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CODERIVATIVE CALCULATION RELATED TO A PARAMETRIC AFFINE VARIATIONAL INEQUALITY PART 1: BASIC CALCULATIONS

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Dedicated to Nguyen Van Hien on the occasion of his sixty-fifth birthday

Abstract. Consider a parametric affine variational inequality $0 \in Mx + q + N(x; \Delta(A, b))$, denoted by $\mathrm{AVI}(M, q, A, b)$, for which the pair $(q, b) \in \mathbb{R}^n \times \mathbb{R}^m$ describes the linear perturbations. Here the matrices $M \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{m\times n}$ are the given data, $\Delta(A, b) = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is a polyhedral convex constraint set, and $N(x; \Delta(A, b))$ denotes the normal cone to $\Delta(A, b)$ at x. We study the normal coderivative of the normal-cone operator $(x, b) \mapsto N(x; \Delta(A, b))$. In the second part of this paper [20], combining the obtained results with some theorems from Mordukhovich [11], Levy and Mordukhovich [10], Yen and Yao [21], we get sufficient conditions for the Aubin property (the Lipschitz-like property) and the local metric regularity in Robinson's sense of the solution map $(q, b) \mapsto S(q, b)$ of the problem $\mathrm{AVI}(M, q, A, b)$ and of the solution map $(w, b) \mapsto S(w, b)$ of the problem $0 \in f(x, w) + N(x; \Delta(A, b))$ where $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$ is a given C^1 vector function. Our investigation complements the well-known work of Dontchev and Rockafellar [3] where the Aubin property of the solution maps $q \mapsto S(q, b)$ and $w \mapsto S(w, b)$ (b is fixed) was established via a critical face condition.

1. Introduction

Necessary optimality conditions of a quadratic programming problem can be written as an affine variational inequality (AVI for brevity); see [8, Chap. 5] for more details. In the terminology of Robinson [16], AVI is a linear generalized equation. By definition, *affine variational inequality* is the problem of finding an x satisfying the inclusion

$$0 \in Mx + q + N(x; \Delta(A, b)), \tag{1.1}$$

which is denoted by $\mathrm{AVI}(M, q, A, b)$ and which depends on the data quadruplet $\{M, q, A, b\}$ with the pair $(q, b) \in \mathbb{R}^n \times \mathbb{R}^m$ describing the linear perturbations

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in the model. Here the matrices $M \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}$ are the given data, $\Delta(A, b) = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is a polyhedral convex constraint set,

$$N(x; \Delta(A, b)) := \{v \in \mathbb{R}^n : \langle v, u - x\rangle \leqslant 0 \text{ for all } u \in \Delta(A, b)\}$$

is the *normal cone* to $\Delta(A, b)$ at $x \in \Delta(A, b)$, and $\langle v, u\rangle$ denotes the scalar product of v and u. By convention, $N(x; \Delta(A, b)) = \emptyset$ whenever $x \notin \Delta(A, b)$. We abbreviate the solution set of (1.1) to $S(q, b)$. Thus, $x \in S(q, b)$ means $x \in \Delta(A, b)$ and

$$\langle Mx + q, u - x\rangle \geqslant 0 \quad \forall u \in \Delta(A, b).$$

In the case $A = -E$ with E denoting the unit matrix in $\mathbb{R}^{n\times n}$ and $b = 0$, x solves (1.1) if and only if

$$Mx + q \geqslant 0, \quad x \geqslant 0, \quad \langle Mx + q, x\rangle = 0. \tag{1.2}$$

System (1.2) of $2n$ linear inequalities and one nonlinear equality is called the *linear complementarity problem.*

Solution existence theorems for AVIs were established by Gowda and Pang [4] (see also [8, Chap. 6]). Solution stability of parametric AVIs is a subject of a large number of research papers. To our knowledge, the work of Robinson [16] establishing an upper Lipschitz continuity property of the solution map of AVI(M, q, A, b) where (A, b) is fixed and (M, q) is perturbed and the work of Dontchev and Rockafellar [3], where the Mordukhovich criterion [11] involving coderivatives of multifunctions was used effectively for obtaining the Aubin property of the solution map $q \mapsto S(q, b)$ (the triplet (M, A, b) is fixed), are among the most important papers in this topic. A new proof for the just mentioned stability theorem of Robinson is given in [8, Chap. 7]. In [3], the authors also studied the Aubin property of the solution map $w \mapsto S(w, b)$ of the problem

$$0 \in f(x, w) + N(x; \Delta(A, b)), \tag{1.3}$$

where $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$ is a given C^1 vector function. Other stability results for AVIs can be found in [8, 9, 17], and the references therein. Basic results on solution stability of (1.2), where M and q are subject to perturbations, can be found in [2, Chap. 7]. New developments and applications of the results of [3] can be found in [5, 6] (the constraint set remains fixed).

For a multifunction $F : X \rightrightarrows Y$ between normed spaces, the set $\operatorname{gph} F := \{(x, y) \in X \times Y : y \in F(x)\}$ is called the *graph* of F. One says that F *has a locally closed graph* around a point $(x_0, y_0) \in \operatorname{gph} F$ if there exists a closed ball B in $X\times Y$ of positive radius with the center (x_0, y_0) such that $B\cap(\operatorname{gph} F)$ is a closed subset of $X \times Y$. The norm in the product space is given by $\|(x, y)\| = \|x\| + \|y\|$.

Specializing the notions of Aubin property (known also as the pseudo-Lipschitz property, the Lipschitz-like property) of multifunctions [1, 3, 13] and the local metric regularity of implicit multifunctions [21] (which has the origin in the work of Robinson [15]) to the solution maps $(q, b) \mapsto S(q, b)$ of (1.1) and $(w, b) \mapsto S(w, b)$ of (1.3), we have the following concepts.

