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QUASI-EQUILIBRIUM INCLUSION PROBLEMS OF THE BLUM-OETTLI TYPE AND RELATED PROBLEMS

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Dedicated to Nguyen Van Hien on the occasion of his sixty-fifth birthday

Abstract. Several quasi-equilibrium inclusion problems of the Blum-Oettli type are formulated and sufficient conditions on the existence of solutions are shown. As special cases, we obtain several results on the existence of solutions of general vector ideal (resp. proper, Pareto, weak) quasi-optimization problems, of quasivariational inequalities, and of quasivariational inclusion problems.

1. Introduction

Let Y be a topological vector space and let $C \subset Y$ be a cone. We put $l(C) = C \cap (-C)$. If $l(C) = \{0\}$, then C is said to be a pointed cone. For a given subset $A \subset Y$, one can define efficient points of A with respect to C in different senses as: ideal, Pareto, proper, weak,...(see [6]). The set of these efficient points is denoted by $\alpha\text{Min}(A/C)$ with $\alpha = \text{I}; \alpha = \text{P}\ ; \alpha = \ \text{Pr}\ ; \alpha = \text{W}; \ldots$ for the case of ideal, Pareto, proper, weak efficient points, respectively. Let D be a subset of another topological vector space X. By 2^D we denote the family of all subsets in D. For a given multivalued mapping $f : D \to 2^Y$, we consider the problem of finding $\bar{x} \in D$ such that

$$(GVOP)_\alpha \qquad f(\bar{x}) \cap \alpha\text{Min}(f(D)/C) \neq \emptyset.$$

This is called a general vector α optimization problem corresponding to D, f and C. The set of such points $\bar{x}$ is said to be a solution set of $(GVOP)_\alpha$. The elements of $\alpha\text{Min}(f(D)/C)$ are called α optimal values of $(GVOP)_\alpha$.

Now, let X, Y and Z be topological vector spaces, let $D \subset X, K \subset Z$ be nonempty subsets and let $C \subset Y$ be a cone. Given the following multivalued

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mappings

$$S : D \to 2^D, \qquad P : D \to 2^K, T : D \times D \to 2^K, \qquad F : K \times D \times D \to 2^Y,$$

we are interested in the problem of finding $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } F(y, \bar{x}, \bar{x}) \cap \alpha\mathrm{Min}(F(y, \bar{x}, S(\bar{x}))/C) \neq \emptyset, \text{for all } y \in P(\bar{x}).$$

This is called a general vector α quasi-optimization problem depending on a parameter (α is respectively one of qualifications: ideal, Pareto, proper, weak). Such a point $\bar{x}$ is said to be a solution of $(GVQOP)_\alpha$. The above multivalued mappings S, P, and F are said to be respectively a constraint, a parameter potential, and an utility mapping. These problems also play a central role in the vector optimization theory concerning multivalued mappings and have many relations to the following problems.

(UIQEP), upper ideal quasi-equilibrium problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } F(y, \bar{x}, x) \subset C, \text{for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

(LIQEP), lower ideal quasi-equilibrium problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } F(y, \bar{x}, x) \cap C \neq \emptyset, \text{ for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

(UPQEP), upper Pareto quasi-equilibrium problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } F(y, \bar{x}, x) \not\subset -(C \backslash l(C)), \text{ for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

(LPQEP), lower Pareto quasi-equilibrium problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } F(y, \bar{x}, x) \cap -(C \backslash l(C)) = \emptyset, \text{ for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

(UWQEP), upper weakly quasi-equilibrium problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } F(y, \bar{x}, x) \not\subset (-\mathrm{int}C), \text{ for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

(LWQEP), lower weakly quasi-equilibrium problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } F(y, \bar{x}, x) \cap (-\mathrm{int}C) = \emptyset, \text{ for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

In general, we call the above problems γ quasi-equilibrium problems involving D, K, S, T, F with respect to C, where γ is one of the following qualifications: upper ideal, lower ideal, upper Pareto, lower Pareto, upper weakly, lower weakly. These problems generalize many well-known problems in the optimization theory as quasi-equilibrium problems, quasivariational inequalities, fixed point problems, complementarity problems, saddle point problems, minimax problems as well as different others which have been studied by many authors, for examples, Park [11] Chan and Pang [2], Parida and Sen [10], Gurraggio and Tan [4] etc. for quasi-equilibrium problems and quasivariational problems, Blum and Oettli [1], Lin, Yu and Kassay [5], Tan [12], Minh and Tan [8], Fan [3] for equilibrium and variational inequality problems and by some others in the references therein. One can easily see that the above problems also have many relations with the following quasivariational inclusion problems which have been considered in Tan [12], Luc and Tan [7] and Minh and Tan [8].

