gamma(t-s)}\|u(s)\|^2_{\mathcal{D}_0^1} ds \\ &- 2\int_0^t e^{-2\gamma(t-s)} I_1(u(s))ds - 2\int_0^t e^{-2\gamma(t-s)} I_2(u(s))ds.\end{aligned} \tag{4.22}$$

Hence

$$\begin{aligned}\|S(t_m)u_m\|^2 &= \|S(T)S(t_m - T)u_m\|^2 \\ &= e^{-2\gamma T}\|S(t_m - T)u_m\|^2 - 2\int_0^T e^{-2\gamma(T-s)}\|S(s)S(t_m - T)u_m\|^2_{\mathcal{D}_0^1} ds \\ &\quad - 2\int_0^T e^{-2\gamma(T-s)} I_1(S(s)S(t_m - T)u_m)ds \\ &\quad - 2\int_0^T e^{-2\gamma(T-s)} I_2(S(s)S(t_m - T)u_m)ds.\end{aligned} \tag{4.23}$$

For m large enough, the first term on the right hand side of (4.23) can be estimated as

$$e^{-2\gamma T}\|S(t_m - T)u_m\|^2 \leqslant R_1^2 e^{-2\gamma T}. \tag{4.24}$$

For every $s \in [0, T]$, since $S(s)$ is weakly continuous in $L^2(\Omega)$, we have

$$S(s)(S(t_m - T)u_m) \rightharpoonup S(s)\psi^* \text{ in } L^2(\Omega).$$

Analogously to (4.14) and by the uniqueness of weak convergence we can assume that

$$S(t)S(t_m - T)u_m \rightharpoonup S(t)\psi^* \text{ in } L^p(0,T;L^p(\Omega)), \tag{4.25}$$

$$S(t)S(t_m - T)u_m \rightharpoonup S(t)\psi^* \text{ in } L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma)), \tag{4.26}$$

$$S(t)S(t_m - T)u_m \to S(t)\psi^* \text{ in } L^2(0,T;L^2(\Omega)), \tag{4.27}$$

$$S(t)S(t_m - T)u_m \to S(t)\psi^* \text{ a.e. in } Q_T. \tag{4.28}$$

Since the norm

$$\Big(\int_0^T e^{-2\gamma(T-s)}\|u(s)\|^2_{\mathcal{D}_0^1} ds\Big)^{1/2}$$

is equivalent to the norm in $L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma))$, from (4.26) we have

$$\int_0^T e^{-2\gamma(T-s)}\|S(s)\psi^*\|^2_{\mathcal{D}_0^1} ds \leqslant \liminf_{m\to\infty} \int_0^T e^{-2\gamma(T-s)}\|S(s)S(t_m - T)u_m)\|^2_{\mathcal{D}_0^1} ds.$$

Therefore

$$\begin{aligned} &\limsup_{m\to\infty}\left(-2\int_0^T e^{-2\gamma(T-s)}\|S(s)S(t_m - T)u_m)\|^2_{\mathcal{D}_0^1} ds\right)\\ &\leqslant\ -2\int_0^T e^{-2\gamma(T-s)}\|S(s)\psi^*\|^2_{\mathcal{D}_0^1} ds. \end{aligned} \tag{4.29}$$

Putting

$$\begin{aligned} v_m(s) &= e^{-\gamma(T-s)}\,((f(S(s)S(t_m - T)u_m)\,S(s)S(t_m - T)u_m + C)^{1/2},\\ v(s) &= e^{-\gamma(T-s)}\,((f(S(s)\psi^*)S(s)\psi^* + C)^{1/2}, \end{aligned}$$

it follows from (4.25) and (4.28) that (v_m) is bounded in $L^2(0,T;L^2(\Omega))$, $v \in L^2(0,T;L^2(\Omega))$ and v_m converges almost everywhere to v in Q_T. Thus, by Lemma 1.3 [10, p. 25] , $v_m \rightharpoonup v \in L^2(0,T;L^2(\Omega))$. Hence

$$\liminf_{m\to\infty} \|v_m\|^2_{L^2(0,T;L^2(\Omega))} \geqslant \|v\|^2_{L^2(0,T;L^2(\Omega))}.$$

Therefore

$$\begin{aligned} &\limsup_{m\to\infty}\left(-2\int_0^T e^{-2\gamma(T-s)} I_1(S(s)S(t_m - T)u_m) ds\right)\\ &\leqslant\ -2\int_0^T e^{-2\gamma(T-s)} I_1(S(s)\psi^*) ds. \end{aligned} \tag{4.30}$$

From (4.27) we have

$$\begin{aligned} &\lim_{m\to\infty}\left(-2\int_0^T e^{-2\gamma(T-s)} I_2(S(s)S(t_m - T)u_m) ds\right)\\ &=\ -2\int_0^T e^{-2\gamma(T-s)} I_2(S(s)\psi^*) ds. \end{aligned} \tag{4.31}$$

It follows from (4.23), (4.24), (4.29), (4.30) and (4.31) that

$$\begin{aligned}
&\limsup_{m\to\infty} \|S(t_m)u_m\|^2 \\
&\leqslant R_1^2 e^{-2\gamma T} - 2\int_0^T e^{-2\gamma(T-s)}\|S(s)\psi^*\|^2_{\mathcal{D}_0^1} ds \\
&\quad - 2\int_0^T e^{-2\gamma(T-s)} I_1(S(s)\psi^*)ds - 2\int_0^T e^{-2\gamma(T-s)} I_2(S(s)\psi^*)ds.
\end{aligned} \tag{4.32}$$