Definition 1.1. (i) The solution map $S(q,b)$ of (1.1) is said to have the *Aubin property* around $(q_0,b_0,x_0)\in \operatorname{gph} S$ if there exist neighborhoods U_1 of q_0, U_2 of b_0, V of x_0 and a constant $\ell>0$ such that

$$S(q',b')\cap V\subset S(q,b)+\ell(\|q'-q\|+\|b'-b\|)B_{\mathbb{R}^n}\quad \forall (q',b'),(q,b)\in U_1\times U_2,$$

where $B_{\mathbb{R}^n}$ stands for the closed unit ball in $\mathbb{R}^n$.
(ii) The solution map $S(w,b)$ of (1.3) is said to have the *Aubin property* around $(w_0,b_0,x_0)\in \operatorname{gph} S$ if there exist neighborhoods W of w_0, U of b_0, V of x_0 and a constant $\ell>0$ such that

$$S(w',b')\cap V\subset S(w,b)+\ell(\|w'-w\|+\|b'-b\|)B_{\mathbb{R}^n}\quad \forall (w',b'),(w,b)\in W\times U.$$

Definition 1.2. (i) The solution map $S(q,b)$ of (1.1) is locally-metrically regular in Robinson's sense around a point $\omega_0=(x_0,q_0,b_0,0_{\mathbb{R}^n})$ satisfying $0\in Mx_0+q_0+N(x_0;\Delta(A,b_0))$ if there exist constants $\gamma>0$, $\mu>0$, and neighborhoods V of x_0, U_1 of q_0, U_2 of b_0 such that

$$(1.4)\begin{cases}\operatorname{dist}(x,S(q,b))\leqslant\gamma\operatorname{dist}(0,Mx+q+N(x;\Delta(A,b)))\\ \text{whenever } x\in V,\ q\in U_1,\ b\in U_2,\ \operatorname{dist}(0,Mx+q+N(x;\Delta(A,b)))<\mu.\end{cases}$$

Here $\operatorname{dist}(u,\Omega):=\inf\{\|u-\omega\| : \omega\in\Omega\}$ denotes the distance from a point u to a set $\Omega\subset\mathbb{R}^n$.
(ii) The solution map $S(w,b)$ of (1.3) is locally-metrically regular in Robinson's sense around a point $\omega_0=(x_0,w_0,b_0,0_{\mathbb{R}^n})$ satisfying $0\in f(x_0,w_0)+N(x_0;\Delta(A,b_0))$ if there exist constants $\gamma>0$, $\mu>0$, and neighborhoods V of x_0, W of w_0, U of b_0 such that

$$(1.5)\begin{cases}\operatorname{dist}(x,S(w,b))\leqslant\gamma\operatorname{dist}(0,f(x,w)+N(x;\Delta(A,b)))\\ \text{whenever } x\in V,\ w\in W,\ b\in U,\ \operatorname{dist}(0,f(x,w)+N(x;\Delta(A,b)))<\mu.\end{cases}$$

The Aubin property and the local metric regularity are important features of implicit multifunctions. For the case of inverse multifunctions, they are equivalent (see for instance [14, 11]). In general, the equivalence does not hold true [7].

Our aim in this paper is to find adequate conditions for having the Aubin property and the local metric regularity of the solution maps of *parametric variational inequalities with moving convex polyhedral constraint sets.* Namely, by studying the normal coderivative of the normal-cone operator

$$(1.6)\qquad (x,b)\mapsto N(x;\Delta(A,b))$$

and using some results from Mordukhovich [11], Levy and Mordukhovich [10], Yen and Yao [21] we will get sufficient conditions for the Aubin property and the local metric regularity in Robinson's sense of the solution map $(q,b)\mapsto S(q,b)$ of (1.1) and of the solution map $(w,b)\mapsto S(w,b)$ of (1.3), which were described in Definitions 1.1 and 1.2. Our investigation complements the study of [3] where the Aubin property of the solution maps $q\mapsto S(q,b)$ and $w\mapsto S(w,b)$ (the parameter b is fixed) was established via a critical face condition.

The inclusion (1.1) can be rewritten as

$$0 \in F(x, y), \tag{1.7}$$

with $y := (q, b) \in \mathbb{R}^n \times R^m$, $F(x, y) := F_1(x, q) + F_2(x, b)$, $F_1(x, q) = Mx + q$, $F_2(x, b) = N(x; \Delta(A, b))$. Then, the solution map $S(q, b)$ coincides with the *implicit multifunction* $G(y) = \{x \in \mathbb{R}^n : 0 \in F(x, y)\}$ defined by (1.7).

The rest of this first part of the paper has three sections. Section 2 recalls some basic notions concerning normal cones to sets and coderivatives of multifunctions from [13]. In Section 3, we obtain a formula for the normal coderivative [13] (called also the limiting coderivative, or the coderivative in the sense of Mordukhovich) of the multifunction $x \mapsto N(x; \Delta(A, b))$ at a point (x, v) in its graph, which is equivalent to the formula established by Dontchev and Rockafellar in [3, Proof of Theorem 2]. It seems to us that the new formula is more convenient for practical computations. Besides, our proof is more elementary and direct: we do not use the Reduction Lemma [3, p. 1090] and other advanced techniques of [3]. In Section 4, combining the method of proof with a suitable trick, we estimate the normal coderivative of the multifunction $F_2 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ at a given point $(x, b, v) \in \operatorname{gph} F_2$.

In Part 2, we use a sum rule in [12] and the above-mentioned coderivative estimate for F_2 to study the normal coderivative of the multifunction $F = F_1 + F_2$. Then, combining the Mordukhovich criterion [11, 19] for the Aubin property of multifunction with an upper estimate for the normal coderivative of a implicit multifunction given by Levy and Mordukhovich [10], we obtain sufficient conditions for the Aubin property of the solution map S around the point $(q_0, b_0, x_0) \in \operatorname{gph} S$. Furthermore, by the upper estimate for the normal coderivative of $F = F_1 + F_2$ and [21, Theorem 3.1] we obtain sufficient conditions for the local metric regularity in Robinson's sense of the solution map $S(q, b)$ around a point $(q_0, b_0, x_0) \in \operatorname{gph} S$. Sufficient conditions for the Aubin property and the local metric regularity in Robinson's sense of the solution map $(w, b) \mapsto S(w, b)$ of (1.3) can be established in a similar way.

2. Normal cones to sets and coderivatives of multifunctions

Let X, Y be Euclidean spaces whose inner products and norms are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. For a subset $\Omega \subset X$, the symbols $\overline{\Omega}$, $\operatorname{int} \Omega$, and $\operatorname{cone} \Omega$ respectively denote the closure of Ω, the interior of Ω, and the cone generated by Ω. The set of the metric projections of $u \in X$ on the closure of Ω is denoted by $\Pi(u, \Omega)$, i.e.,

$$\Pi(u, \Omega) = \{x \in \overline{\Omega} : \|x - u\| = \operatorname{dist}(u, \Omega)\}.$$

If $M \subset X$ is a cone, then the negative dual cone to M is denoted by M^*. The closed ball centered at x with radius ρ and the closed unit balls in X are denoted respectively by $B_\rho(x)$ and B_X. Given a point $x_0 \in X$, we abbreviate the collection of all the neighborhoods of x_0 to $\mathcal{N}(x_0)$. If A is a matrix, then A^T denotes the transpose of A.