(UQVIP), upper quasivariational inclusion problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } F(y, \bar{x}, x) \subset F(y, \bar{x}, \bar{x}) + C, \text{for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

(LQVIP), lower quasivariational inclusion problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } F(y, \bar{x}, \bar{x}) \subset F(y, \bar{x}, x) - C, \text{for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

The purpose of this paper is to give some sufficient conditions on the existence of solutions to the above γ quasi-equilibrium problems involving D, K, S, T, F with respect to $(-C)$, where F is of the form $F(y, x, x') = G(y, x', x) - H(y, x, x')$ with $G, H : K \times D \times D \to 2^Y$ being two different multivalued mappings. We also call them quasi-equilibrium problems of the Blum-Oettli Type.

2. Preliminaries and definitions

Throughout this paper, we denote by X, Y and Z real Hausdorff topological vector spaces. The space of real numbers is denoted by R. Given a subset $D \subset X$, we consider a multivalued mapping $F : D \to 2^Y$. The effective domain of F is denoted by

$$\text{dom} F = \{x \in D / F(x) \neq \emptyset\}.$$

Further, let Y be a topological vector space with a cone C. We introduce new definitions of C-continuities.

Definition 2.1. Let $F : D \to 2^Y$ be a multivalued mapping.

(i) F is said to be upper (resp. lower) C–continuous at $\bar{x} \in \text{dom } F$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that

$$F(x) \subset F(\bar{x}) + V + C$$

$$(F(\bar{x}) \subset F(x) + V - C, \text{ respectively})$$

holds for all $x \in U \cap \text{dom} F$.

(ii) If F is simultaneously upper C–continuous and lower C–continuous at $\bar{x}$, then we say that it is C–continuous at $\bar{x}$.

(iii) If F is upper, lower,..., C–continuous at any point of dom$F$, we say that it is upper, lower,..., C–continuous on D.

(iv) In the case $C = \{0\}$ in Y, we shall only say that F is upper, lower continuous instead of upper, lower 0-continuous. The mapping F is continuous if it is simultaneously upper and lower continuous.

Definition 2.2. Let $F : D \times D \to 2^Y$ be a multivalued mapping with nonempty values. We say that

(i) F is upper C-monotone if

$$F(x, y) \subset -F(y, x) - C$$

holds for all $x, y \in D$.

(ii) F is lower C-monotone if for any $x, y \in D$ we have

$$(F(x,y) + F(y,x)) \cap (-C) \neq \emptyset.$$

Definition 2.3. Let $F : K \times D \times D \to 2^Y, T : D \times D \to 2^K$ be multivalued mappings with nonempty values. We say that

(i) F is diagonally upper (T, C)-quasiconvex in the third variable on D if for any finite $x_i \in D, t_i \in [0,1], i = 1, ..., n, \sum_{i=1}^{n} t_i = 1, x_t = \sum_{i=1}^{n} t_i x_i$, there exists $j \in \{1, 2, \ldots, n\}$ such that

$$F(y, x_t, x_j) \subset F(y, x_t, x_t) + C, \text{ for all } y \in T(x_t, x_j).$$

(ii) F is diagonally lower (T, C)-quasiconvex in the third variable on D if for any finite $x_i \in D, t_i \in [0,1], i = 1, ..., n, \sum_{i=1}^{n} t_i = 1, x_t = \sum_{i=1}^{n} t_i x_i$, there exists $j \in \{1, 2, \ldots, n\}$ such that

$$F(y, x_t, x_t) \subset F(y, x_t, x_j) - C, \text{ for all } y \in T(x_t, x_j).$$

To prove the main results we shall need the following theorem:

Theorem 2.4. ([13]) *Let D be a nonempty convex compact subset of X and $F : D \to 2^D$ be a multivalued mapping satisfying the following conditions:*

1) *For all $x \in D, x \notin F(x)$ and $F(x)$ is convex;*

2) *For all $y \in D, F^{-1}(y)$ is open in D.*

Then there exists $\bar{x} \in D$ such that $F(\bar{x}) = \emptyset$.