Applying (4.22) with $\psi = S(T)\psi^*$ we have

$$\begin{aligned}
&\|\psi\|^2 = \|S(T)\psi^*\|^2 \\
&= \|\psi^*\|^2 e^{-2\gamma T} - 2\int_0^T e^{-2\gamma(T-s)}\|S(s)\psi^*\|^2_{\mathcal{D}_0^1} ds \\
&\quad - 2\int_0^T e^{-2\gamma(T-s)} I_1(S(s)\psi^*)ds - 2\int_0^T e^{-2\gamma(T-s)} I_2(S(s)\psi^*)ds.
\end{aligned} \tag{4.33}$$

Combining (4.32) with (4.33) we get

$$\begin{aligned}
\limsup_{m\to\infty} \|S(t_m)u_m\|^2 &\leqslant \|\psi\|^2 + R_1^2 e^{-2\gamma T} - \|\psi^*\|^2 e^{-2\gamma T} \\
&\leqslant \|\psi\|^2 + R_1^2 e^{-2\gamma T}.
\end{aligned}$$

Letting $T \to \infty$ we have

$$\limsup_{m\to\infty} \|S(t_m)u_m\|^2 \leqslant \|\psi\|^2. \tag{4.34}$$

From (4.20) and (4.34) we get

$$\lim_{m\to\infty} \|S(t_m)u_m\|^2 = \|\psi\|^2.$$

This together with the fact that $S(t_m)u_m \rightharpoonup \psi$, implies that $S(t_m)u_m \to \psi$ strongly in $L^2(\Omega)$, i.e. $S(t)$ is asymptotically compact. Lemma 4.2 is proved. □

Combining the above results, by Theorem 2.1, we have the following

Theorem 4.2. *Under the conditions* (4.1) – (4.2), *the semigroup associated to* (1.1) *possesses a compact connected global attractor* $\mathcal{A} = \omega(\mathcal{B}_0)$ *in* $L^2(\Omega)$.

5. Remarks on the case of an unbounded domain

In this section we discuss the case of an unbounded domain $\Omega \subset \mathbb{R}^N, N \geqslant 2$, for the weight function $\sigma(x)$ we assume that it satisfies the condition $(\mathcal{H}_\beta^\infty)$. From Subsection 2.1 we see that, for condition $(\mathcal{H}_\beta^\infty)$, the operator $A = -\mathrm{div}(\sigma(x)\nabla)$ has the same properties as in the case of a bounded domain (in particular, A is still a sectorial operator in $L^2(\Omega)$). On the other hand, we still have the continuous embedding $\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^{2_\alpha^*}(\Omega)$, and in particular the embedding $\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^2(\Omega)$ is compact. Therefore, we may apply the methods used in the case of a bounded domain to this case with some small changes on the conditions of the nonlinear term $f(u)$.

More precisely, in order to prove the existence of global attractors in $\mathcal{D}_0^1(\Omega,\sigma)$, we assume that the nonlinear term $f(u)$ satisfies the following conditions

$$(5.1)\qquad |f(u)-f(v)| \leqslant C|u-v|(1+|u|^\rho+|v|^\rho),\ 0\leqslant \rho \leqslant \frac{2-\alpha}{N-2+\alpha},$$

$$(5.2)\qquad f(0)=0,\ f(u)u\geqslant -\mu u^2,\ F(u)\geqslant -\frac{1}{2}\mu u^2,\ \mu<\lambda_1.$$

We may now repeat the arguments used in Section 3 to conclude the following

Theorem 5.1. *Under the conditions* (H_β^∞) *and* (5.1) – (5.2)*, problem* (1.1) *defines a semigroup* $S(t):\mathcal{D}_0^1(\Omega,\sigma)\to\mathcal{D}_0^1(\Omega,\sigma)$*, which possesses a compact connected global attractor* $\mathcal{A}=W^u(E)$ *in* $\mathcal{D}_0^1(\Omega,\sigma)$*.*

In the case

$$(5.3)\qquad f(u)=k|u|^\rho u+h(u),\ 0\leqslant\rho\leqslant\frac{4-2\alpha}{N-2+\alpha},$$

where $k>0$ and $h\in C(\mathbb{R})$ with $|h(u)|\leqslant C(1+|u|^\gamma)$ for $\gamma\in[0,\rho+1)$, and f satisfies the condition

$$(5.4)\qquad (f(u)-f(v))(u-v)\geqslant -C(u-v)^2,\ C>0,$$

we can repeat the arguments in Section 4 to obtain

Theorem 5.2. *Under the conditions* (H_β^∞) *and* (5.3) – (5.4)*, problem* (1.1) *defines a semigroup* $S(t):L^2(\Omega)\to L^2(\Omega)$*, which possesses a compact connected global attractor* $\mathcal{A}=\omega(\mathcal{B}_0)$ *in* $L^2(\Omega)$*.*

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Department of Mathematics, Hanoi University of Education
136 Xuan Thuy, Cau Giay, Hanoi, Vietnam

E-mail address: anhctmath@hnue.edu.vn

E-mail address: hungpq@hnue.edu.vn