For a multifunction $\Phi : X \rightrightarrows Y$, the expression $\operatorname*{Lim\,sup}_{x\to\bar{x}} \Phi(x)$ denotes the sequential Kuratowski-Painlevé upper limit of $\Phi(x)$ as $x \to \bar{x}$, that is

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} \Phi(x) = \{\xi \in Y : \quad \exists \text{ sequences } x_k \to \bar{x},\ \xi_k \to \xi, \text{ with } \xi_k \in \Phi(x_k) \text{ for all } k = 1, 2, \dots\}.$$

Following [13], we now define normal cones to sets and coderivatives of multifunctions.

The set $\widehat{N}_\varepsilon(x;\Omega)$ of the Fréchet ε-normals to Ω at $x \in \overline{\Omega}$ is given by

$$\widehat{N}_\varepsilon(x;\Omega) = \left\{ v \in X \,:\, \limsup_{u \xrightarrow{\Omega} x} \frac{\langle v, u - x\rangle}{\|u - x\|} \leqslant \varepsilon \right\}, \tag{2.1}$$

where the notation $u \xrightarrow{\Omega} x$ means $u \to x$ and $u \in \Omega$. For $\varepsilon = 0$, the set in (2.1) is a closed convex cone which is called the *Fréchet normal cone* to Ω at x and is denoted by $\widehat{N}(x;\Omega)$. One puts $\widehat{N}_\varepsilon(x;\Omega) = \emptyset$ for all $\varepsilon \geqslant 0$ whenever $x \notin \overline{\Omega}$. The cone

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x\to\bar{x},\ \varepsilon\downarrow 0} \widehat{N}_\varepsilon(x;\Omega) \tag{2.2}$$

is said to be the *normal cone* in the sense of Mordukhovich to Ω at $\bar{x}$. If $\bar{x} \notin \overline{\Omega}$, then one puts $N(\bar{x};\Omega) = \emptyset$. If Ω is locally closed around $\bar{x}$, then

$$N(\bar{x};\Omega) = \operatorname*{Lim\,sup}_{x\to\bar{x}} [\operatorname{cone}(x - \Pi(x,\Omega))] \tag{2.3}$$

(see [13, Theorem 1.6]) and

$$N(\bar{x};\Omega) = \operatorname*{Lim\,sup}_{x\to\bar{x}} \widehat{N}(x;\Omega). \tag{2.4}$$

Note that in [11, 12] the normal cone $N(\bar{x};\Omega)$ was defined by (2.3). From (2.1) and (2.2) it follows that $\widehat{N}(\bar{x};\Omega) \subset N(\bar{x};\Omega)$. If Ω is a convex set, then

$$\widehat{N}(\bar{x};\Omega) = N(\bar{x};\Omega) = \{v \in C \,:\, \langle v, u - \bar{x}\rangle \leqslant 0 \ \text{ for all } \ u \in \Omega\}.$$

The multifunction $D^*\Phi(\bar{x},\bar{y}) : Y \rightrightarrows X$ defined by

$$D^*\Phi(\bar{x},\bar{y})(y^*) := \{x^* \in X \,:\, (x^*, -y^*) \in N((\bar{x},\bar{y}); \operatorname{gph}\Phi)\} \tag{2.5}$$

is said to be the *normal coderivative* (called also the *limiting coderivative* and the *coderivative in the sense of Mordukhovich*) of Φ at $(\bar{x},\bar{y})$. We put $D^*\Phi(\bar{x},\bar{y})(y^*) = \emptyset$ whenever $(\bar{x},\bar{y}) \notin \overline{\operatorname{gph}\Phi}$.

3. NORMAL CODERIVATIVE OF THE MULTIFUNCTION $x \mapsto N(x; \Delta(A,b))$

From now on, we shall employ the notation of Section 1. Given any $b \in \mathbb{R}^n$, we now establish several lemmas which lead to a formula for calculating the normal coderivative of the multifunction $F_3(x) := N(x; \Delta(A,b))$ at a point $(x,v) \in \Omega_3$, where $\Omega_3 := \operatorname{gph} F_3$. For simplicity of notation, in this section we set $C = \Delta(A,b)$.

We first compute the Fréchet normal cone $\widehat{N}((x,v);\Omega_3)$, where $(x,v) \in \Omega_3$ is given arbitrarily. The last inclusion means $x \in C$ and $v \in N(x;C)$. By definition, $(x^*,v^*) \in \widehat{N}((x,v);\Omega_3)$ if and only if

$$\limsup_{(\widetilde{x},\widetilde{v}) \xrightarrow{\Omega_3} (x,v)} \frac{\langle x^*, \widetilde{x}-x\rangle + \langle v^*, \widetilde{v}-v\rangle}{\|\widetilde{x}-x\| + \|\widetilde{v}-v\|} \leqslant 0. \tag{3.1}$$

Let $J = \{1,\ldots,m\}$. For each $x \in C$, the active index set of x is given by $I(x) = \{i \in J \,:\, A_i x = b_i\}$, where A_i denotes the i-th row of A and b_i is the i-th component of b. For every subset $I \subset J$, we put $\bar{I} = J \setminus I$ and let A_I (resp., $A_{\bar{I}}$) be the matrix composed by the rows $A_i, i \in I$, of A (resp., the rows $A_i, i \in \bar{I}$). The *pseudo-face* $\mathcal{F}_I$ of $C = \Delta(A,b)$ corresponding to an index set I is defined by

$$\mathcal{F}_I = \{x \in \mathbb{R}^n \,:\, A_I x = b_I,\ A_{\bar{I}} x < b_{\bar{I}}\}.$$

If $x, \widetilde{x} \in \mathcal{F}_I$ then

$$T(\widetilde{x};C) = T(x;C) = \{u \in \mathbb{R}^n \,:\, A_I u \leqslant 0\}$$

where

$$T(x;C) = \operatorname{cone}(\Delta(A,b) - x) = \big(N(x;C)\big)^*$$

is the tangent cone to the polyhedral convex set C at x (see, e.g., [18]). By the Farkas lemma [18, p. 200], from the above formula for $T(\widetilde{x};C)$ and $T(x;C)$ we have

$$N(\widetilde{x};C) = N(x;C) = \operatorname{pos}\{A_i^T \,:\, i \in I\},$$

where $\operatorname{pos}\{A_i^T \,:\, i \in I\}$ denotes the convex cone generated by the column vectors $\{A_i^T \,:\, i \in I\}$. In the sequel, it is convenient for us to abbreviate $T(\widetilde{x};C)$, for any $\widetilde{x} \in \mathcal{F}_I$, to $T(\mathcal{F}_I;C)$. A set $Q \subset \mathbb{R}^n$ is said to be a *closed face* of C if there exists $I \subset J$ such that

$$Q = \overline{\mathcal{F}}_I := \{x \in \mathbb{R}^n \,:\, A_I x = b_I,\ A_{\bar{I}} x \leqslant b_{\bar{I}}\}.$$

This definition is equivalent to the following one: $Q \subset \mathbb{R}^n$ is a closed face of C if there exist $\bar{x} \in C$ and $\bar{v} \in N(\bar{x};C) = \operatorname{pos}\{A_i^T \,:\, i \in I(\bar{x})\}$ such that $Q = \{x \in C \,:\, \langle \bar{v}, x - \bar{x}\rangle = 0\}$. Clearly, if C is a cone (that is the case where $b = 0$), then Q is a closed face of C if and only if there exists $\bar{v} \in C^*$ such that $Q = \{x \in C \,:\, \langle \bar{v}, x\rangle = 0\}$.