3. Main results

Let $D \subset X$, $K \subset Z$ be nonempty convex compact subsets, $C \subset Y$ be a convex closed pointed cone. We assume implicitly that multivalued mappings S, T and G, H are as in Introduction. In the sequel, we always suppose that the multivalued mapping S has nonempty convex values and $S^{-1}(x)$ is open for any $x \in D$. We have

Theorem 3.1. *Assume that*

1) *For any $x' \in D$, the set*

$$A_1(x') = \{x \in D | (G(y, x, x') - H(y, x', x)) \not\subset -C, \text{ for some } y \in T(x, x')\}$$

is open in D;

2) *The multivalued mapping $G + H$ is diagonally upper (T, C)-quasiconvex in the third variable;*

3) *For any fixed $y \in K$, the multivalued mapping $G(y, ., .) : D \times D \to 2^Y$ is upper C-monotone;*

4) *$(G(y, x, x) + H(y, x, x)) \subset C$ for all $(y, x) \in K \times D$.*

Then there exists $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } (G(y, x, \bar{x}) - H(y, \bar{x}, x)) \subset -C, \text{for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

Proof. We define the multivalued mapping $M_1 : D \to 2^D$ by

$$M_1(x) = \{x' \in D | (G(y, x', x) - H(y, x, x')) \not\subset -C, \text{ for some } y \in T(x, x')\}.$$

Observe that if for some $\bar{x} \in D, \bar{x} \in S(\bar{x})$, one has $M_1(\bar{x}) \cap S(\bar{x}) = \emptyset$, then

$$(G(y, x, \bar{x}) - H(y, \bar{x}, x)) \subset -C, \text{for all } x \in S(\bar{x}), y \in T(\bar{x}, x)$$

and hence the proof is complete. Thus, our aim is to show the existence of such a point $\bar{x}$. Consider the multivalued mapping Q from D to itself defined by

$$Q(x) = \begin{cases} coM_1(x) \cap S(x), & \text{if } x \in S(x), \\ S(x), & \text{otherwise,} \end{cases}$$

where the multivalued mapping $coM_1 : D \to 2^D$ is defined by $coM_1(x) = co(M_1(x))$ with $co(B)$ denoting the convex hull of the set B. We now show that Q satisfies all conditions of Theorem 2.4 in Section 2. It is easy to see that for any $x \in D, Q(x)$ is convex and

$$\begin{aligned} Q^{-1}(x) &= [(coM_1)^{-1}(x)) \cap S^{-1}(x)] \cup [S^{-1}(x) \setminus \{x\})] \\ &= [coA_1(x) \cap S^{-1}(x)] \cup [S^{-1}(x) \setminus \{x\})] \end{aligned}$$

is open in D.

Further, we claim that $x \notin Q(x)$ for all $x \in D$. Indeed, suppose to the contrary that there exists a point $\bar{x} \in D$ such that $\bar{x} \in Q(\bar{x}) = coM_1(\bar{x}) \cap S(\bar{x})$. In particular, $\bar{x} \in coM_1(\bar{x})$, we then conclude that there exist $x_1, ..., x_n \in M_1(\bar{x})$ such that $\bar{x} = \sum_{i=1}^n t_i x_i, x_i \in M_1(\bar{x}), t_i \geq 0, \sum_{i=1}^n t_i = 1$. By the definition of M_1 we can see that

$$(1) \qquad (G(y_i, x_i, \bar{x}) - H(y_i, \bar{x}, x_i)) \not\subset -C,$$

for some $y_i \in T(\bar{x}, x_i)$, and for all $i = 1, ..., n$. Since the multivalued mapping $G + H$ is diagonally upper (T, C)-quasiconvex in the third variable, there exists $j \in \{1, ..., n\}$ such that

$$(2) \qquad G(y, \bar{x}, x_j) + H(y, \bar{x}, x_j) \subset C + G(y, \bar{x}, \bar{x}) + H(y, \bar{x}, \bar{x}) \subset C,$$

for all $y \in T(\bar{x}, x_j)$.

Since G is upper C-monotone, we deduce

$$(3) \qquad G(y, x_j, \bar{x}) \subset (-C - G(y, \bar{x}, x_j)), \text{for } y \in T(\bar{x}, x_j).$$

A combination of (2) and (3) gives

$$\begin{aligned} (G(y, x_j, \bar{x}) - H(y, \bar{x}, x_j)) &\subset (-C - \{G(y, \bar{x}, x_j) + H(y, \bar{x}, x_j)\}) \\ &\subset -C - C = -C, \text{for all } y \in T(\bar{x}, x_j). \end{aligned}$$

This contradicts (1). Applying Theorem 2.4 in Section 2, we conclude that there exists a point $\bar{x} \in D$ with $Q(\bar{x}) = \emptyset$. If $\bar{x} \notin S(\bar{x})$, then $Q(\bar{x}) = S(\bar{x}) = \emptyset$, which is impossible. Therefore, we deduce $\bar{x} \in S(\bar{x})$, and $Q(\bar{x}) = coM_1(\bar{x})) \cap S(\bar{x}) = \emptyset$. This implies $M_1(\bar{x}) \cap S(\bar{x}) = \emptyset$ and hence

$$\bar{x} \in S(\bar{x}), \quad (G(y, x, \bar{x}) - H(y, \bar{x}, x)) \subset -C, \text{for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

The proof is complete. $\square$

Theorem 3.2. *Assume that*

1) *For any $x' \in D$, the set*

$$A_2(x') = \{x \in D | (G(y,x,x') - H(y,x',x)) \cap (-C) \neq \emptyset, \text{ for some } y \in T(x,x')\}$$

is open in D;

2) *The multivalued mapping $G+H$ is diagonally lower (T,C)-quasiconvex in the third variable;*

3) *For any fixed $y \in K$, the multivalued mapping $G(y,.,.) : D \times D \to 2^Y$ is upper C-monotone;*

4) $(G(y,x,x) + H(y,x,x)) \subset C$ *for all* $(y,x) \in K \times D$.

Then there exists $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } (G(y,x,\bar{x}) - H(y,\bar{x},x)) \cap (-C) \neq \emptyset, \text{for all } x \in S(\bar{x}), y \in T(\bar{x},x).$$

Proof. The proof proceeds exactly as the one of Theorem 3.1 with M_1 replaced by

$$M_2(x) = \{x' \in D| \, (G(y,x',x) - H(y,x,x') \cap (-C) = \emptyset,$$

for some $y \in T(x,x')\}$. Similarly, as in (1) we obtain

$$(G(y_i,x_i,\bar{x}) - H(y_i,\bar{x},x_i)) \cap (-C) = \emptyset, \text{for } i = 1,...,n, y_i \in T(\bar{x},x_i). \tag{4}$$

Since the multivalued mapping $G+H$ is diagonally lower (T,C)-quasiconvex in the third variable, there exists $j \in \{1,...,n\}$ such that

$$(G(y,\bar{x},x_j) + H(y,\bar{x},x_j)) \cap C \neq \emptyset, \text{for all } y \in T(\bar{x},x_j).$$

Since G is upper C-monotone, we deduce

$$(G(y,x_j,\bar{x}) \subset (-C - G(y,\bar{x},x_j)), \text{for } y \in T(\bar{x},x_j).$$

Therefore, we have

$$G(y,\bar{x},x_j) + H(y,\bar{x},x_j) \subset (-C - \{G(y,x_j,\bar{x}) - H(y,\bar{x},x_j))\}$$

and then

$$\emptyset \neq (G(y,\bar{x},x_j) + H(y,\bar{x},x_j) \cap C \subset C \cap (-C - \{(G(y,x_j,\bar{x}) - H(y,\bar{x},x_j))\}).$$

This implies

$$(G(y,x_j,\bar{x}) - H(y,\bar{x},x_j)) \cap (-C) \neq \emptyset, \text{for all } y \in T(\bar{x},x_j).$$

This contradicts (4). Further, we can argue as in the proof of Theorem 3.1. □

Theorem 3.3. *Assume that*

1) *For any $x' \in D$, the set*

$$A_3(x') = \{x \in D| \quad (G(y,x,x') - H(y,x',x) \subset (C \backslash \{0\}) \quad \text{for some} \quad y \in T(x,x')\}$$

is open in D;

2) *The multivalued mapping $G+H$ is diagonally lower (T,C)-quasiconvex in the third variable;*

3) *For any fixed $y \in K$, the multivalued mapping $G(y,.,.) : D \times D \to 2^Y$ is upper C-monotone;*

4) $(G(y,x,x) + H(y,x,x)) \cap (-C \setminus \{0\}) = \emptyset$ *for all* $(y,x) \in K \times D$.

Then there exists $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } (G(y,x,\bar{x}) - H(y,\bar{x},x)) \not\subset (C \setminus \{0\}), \text{ for all } x \in S(\bar{x}), y \in T(\bar{x},x).$$

Proof. The proof proceeds exactly as the one of Theorem 3.1 with M_1 replaced by

$$M_3(x) = \{x' \in D | \, (G(y,x',x) - H(y,x,x')) \subset C \setminus \{0\}, \text{ for some } y \in T(x,x')\}.$$

Similarly, as in (1) we obtain

$$(G(y_i,x_i,\bar{x}) - H(y_i,\bar{x},x_i)) \subset C \setminus \{0\}, \text{ for } i = 1,\dots,n, y_i \in T(\bar{x},x_i). \tag{5}$$