Lemma 3.1. *If* $(x^*,v^*) \in \widehat{N}((x,v);\Omega_3)$ *then*

$$v^* \in T(x;C) \cap v^{\perp} \tag{3.2}$$

and

$$x^* \in \left(T(x;C) \cap v^\perp\right)^*, \tag{3.3}$$

where $v^\perp := \{u \in \mathbb{R}^n \,:\, \langle u, v\rangle = 0\}$.

Proof. Let $(x^*, v^*) \in \widehat{N}((x,v);\Omega_3)$. For $\widetilde{x} = x$, $v' \in N(x;C)$, $t > 0$, $\widetilde{v} = v + t(v' - v) \in F_3(x) = N(x;C)$, by (3.1) we have

$$\langle v^*, v' - v\rangle \leqslant 0 \quad \forall v' \in N(x;C). \tag{3.4}$$

Substituting $v' = 2v$ and $v' = \frac{1}{2}v$ into (3.4) gives $\langle v^*, v\rangle = 0$. Hence, by (3.4) we get $\langle v^*, v'\rangle \leqslant 0$ for every $v' \in N(x;C)$. As $N(x;C)^* = T(x;C)$, it follows that (3.2) is valid.

Let $I = I(x)$ and $\bar{I} = J \setminus I$. Given any $\xi \in T(x;C) \cap v^\perp$, to get (3.3) it suffices to show that $\langle x^*, \xi\rangle \leqslant 0$. Put $\widetilde{x}_t = x + t\xi$. As $A_I\xi \leqslant 0$, $A_I x \leqslant b_I$, and $A_{\bar{I}}x < b_{\bar{I}}$, there exists $\delta > 0$ such that $A(x + t\xi) \leqslant b$ for all $t \in (0,\delta)$. This means that $x_t \in C$ for every $t \in (0,\delta)$. Since $\langle v, \xi\rangle = 0$ and $v \in N(x,C)$, we have

$$\langle v, x' - \widetilde{x}_t\rangle = \langle v, x' - x\rangle + t\langle v, \xi\rangle = \langle v, x' - x\rangle \leqslant 0$$

for every $x' \in C$; so $v \in N(\widetilde{x}_t, C)$ for all $t \in (0,\delta)$. We now see that $(\widetilde{x}_t, v) \xrightarrow{\Omega_3} (x,v)$ as $t \to 0^+$. Substituting $(\widetilde{x}, \widetilde{v}) := (\widetilde{x}_t, v)$ into (3.1) and passing to the limit as $t \to 0^+$, we obtain the desired inequality $\langle x^*, \xi\rangle \leqslant 0$. □

The next lemma shows that (3.2) and (3.3) are not only necessary but also sufficient conditions for having $(x^*, v^*) \in \widehat{N}((x,v);\Omega_3)$. This is a known fact [3, Proof of Theorem 1], but the proof we provide here is new.

Lemma 3.2. *Any pair* (x^*, v^*) *which satisfies the conditions* (3.2) *and* (3.3) *must belong to* $\widehat{N}((x,v);\Omega_3)$.

Proof. Let (x^*, v^*) be such that (3.2) and (3.3) hold. Given any sequence $(\widetilde{x}_k, \widetilde{v}_k) \xrightarrow{\Omega_3} (x,v)$ as $k \to \infty$, we have to show that

$$\limsup_{k\to\infty} \frac{\langle x^*, \widetilde{x}_k - x\rangle + \langle v^*, \widetilde{v}_k - v\rangle}{\|\widetilde{x}_k - x\| + \|\widetilde{v}_k - v\|} \leqslant 0. \tag{3.5}$$

By considering a subsequence, if necessary, we may assume that all the vectors $\widetilde{x}_k$ belong to a pseudo-face

$$\mathcal{F}_{I_0} = \{x' \in \mathbb{R}^n \,:\, A_{I_0}x' = b_{I_0},\ A_{\bar{I}_0}x' < b_{\bar{I}_0}\}$$

which has x in its topological closure (hence $I_0 \subset I := I(x)$). Note that

$$\widetilde{v}_k \in N(\widetilde{x}_k;C) = \mathrm{pos}\{A_i^T \,:\, i \in I_0\} \subset \mathrm{pos}\{A_i^T \,:\, i \in I\}.$$

Let $\widetilde{v}_k = \sum_{i\in I}\lambda_i^k A_i^T$, where $\lambda_i^k \geqslant 0$ for all i (we put $\lambda_i^k = 0$ whenever $i \in I \setminus I_0$). Observe that

$$\langle v^*, \widetilde{v}_k - v\rangle = \langle v^*, \widetilde{v}_k\rangle = \sum_{i\in I}\lambda_i^k\langle v^*, A_i^T\rangle \leqslant 0, \tag{3.6}$$

because $v^* \in T(x;C)$ by our assumption and $A_i^T \in N(x;C)$ for every $i \in I$.

If $\langle x^*, \widetilde{x}_k - x\rangle \leqslant 0$ for all k large enough, then (3.5) follows immediately from (3.6).

We now suppose that there is a subsequence $\{k_j\} \subset \{k\}$ such that the strict inequality $\langle x^*, \widetilde{x}_{k_j} - x\rangle > 0$ occurs for each index k_j. Then we have

$$\begin{aligned} &\frac{\langle x^*, \widetilde{x}_{k_j} - x\rangle + \langle v^*, \widetilde{v}_{k_j} - v\rangle}{\|\widetilde{x}_{k_j} - x\| + \|\widetilde{v}_{k_j} - v\|} \\ &= \frac{\langle x^*, \widetilde{x}_{k_j} - x\rangle}{\|\widetilde{x}_{k_j} - x\| + \|\widetilde{v}_{k_j} - v\|} + \frac{\langle v^*, \widetilde{v}_{k_j} - v\rangle}{\|\widetilde{x}_{k_j} - x\| + \|\widetilde{v}_{k_j} - v\|} \\ &\leqslant \left\langle x^*, \frac{\widetilde{x}_{k_j} - x}{\|\widetilde{x}_{k_j} - x\|}\right\rangle + \frac{\langle v^*, \widetilde{v}_{k_j} - v\rangle}{\|\widetilde{x}_{k_j} - x\| + \|\widetilde{v}_{k_j} - v\|}. \end{aligned} \tag{3.7}$$