Since the multivalued mapping $G + H$ is diagonally lower (T,C)-quasiconvex in the third variable, there exists $j \in \{1,...,n\}$ such that

$$(G(y,\bar{x},x_j) + H(y,\bar{x},x_j)) \cap (C + G(y,\bar{x},\bar{x}) + H(y,\bar{x},\bar{x})) \neq \emptyset, \forall\, y \in T(\bar{x},x_j).$$

Since G is upper C-monotone, we then have

$$(G(y,\bar{x},x_j) + H(y,\bar{x},x_j)) \subset (-C - \{G(y,x_j,\bar{x}) - H(y,\bar{x},x_j)\}),$$

for all $y \in T(\bar{x},x_j)$.
This implies

$$(C + G(y,\bar{x},\bar{x}) + H(y,\bar{x},\bar{x})) \cap (-C - \{G(y,x_j,\bar{x}) - H(y,\bar{x},x_j)\}) \neq \emptyset,$$

for all $y \in T(\bar{x},x_j)$. Together with (5) we get

$$(G(y_j,\bar{x},\bar{x}) + H(y_j,\bar{x},\bar{x})) \cap -(C \setminus \{0\}) \neq \emptyset,$$

which is impossible by Assumption 4.

The rest of the proof can be done as in proving Theorem 3.1. □

Theorem 3.4. *Assume that*

1) *For any $x' \in D$, the set*

$$A_4(x') = \{x \in D | (G(y,x,x') - H(y,x',x)) \cap (C \setminus \{0\}) \neq \emptyset \textit{ for some } y \in T(x,x')\}$$

is open in D;

2) *The multivalued mapping $G + H$ is diagonally upper (T,C)-quasiconvex in the third variable with $G(y,x,x) + H(y,x,x) \subset C$, for any $(y,x) \in D \times K$;*

3) *For any fixed $y \in K$, the multivalued mapping $G(y,.,.) : D \times D \to 2^Y$ is upper C-monotone;*

4) *$(G(y,x,x) + H(y,x,x)) \cap (-C \setminus \{0\}) = \emptyset$ for all $(y,x) \in K \times D$.*

Then there exists $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } (G(y,x,\bar{x}) - H(y,\bar{x},x)) \cap (C \setminus \{0\}) = \emptyset, \text{for all } x \in S(\bar{x}), y \in T(\bar{x},x).$$

Proof. The proof proceeds exactly as the one of Theorem 3.1 with M_1 replaced by

$$M_4(x) = \{x' \in D| \quad (G(y,x',x) - H(y,x,x') \cap (C \setminus \{0\}) \neq \emptyset, \text{ for some } y \in T(x,x')\}.$$

Similarly, as in (1) we obtain

$$(G(y_i,x_i,\bar{x}) - H(y_i,\bar{x},x_i)) \cap (C \setminus \{0\}) \neq \emptyset, \tag{6}$$

for $i = 1,...,n, y_i \in T(\bar{x},x_i)$.

Since the multivalued mapping $G+H$ is diagonally upper (T,C)-quasiconvex in the third variable, there exists $j \in \{1,...,n\}$ such that

$$G(y,\bar{x},x_j) + H(y,\bar{x},x_j) \subset C + G(y,\bar{x},\bar{x}) + H(y,\bar{x},\bar{x}), \tag{7}$$

for all $y \in T(\bar{x},x)$.

Since G is upper C-monotone,

$$(G(y,x_j,\bar{x}) - H(y,\bar{x},x_j)) \subset (-C - \{G(y,\bar{x},x_j) + H(y,\bar{x},x_j)\}),$$

for all $y \in T(\bar{x},x_j)$ and then together with (7), we deduce

$$(G(y,x_j,\bar{x}) - H(y,\bar{x},x_j)) \subset (-C - \{G(y,\bar{x},\bar{x}) + H(y,\bar{x},\bar{x})\}), \tag{8}$$

for all $y \in T(\bar{x},x_j)$.

A combination of (6) and (8) gives

$$(C \setminus \{0\}) \cap (-C - \{G(y_j,\bar{x},\bar{x}) + H(y_j,\bar{x},\bar{x})\}) \neq \emptyset.$$

It follows that

$$(G(y_j,\bar{x},\bar{x}) + H(y_j,\bar{x},\bar{x})) \cap -(C \setminus \{0\}) \neq \emptyset.$$

This is impossible by Assumption 4.

Further, we continue the proof as in Theorem 3.1. □

Theorem 3.5. *Assume that*

1) *For any $x' \in D$, the set*

$$A_5(x') = \{x \in D | \quad (G(y,x,x') - H(y,x',x)) \subset \mathrm{int}C \text{ for some } y \in T(x,x')\}$$

is open in D;

2) *The multivalued mapping $G+H$ is diagonally lower (T,C)-quasiconvex in the third variable with $G(y,x,x)+H(y,x,x) \subset C$, for any $(y,x) \in D \times K$;*

3) *For any fixed $y \in K$, the multivalued mapping $G(y,.,.) : D \times D \to 2^Y$ is upper C-monotone.*

4) *$(G(y,x,x)+H(y,x,x)) \cap -\mathrm{int}C = \emptyset$ for all $(y,x) \in K \times D$.*

Then there exists $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and } (G(y,x,\bar{x}) - H(y,\bar{x},x)) \not\subset \mathrm{int}C, \text{ for all } x \in S(\bar{x}), y \in T(\bar{x},x).$$

Proof. The proof proceeds exactly as the one of Theorem 3.1 with M_1 replaced by

$$M_5(x) = \{x' \in D | \quad (G(y,x',x) - H(y,x,x') \subset \mathrm{int}C, \text{ for some } y \in T(x,x')\}.$$

Similarly, as in (1) we obtain

$$(G(y_i,x_i,\bar{x}) - H(y_i,\bar{x},x_i)) \subset \mathrm{int}C, \text{ for } i=1,...,n, \ y_i \in T(\bar{x},x_i). \tag{9}$$

Since the multivalued mapping $G+H$ is diagonally lower (T,C)-quasiconvex in the third variable, there exists $j \in \{1,...,n\}$ such that

$$G(y,\bar{x},x_j) + H(y,\bar{x},x_j) \cap (C + G(y,\bar{x},\bar{x}) + H(y,\bar{x},\bar{x})) \neq \emptyset, \forall\, y \in T(\bar{x},x_j).$$

Since G is upper C-monotone, we then have

$$G(y,\bar{x},x_j) + H(y,\bar{x},x_j) \subset (-C - \{G(y,x_j,\bar{x}) - H(y,\bar{x},x_j)\})$$

$$\subset (-C - \mathrm{int}C) = -\mathrm{int}C \text{ for all } y \in T(\bar{x}, x_j).$$

Together with (9), we conclude

$$(C + G(y_i, \bar{x}, \bar{x}) + H(y_i, \bar{x}, \bar{x})) \cap -\mathrm{int}C \neq \emptyset.$$

It is impossible by Assumption 4.

Further, we continue the proof as in Theorem 3.1. □

Theorem 3.6. *Assume that*

1) *For any $x' \in D$, the set*

$$A_6(x') = \{x \in D| \quad (G(y, x, x') - H(y, x', x)) \cap \mathrm{int}C \neq \emptyset \textit{ for some } y \in T(x, x')\}$$

is open in D;

2) *The multivalued mapping $G + H$ is diagonally upper (T, C)-quasiconvex in the third variable;*

3) *For any fixed $y \in K$, the multivalued mapping $G(y, ., .) : D \times D \to 2^Y$ is upper C-monotone.*

4) *$(G(y, x, x) + H(y, x, x)) \subset C$ for all $(y, x) \in K \times D$.*

Then there exists $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \textit{ and } (G(y, x, \bar{x}) - H(y, \bar{x}, x)) \cap \mathrm{int}C = \emptyset, \textit{ for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

Proof. The proof proceeds exactly as the one of Theorem 3.1 with M_1 replaced by

$$M_6(x) = \{x' \in D| \quad (G(y, x', x) - H(y, x, x')) \cap \mathrm{int}C \neq \emptyset, \text{ for some } y \in T(x, x')\}.$$

Similarly, as in (1) we obtain

$$(10) \qquad (G(y_i, x_i, \bar{x}) - H(y_i, \bar{x}, x_i)) \cap \mathrm{int}C \neq \emptyset, \text{ for } \quad i = 1, ..., n, \; y_i \in T(\bar{x}, x_i).$$

Since the multivalued mapping $G + H$ is diagonally upper (T, C)-quasiconvex in the third variable, there exists $j \in \{1, ..., n\}$ such that

$$(G(y, \bar{x}, x_j) + H(y, \bar{x}, x_j)) \subset (C + G(y, \bar{x}, \bar{x}) + H(y, \bar{x}, \bar{x})), \text{for all } y \in T(\bar{x}, x_j).$$

Remarking that

$$(G(y, \bar{x}, \bar{x}) + H(y, \bar{x}, \bar{x})) \subset C,$$

we obtain

$$(11) \qquad (G(y, \bar{x}, x_j) + H(y, \bar{x}, x_j)) \subset C, \text{for all } y \in T(\bar{x}, x_j).$$

Since G is upper C-monotone, we then have

$$(G(y, x_j, \bar{x}) - H(y, \bar{x}, x_j)) \subset (-C - \{G(y, \bar{x}, x_j) + H(y, \bar{x}, x_j)\}),$$

for all $y \in T(\bar{x}, x_j)$.