There is no loss of generality in assuming that

$$\frac{\widetilde{x}_{k_j} - x}{\|\widetilde{x}_{k_j} - x\|} \to \xi \in T(x; C).$$

On one hand, we have $\langle \xi, v\rangle \leqslant 0$ because $v \in N(x; C)$. On the other hand, the inclusion $\widetilde{v}_{k_j} \in N(\widetilde{x}_{k_j}; C)$ implies

$$\left\langle \widetilde{v}_{k_j}, \frac{x - \widetilde{x}_{k_j}}{\|x - \widetilde{x}_{k_j}\|}\right\rangle \leqslant 0.$$

Letting $k_j \to \infty$ and recalling that $\widetilde{v}_{k_j} \to v$, from the last inequality we get $\langle v, -\xi\rangle \leqslant 0$. Thus $\langle v, \xi\rangle = 0$, and we see that $\xi \in T(x; C) \cap v^\perp$. Taking account of (3.7), (3.6), and (3.3), we obtain

$$\begin{aligned} &\limsup_{k_j \to \infty} \frac{\langle x^*, \widetilde{x}_{k_j} - x\rangle + \langle v^*, \widetilde{v}_{k_j} - v\rangle}{\|\widetilde{x}_{k_j} - x\| + \|\widetilde{v}_{k_j} - v\|} \\ &\leqslant \lim_{k_j \to \infty} \left\langle x^*, \frac{\widetilde{x}_{k_j} - x}{\|\widetilde{x}_{k_j} - x\|}\right\rangle + \limsup_{k_j \to \infty} \frac{\langle v^*, \widetilde{v}_{k_j} - v\rangle}{\|\widetilde{x}_{k_j} - x\| + \|\widetilde{v}_{k_j} - v\|} \\ &\leqslant \langle x^*, \xi\rangle \leqslant 0 \end{aligned}$$

which establishes (3.5) and completes the proof. □

We are now in a position to compute the normal cone in the sense of Mordukhovich to Ω_3 at a point $(x, v) \in \Omega_3 = \operatorname{gph} F_3$.

Theorem 3.3. (Normal cone in the sense of Mordukhovich; the case where b is fixed). *For any pair $(x, v) \in \Omega_3$, it holds*

$$N\big((x, v); \Omega_3\big) = \bigcup_{(I', Q)} (Q^* \times Q) \tag{3.8}$$

with the union being taken upon the family of the pairs (I', Q) where

$$I' \subset I(x) := \{i \in J \,:\, A_i x = b_i\}$$

satisfying

$$v \in \operatorname{pos}\{A_i^T \,:\, i \in I'\} \tag{3.9}$$

and Q is a closed face of the polyhedral convex cone $T(\mathcal{F}_{I'}; C) \cap v^\perp$.

Proof. If $\mathcal{F}_{I'}$ is a pseudo-face of C having x in its topological closure, then we must have $I' \subset I(x)$. Indeed, if $x \in \overline{\mathcal{F}}_{I'}$ and there is some $i \in I' \setminus I(x)$, then there exists a sequence $x_k \xrightarrow{\mathcal{F}_{I'}} x$ such that $A_i x_k = b_i$ for all k. Hence $A_i x = b_i$. This inequality is an absurd, because $i \in J \setminus I(x)$. Conversely, if $I' \subset I(x)$ and $\mathcal{F}_{I'} \neq \emptyset$, then $x \in \overline{\mathcal{F}}_{I'}$. Indeed, take any $x' \in \mathcal{F}_{I'}$ and put $x_t = (1-t)x + tx'$ for $t \in (0,1)$. It is easy to see that $x_t \in \mathcal{F}_{I'}$ and $x_t \to x$ as $t \to 0^+$.

By definition, $(x^*, v^*) \in N\big((x,v);\Omega_3\big)$ if and only if one can find sequences $(x_k, v_k) \to (x,v)$ and $(x_k^*, v_k^*) \to (x^*, v^*)$ with $v_k \in N(x_k; C)$ and

$$(x_k^*, v_k^*) \in \widehat{N}((x_k, v_k);\Omega_3) \quad \forall k.$$

Since the number of pseudo-faces of C is finite, there must exist an index set $I' \subset J$ and a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \in \mathcal{F}_{I'}$ for each k_j. As $x_{k_j} \to x$, we have $I' \subset I(x)$. According to Lemmas 3.1 and 3.2, the inclusion $(x_{k_j}^*, v_{k_j}^*) \in \widehat{N}((x_{k_j}, v_{k_j});\Omega_3)$ means

$$\begin{aligned}(x_{k_j}^*, v_{k_j}^*) &\in \Big(T(x_{k_j};C)\cap v_{k_j}^{\perp}\Big)^* \times \Big(T(x_{k_j};C)\cap v_{k_j}^{\perp}\Big)\\ &= \Big(T(\mathcal{F}_{I'};C)\cap v_{k_j}^{\perp}\Big)^* \times \Big(T(\mathcal{F}_{I'};C)\cap v_{k_j}^{\perp}\Big).\end{aligned} \tag{3.10}$$

Due to the condition $v_{k_j} \in N(x_{k_j};C)$, we have $\langle v_{k_j}, u\rangle \leqslant 0$ for every $u \in T(\mathcal{F}_{I'};C)$. Thus $T(\mathcal{F}_{I'};C)\cap v_{k_j}^{\perp}$ is a closed face of the polyhedral convex cone $T(\mathcal{F}_{I'};C)$. Of course, by using a subsequence (if it is necessary), we may assume that

$$T(\mathcal{F}_{I'};C)\cap v_{k_j}^{\perp} = Q \quad \forall k_j,$$

where Q is a closed face of $T(\mathcal{F}_{I'};C)$. Passing to the limit as $k_j \to \infty$, from (3.10) we obtain

$$(x^*, v^*) \in Q^* \times Q. \tag{3.11}$$

Since $v_{k_j} \to v$ as $k_j \to \infty$, it holds

$$Q \subset T(\mathcal{F}_{I'};C)\cap v^{\perp} \tag{3.12}$$

and, moreover, Q is a closed face of the polyhedral convex cone on the right-hand side of (3.12). Since $v_{k_j} \in N(x_{k_j};C) = \mathrm{pos}\{A_i^T \,:\, i \in I'\}$ for all k_j, and the latter cone is closed, we must have (3.9). We have shown that $N\big((x,v);\Omega_3\big)$ is contained in the set on the right-hand side of (3.8).

Conversely, suppose that the inclusion (3.11) is valid for an index set $I' \subset I(x)$ satisfying (3.9) and a closed face Q of the polyhedral convex cone $T(\mathcal{F}_{I'};C)\cap v^{\perp}$. Since $\mathcal{F}_{I'} \neq \emptyset$, we can find a sequence $\{x_k\} \subset \mathcal{F}_{I'}$ converging to x. From our assumption it follows that Q is a closed face of the polyhedral convex cone $T(\mathcal{F}_{I'};C)$. Hence we can find an $\bar{v} \in K := \mathrm{pos}\{A_i^T \,:\, i \in I'\}$ such that $Q = T(\mathcal{F}_{I'};C)\cap \bar{v}^{\perp}$. Choose a sequence $\{t_k\} \subset (0,1)$ such that $t_k \to 0^+$ as $k \to \infty$. By the convexity of K,

$$v_k := (1-t_k)v + t_k\bar{v} \in K \quad \forall k.$$

From what which has already been said, we have $v_k \in N(x_k; C)$ for all k, $v_k \to v$ as $k \to \infty$, and

$$Q = T(\mathcal{F}_{I'}; C) \cap v_k^{\perp} \quad \forall k.$$

Then, the inclusion (3.11) and Lemma 3.2 show that $(x^*, v^*) \in \widehat{N}((x_k, v_k); C)$ for all k. This yields $(x^*, v^*) \in N\big((x, v); \Omega_3\big)$ and establishes equality (3.8). □

In [3], the cone $N\big((x, v); \Omega_3\big)$ is described as follows.

Theorem 3.4. (The normal cone $N\big((x, v); \Omega_3\big)$; Dontchev-Rockafellar's description). *For any pair $(x, v) \in \Omega_3$, let*

$$K(x, v) = T(x; C) \cap v^{\perp}.$$

It holds

$$N\big((x, v); \Omega_3\big) = \bigcup_{(K_1, K_2)} \left[(K_1 - K_2)^* \times (K_1 - K_2)\right], \tag{3.13}$$

where the union is taken upon the set of all the pairs (K_1, K_2) of closed faces of the polyhedral convex cone $K(x, v)$ satisfying the relation $K_2 \subset K_1$.

Proof. In our notation, a result in [3, p. 1093] asserts that

$$\widehat{N}((x', v'); \Omega_3) = (K(x', v'))^* \times K(x', v')$$

for any pair $(x', v') \in \Omega_3$. In [3, p. 1092], the authors observed that there exists a neighborhood $U \subset \mathbb{R}^n \times \mathbb{R}^n$ of (x, v) such that

$$N\big((x, v); \Omega_3\big) = \bigcup_{(x', v') \in U \cap \Omega_3} \big[(K(x', v'))^* \times K(x', v')\big]. \tag{3.14}$$

Moreover, they showed that every set $K(x', v')$ figured in (3.14) can be represented in the form $K_1 - K_2$ where K_1, K_2 are closed faces of the cone $K(x, v)$ satisfying the relation $K_2 \subset K_1$. Conversely, any cone $K_1 - K_2$ of this form describes a set $K(x', v')$ participating in (3.14). Based on the preceding proof, it is not difficult to see that every cone $K(x', v')$ in (3.14) corresponds to a closed face Q defined in Theorem 3.3. Thus, the formulae (3.13) and (3.8) are equivalent. □

Remark 3.5. Listing all the pairs (K_1, K_2) of closed faces of $K(x, v)$ satisfying $K_2 \subset K_1$ seems to be a difficult task. Instead of (3.13), we would prefer using (3.8) which offers an explicit calculation of the normal cone $N\big((x, v); \Omega_3\big)$.

Remark 3.6. Despite the difficulty mentioned in the preceding remark, (3.13) shows that *in order to get complete information about the nonconvex cone $N\big((x, v); \Omega_3\big)$ one only needs to know the convex cone $K(x, v)$.* In other words, the nonconvex, complicated cone $N\big((x, v); \Omega_3\big)$ allows a complete description via the convex, much simpler, cone $K(x, v)$. This is an amazing fact about Mordukhovich normal cones in the case under consideration.

The normal coderivative of the multifunction F_3 at a given point in its graph can be computed easily by employing Theorem 3.3.

Theorem 3.7. (Normal coderivative; the case where b is fixed). *For any* $(x, v) \in \operatorname{gph} F_3$ *and* $v^* \in \mathbb{R}^n$, *the set* $D^*F_3(x, v)(v^*)$ *consists of all* $x^* \in \mathbb{R}^n$ *such that*

$$(x^*, -v^*) \in Q^* \times Q \tag{3.15}$$

for an index set $I' \subset I = I(x)$ *satisfying condition* (3.9) *and a closed face* Q *of the polyhedral convex cone* $T(\mathcal{F}_{I'}; C) \cap v^\perp$.

4. Normal coderivative of the multifunction $(x, b) \mapsto N(x; \Delta(A, b))$

Given any $b \in \mathbb{R}^n$ and $x \in \Delta(A, b)$, we want to calculate the normal coderivative of the multifunction $F_2(x, b) := N(x; \Delta(A, b))$ at $(x, b, v) \in \Omega_2$, where $\Omega_2 := \operatorname{gph} F_2$.

First, let us establish some facts about the Fréchet normal cone to Ω_2 at $(x, b, v) \in \Omega_2$.

Lemma 4.1. *If* $(x^*, b^*, v^*) \in \widehat{N}((x, b, v); \Omega_2)$ *then*

$$(x^*, v^*) \in \Big(T(x; \Delta(A, b)) \cap v^\perp\Big)^* \times \Big(T(x; \Delta(A, b)) \cap v^\perp\Big), \tag{4.1}$$

$$x^* = -A_I^T b_I^* \tag{4.2}$$

and

$$b_{\bar{I}}^* = 0, \tag{4.3}$$

where $I = I_{A,b}(x) := \{i \in J : A_i x = b_i\}$, $\bar{I} = J \setminus I$. *Moreover, if* $v = \sum_{i \in I} \lambda_i A_i^T$ *with* $\lambda_i \geqslant 0$ *for all* $i \in I$, *and* $I_0 := \{i \in I : \lambda_i = 0\}$, *then*

$$b_{I_0}^* \leqslant 0. \tag{4.4}$$

Proof. Suppose that $(x, b, v) \in \Omega_2$. Let $I_{A,b}(x), I, \bar{I}$ be defined as in the formulation of the lemma. If $(x^*, b^*, v^*) \in \widehat{N}((x, b, v); \Omega_2)$ then

$$\limsup_{(\widetilde{x}, \widetilde{b}, \widetilde{v}) \xrightarrow{\Omega_2} (x, b, v)} \frac{\langle x^*, \widetilde{x} - x\rangle + \langle b^*, \widetilde{b} - b\rangle + \langle v^*, \widetilde{v} - v\rangle}{\|\widetilde{x} - x\| + \|\widetilde{b} - b\| + \|\widetilde{v} - v\|} \leqslant 0. \tag{4.5}$$

Taking $\widetilde{b} = b$, from the last expression and Lemma 3.1 we get (4.1).