Taking account of (11), we conclude that

$$(G(y, x_j, \bar{x}) - H(y, \bar{x}, x_j)) \subset -C \text{ for all } y \in T(\bar{x}, x_j).$$

A combination of (10) and (11) gives

$$\mathrm{int}C \cap (-C) \neq \emptyset.$$

It is impossible, since C is a pointed cone.

Further, we continue the proof as in Theorem 3.1. □

Remark 1. 1) In the case $G(y,x,x') = \{0\}$ (resp. $H(y,x,x') = \{0\}$) for all $(y,x,x') \in K \times D \times D$, the above theorems show the existence of solutions of quasi-equilibrium inclusion problems of the Ky Fan (of the Browder-Minty, respectively) type. These also generalize the results obtained by Luc and Tan [7]; Minh and Tan [8, 9] and many other well-known results for vector optimization problems, variational inequalities, equilibrium, quasi-equilibrium problems concerning scalar and vector functions optimization etc.

2) If G and H are single-valued mappings, then we can see that Theorem 3.1 coincides with Theorem 3.2, Theorem 3.3 with Theorem 3.4 and Theorem 3.5 with Theorem 3.6.

Further, the following propositions give sufficient conditions putting on the multivalued mappings T and F such that Conditions 1 of the above theorems are satisfied.

Proposition 3.7. *Let $F : K \times D \to 2^Y$ be a lower C-continuous multivalued mapping with nonempty values and $T : D \to 2^K$ be a lower continuous multivalued mapping with nonempty values. Then the set*

$$A_1 = \{x \in D |\ F(T(x), x) \not\subset -C\}$$

is open in D.

Proof. Let $\bar{x} \in A_1$ be arbitrary. We have $F(T(\bar{x}), \bar{x}) \not\subset -C$. Therefore, there exists $\bar{y} \in T(\bar{x})$ such that $F(\bar{y}, \bar{x}) \not\subset -C$. Since F is lower C-continuous at $(\bar{y}, \bar{x}) \in K \times D$, then for any neighborhood V of the origin in Y one can find neighborhoods U of $\bar{x}, W$ of $\bar{y}$ such that

$$F(\bar{y}, \bar{x}) \subset F(y, x) + V - C, \text{ for all } (y, x) \in W \times U.$$

Since T is lower continuous at $\bar{x}$, one can find a neighborhood $U_0 \subset U$ of $\bar{x}$ such that

$$T(x) \cap W \neq \emptyset, \text{ for all } \quad x \in U_0 \cap D.$$

Hence, for any $x \in U_0 \cap D$ there is $y \in T(x) \cap W$, such that

$$F(\bar{y}, \bar{x}) \subset F(y, x) + V - C.$$

If there is some $x \in U_0 \cap D, y \in T(x), F(y, x) \subset -C$, then we have $F(\bar{y}, \bar{x}) \subset V - C$ for any V. It then follows that $F(\bar{y}, \bar{x}) \subset -C$ and we have a contradiction. So, we have shown that

$$F(T(x), x) \not\subset -C, \text{ for all } x \in U_0 \cap D.$$

This means that $U_0 \cap D \subset A_1$ and then A_1 is open in D. □

Proposition 3.8. *Let $F : K \times D \to 2^Y$ be an upper C-continuous multivalued mapping with nonempty values and $T : D \to 2^K$ be a lower continuous multivalued mapping with nonempty closed values. Then the set*

$$A_2 = \{x \in D |\ F(y, x) \cap (-C) = \emptyset, \text{ for some } y \in T(x)\}$$

is open in D.