Fix any $j \in \bar{I}$. Property (4.3) will be established if we can show that $b_j^* = 0$. Let $\widetilde{b}_i = b_i$ for every $i \in J \setminus \{j\}$ and $\widetilde{b}_j \in (b_j - \varepsilon, b_j + \varepsilon)$, where $\varepsilon = b_j - A_j x > 0$. Obviously,

$$A_i x = \widetilde{b}_i \quad \forall i \in I, \quad A_i x < \widetilde{b}_i \quad \forall i \in \bar{I}.$$

Hence $\widetilde{x} := x$ belongs to $\Delta(A, \widetilde{b})$ and $\widetilde{v} := v$ satisfies the relation

$$\widetilde{v} \in \operatorname{pos}\{A_i : i \in I\} = N(\widetilde{x}; \Delta(A, \widetilde{b})). \tag{4.6}$$

Therefore, from (4.5) it follows that

$$\limsup_{\widetilde{b}_j \to b_j} \frac{b_j^*(\widetilde{b}_j - b_j)}{|\widetilde{b}_j - b_j|} \leqslant 0.$$

Since $\widetilde{b}_j \in (b_j - \varepsilon, b_j + \varepsilon)$ can be chosen arbitrarily, this yields $b_j^* = 0$.

Given any $\widetilde{x} \to x$, we choose $\widetilde{b}_I = A_I\widetilde{x}$, $\widetilde{b}_{\bar{I}} = b_{\bar{I}}$, and $\widetilde{v} = v$. It is clear that (4.6) holds whenever $\widetilde{x}$ is sufficiently close to x. Substituting the chosen triplet $(\widetilde{x}, \widetilde{b}, \widetilde{v})$ into (4.5) and noting that $b_I = A_I x$, we get

$$\limsup_{\widetilde{x} \to x} \frac{\langle x^*, \widetilde{x} - x\rangle + \langle b_I^*, A_I\widetilde{x} - A_I x\rangle}{\|\widetilde{x} - x\| + \|A_I\widetilde{x} - A_I x\|} \leqslant 0.$$

Therefore,

$$\limsup_{\widetilde{x} \to x} \frac{\left\langle x^* + A_I^T b_I^*, \frac{\widetilde{x} - x}{\|\widetilde{x} - x\|}\right\rangle}{1 + \|A_I\left(\frac{\widetilde{x} - x}{\|\widetilde{x} - x\|}\right)\|} \leqslant 0.$$

So we have

$$\frac{\langle x^* + A_I^T b_I^*, w\rangle}{1 + \|A_I w\|} \leqslant 0$$

for any $w \in \mathbb{R}^n$ with $\|w\| = 1$. Clearly, this property implies (4.2).

It remains to verify the second claim of the lemma. Let $v = \sum_{i \in I} \lambda_i A_i^T$ with λ_i being nonnegative for all $i \in I$, and let $I_0 = \{i \in I \,:\, \lambda_i = 0\}$. Fix an index $j \in I_0$. Choose $\widetilde{b}_j \to b_j, \widetilde{b}_j > b_j$, $\widetilde{b}_i = b_i$ for any $i \in J \setminus \{j\}$, $\widetilde{x} = x$, and $\widetilde{v} = v$. Clearly,

$$\widetilde{v} = v \in \mathrm{pos}\{A_i \,:\, i \in I \setminus \{j\}\} = N(\widetilde{x}; \Delta(A, \widetilde{b})).$$

Therefore, by (4.5) we obtain

$$\limsup_{\widetilde{b}_j \to b_j + 0} \frac{b_j^*(\widetilde{b}_j - b_i)}{|\widetilde{b}_j - b_i|} \leqslant 0,$$

which implies the desired inequality $b_j \leqslant 0$. □

The above lemma describes necessary conditions for a triplet (x^*, b^*, v^*) to belong to the Fréchet normal cone $\widehat{N}((x, b, v); \Omega_2)$. We show that the set of necessary conditions is sufficient for having $(x^*, b^*, v^*) \in \widehat{N}((x, b, v); \Omega_2)$ if, instead of (4.4), a little bit tighter condition $b_I^* \leqslant 0$ is satisfied.

Lemma 4.2. *If* $(x, b, v) \in \Omega_2$ *and if* $(x^*, b^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ *is a triplet satisfying* (4.1)–(4.3) *and the additional condition*

$$b_I^* \leqslant 0, \tag{4.7}$$

then $(x^*, b^*, v^*) \in \widehat{N}((x, b, v); \Omega_2)$.

Proof. Given any $(x,b,v)\in\Omega_2$ and $(x^*,b^*,v^*)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n$ satisfying (4.1)–(4.3) and (4.7), we are going to show that $(x^*,b^*,v^*)\in\widehat{N}((x,b,v);\Omega_2)$. To achieve the goal, it suffices to verify the inequality (4.5). Let there be given a sequence $(\widetilde{x}_k,\widetilde{b}_k,\widetilde{v}_k)\xrightarrow{\Omega_2}(x,b,v)$. Since $(\widetilde{x}_k,\widetilde{b}_k)\to(x,b)$, we must have

$$I_{A,b_k}(\widetilde{x}_k)\subset I=I_{A,b}(x)$$

for all k sufficiently large. As

$$\widetilde{v}_k\in\mathrm{pos}\{A_i\,:\,i\in I_{A,b_k}(\widetilde{x}_k)\}\subset\mathrm{pos}\{A_i\,:\,i\in I\}=N(x;\Delta(A,b)),$$

condition (4.1) implies that

$$\langle v^*,\widetilde{v}_k-v\rangle=\langle v^*,\widetilde{v}_k\rangle\leqslant 0. \tag{4.8}$$

Due to (4.2) and (4.3), we have

$$\begin{aligned}\langle x^*,\widetilde{x}_k-x\rangle+\langle b^*,\widetilde{b}_k-b\rangle&=\langle -A_I^Tb_I^*,\widetilde{x}_k-x\rangle+\langle b_I^*,(\widetilde{b}_k)_I-b_I\rangle\\&=\langle b_I^*,A_Ix-A_I\widetilde{x}_k\rangle+\langle b_I^*,(\widetilde{b}_k)_I-b_I\rangle\\&=\langle b_I^*,(\widetilde{b}_k)_I-A_I\widetilde{x}_k\rangle.\end{aligned}$$

Using (4.7) and the inequality $A_I\widetilde{x}_k\leqslant(\widetilde{b}_k)_I$, from this we see that

$$\langle x^*,\widetilde{x}_k-x\rangle+\langle b^*,\widetilde{b}_k-b\rangle\leqslant 0. \tag{4.9}$$

Combining (4.9) with (4.8), we get

$$\limsup_{k\to\infty}\frac{\langle x^*,\widetilde{x}_k-x\rangle+\langle b^*,\widetilde{b}_k-b\rangle+\langle v^*,\widetilde{v}_k-v\rangle}{\|\widetilde{x}_k-x\|+\|\widetilde{b}_k-b\|+\|\widetilde{v}_k-v\|}\leqslant 0$$

which establishes (4.5) and completes the proof. □

In connection with the preceding lemmas, we would like to raise two open questions.

Question 1. *Does the system* (4.1)–(4.4) *imply that* $(x^*,b^*,v^*)\in\widehat{N}((x,b,v);\Omega_2)$*?*

Question 2. *Does the inclusion* $(x^*,b^*,v^*)\in\widehat{N}((x,b,v);\Omega_2)$ *imply* (4.7)*?*

Using Lemma 4.1 we now give an upper estimate for the Mordukhovich normal cone to Ω_2 at $(x,b,v)\in\Omega_2$.

Theorem 4.3. (Normal cone in the sense of Mordukhovich to Ω_2). *For any point $(x,b,v)\in\Omega_2$, if a triplet $(x^*,b^*,v^*)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n$ belongs to the cone $N\big((x,b,v);\Omega_2\big)$, then there exist an index set*

$$I'\subset I_{A,b}(x):=\{i\in J\,:\,A_ix=b_i\}$$

satisfying

$$v\in\mathrm{pos}\{A_i^T\,:\,i\in I'\} \tag{4.10}$$

and a closed face Q of the polyhedral convex cone $T(\mathcal{F}_{I'};C)\cap v^\perp$ such that

$$(x^*,v^*)\in Q^*\times Q, \tag{4.11}$$

$$x^* = -A_{I'}^T b_{I'}^* \tag{4.12}$$

and

$$b_{\bar{I}'}^* = 0, \tag{4.13}$$

where

$$\mathcal{F}_{I'} = \{x \,:\, A_{I'}x = b_{I'},\ A_{\bar{I}'}x < b_{\bar{I}'}\}, \quad \bar{I}' = J \setminus I'.$$

Proof. Suppose that $(x^*, b^*, v^*) \in N\big((x, b, v); \Omega_2\big)$. This inclusion means that there exist sequences $(x_k, b_k, v_k) \to (x, b, v)$ and $(x_k^*, b_k^*, v_k^*) \to (x^*, b^*, v^*)$ such that $v_k \in N(x_k; \Delta(A, b_k))$ and

$$(x_k^*, b_k^*, v_k^*) \in \widehat{N}((x_k, b_k, v_k); \Omega_2) \tag{4.14}$$

for all k. Since

$$I_{A,b_k}(x_k) := \{i \in J \,:\, A_i x_k = (b_k)_i\} \subset J,$$

there must exist a subset $I' \subset J$ such that the equality $I_{A,b_k}(x_k) = I'$ holds for an infinite number of indices k. By considering a subsequence, if necessary, we may assume that $I_{A,b_k}(x_k) = I'$ for all k. The inclusion $I' \subset I$ is valid. Indeed, otherwise there is an index $j \in I' \setminus I$, and we have $A_j x_k = (b_k)_j$ for all k. Passing to the limit, we get $A_j x = b_j$ which is an absurd.

By Lemma 4.1, (4.14) and the equality $I_{A,b_k}(x_k) = I'$ imply that

$$(x_k^*, v_k^*) \in \Big(T(x_k; \Delta(A, b_k)) \cap v_k^{\perp}\Big)^* \times \Big(T(x_k; \Delta(A, b_k)) \cap v_k^{\perp}\Big), \tag{4.15}$$

$$x_k^* = -A_{I'}^T (b_k^*)_{I'}, \tag{4.16}$$

$$(b_k^*)_{\bar{I}'} = 0 \tag{4.17}$$

and

$$(b_k^*)_{I_0'(k)} \leqslant 0, \tag{4.18}$$

where $\bar{I}' = J \setminus I'$, $v_k = \sum_{i \in I'} \lambda_i^k A_i^T$ with $\lambda_i^k \geqslant 0$ being nonnegative for all $i \in I'$, and $I_0'(k) := \{i \in I' \,:\, \lambda_i^k = 0\}$. Since

$$T(x_k; \Delta(A, b_k)) = \{v \,:\, A_{I'}v \leqslant 0\} = T(\mathcal{F}_{I'}; C) \quad \forall k,$$

we can rewrite (4.15) as follows

$$(x_k^*, v_k^*) \in \Big(T(\mathcal{F}_{I'}; C) \cap v_k^{\perp}\Big)^* \times \Big(T(\mathcal{F}_{I'}; C) \cap v_k^{\perp}\Big). \tag{4.19}$$

By letting $k \to \infty$ and using an argument of the proof of Theorem 3.3, from (4.19), (4.16) and (4.17) we deduce the existence of a closed face Q of the polyhedral convex cone $T(\mathcal{F}_{I'}; C) \cap v^{\perp}$ such that (4.11)–(4.13) are satisfied. □

Remark 4.4. Let $v = \sum_{i \in I'} \lambda_i A_i^T$ with $\lambda_i \geqslant 0$ for all $i \in I'$ and let $I'_0 := \{i \in I' : \lambda_i = 0\}$. Concerning the index sets $I'_0(k)$ appeared in (4.18), we observe that they may vary on k. By considering a subsequence, if necessary, we may assume that $I'_0(k) = I'' \subset I'$. But, in general, the condition $v_k \to v$ does not imply that $I'' \subset I_0$. Hence, from (4.18) we may not have $b^*_{I'_0} \leqslant 0$. This explains why the last property cannot be included in the conclusion of the above theorem.

Using Theorem 4.3 we can estimate the values of the normal coderivative of multifunction F_2 as follows.

Theorem 4.5. (Normal coderivative; the case where b is varying). *For any* $(x, b, v) \in \operatorname{gph} F_2$ *and* $v^* \in \mathbb{R}^n$*, if* $(x^*, b^*) \in D^* F_2(x, b, v)(v^*)$ *then there must exist an index set* $I' \subset I_{A,b}(x)$ *satisfying* (4.10) *and a closed face* Q *of the polyhedral convex cone* $T(\mathcal{F}_{I'}; C) \cap v^\perp$ *such that the conditions* (4.12), (4.13) *are satisfied, and*

$$(x^*, -v^*) \in Q^* \times Q.$$

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