Proof. Let $\bar{x} \in A_2$ be arbitrary, $F(\bar{y}, \bar{x}) \cap (-C) = \emptyset$, for some $\bar{y} \in T(\bar{x})$. Since F is upper C-continuous at $(\bar{y}, \bar{x}) \in K \times D$, then for any neighborhood V of the origin in Y one can find neighborhoods U of $\bar{x}, W$of $\bar{y}$ such that

$$F(y, x) \subset F(\bar{y}, \bar{x}) + V + C, \text{ for all } (y, x) \in W \times U.$$

Since T is lower continuous at $\bar{x}$, one can find a neighborhood U_0 of $\bar{x}$ such that

$$T(x) \cap W \neq \emptyset, \text{ for all } x \in U_0 \cap D.$$

Therefore, for any $x \in U_0 \cap D$ there is $y \in T(x) \cap W$, we have

$$F(y, x) \subset F(\bar{y}, \bar{x}_0) + V + C.$$

If there is some $x \in U_0 \cap D, y \in T(x), F(y, x) \cap (-C) \neq \emptyset$, then we have $(F(\bar{y}, \bar{x}) + V + C) \cap (-C) \neq \emptyset$ for any V. It then follows that $F(\bar{y}, \bar{x}) \cap (-C) \neq \emptyset$ and we have a contradiction. So, we have shown that

$$F(T(x), x) \cap (-C) = \emptyset, \text{ for all } x \in U_0 \cap D.$$

This means that $U_0 \cap D \subset A_2$ and then A_2 is open in D. □

Analogously, we can prove the following propositions

Proposition 3.9. *Let $F : K \times D \to 2^Y$ be an upper C-continuous multivalued mapping with nonempty values and $T : D \to 2^K$ be a lower continuous multivalued mapping with nonempty values. Then the set*

$$A_3 = \{x \in D| \quad F(y, x) \subset \text{int}C, \quad \textit{for some} \quad y \in T(x)\}$$

is open in D.

Proposition 3.10. *Let $F : K \times D \to 2^Y$ be a lower C-continuous multivalued mapping with nonempty values and $T : D \to 2^K$ be a lower continuous multivalued mapping with nonempty values. Then the set*

$$A_4 = \{x \in D| \quad F(y, x) \cap \text{int}C \neq \emptyset, \quad \textit{for some} \quad y \in T(x)\}$$

is open in D.

Remark 2. 1) Assume that the multivalued mappings T, G and H are given as in Theorems 3.1-3.6 with nonempty values. In addition, suppose that T is a lower continuous multivalued mapping. For any fixed $x \in D$ if the multivalued mapping $F : K \times D \to D$ defined by

$$F(y, x') = G(y, x, x') - H(y, x', x), \quad (y, x') \in K \times D,$$

is lower, upper, upper and lower C-continuous, then Conditions 1 of Theorems 3.1, 3.2, 3.5 and 3.6 is satisfied, respectively.

2) Assume that there exists a cone $\tilde{C} \subset Y$ such that $\tilde{C}$ is not the whole space Y and $(C \setminus \{0\}) \subset \text{int}\tilde{C}$ and the mapping T is lower continuous, the mapping F defined as above is upper (resp. lower) C-continuous, then Theorem 3.3 (resp. Theorem 3.4) is also true without Condition 1 (apply Theorems 3.5, 3.6 with C replaced by $\tilde{C}$).

To conclude this section, we consider the simple case when G and H are real functions. We can see that Theorems 3.1- 3.6 are extensions of a result by Blum and Oettli to vector and multivalued problems. We have

Theorem 3.11. *Let D, K, S, T be as above with T lower continuous. Let $G, H : K \times D \times D \to R$ be real functions satisfying the following conditions*

1) *For any fixed $(y, x) \in K \times D$ the function $F : D \to R$ defined by $F(x') = G(y, x, x') - H(y, x', x)$ is lower semi-continuous in the usual sense. For any fixed, $y \in K, x_1, x_2 \in D$, the function $g : [0, 1] \to R$ defined by $g(t) = G(y, tx_1 + (1-t)x_2, x_2)$ is upper semi-continuous in the usual sense;*

2) *For any fixed $(y, x) \in K \times D, G(y, x, .), H(y, x, .)$ are convex functions;*

3) *For any fixed $y \in K$ the function $G(y, ., .)$ is monotone (i.e, $G(y, x, x') + G(y, x'x) \leq 0$, for all $x, x' \in D$);*

4) *$G(y, x, x) = H(y, x, x) = 0$ for all $(y, x) \in K \times D$.*

Then there exists $\bar{x} \in D$ such that $\bar{x} \in S(\bar{x})$ and

$$G(y, \bar{x}, x) + H(y, \bar{x}, x) \geq 0, \textit{ for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

Proof. Take $Y = R, C = R_+$, we can see that all assumptions of Theorems 3.1–3.6 are satisfied. Applying any of the theorems, we conclude that there exists $\bar{x} \in D$ with $\bar{x} \in S(\bar{x})$ such that

$$G(y, x, \bar{x}) - H(y, \bar{x}, x) \leq 0, \text{ for all } x \in S(\bar{x}), y \in T(\bar{x}, x).$$

This is equivalent to

$$G(y, \bar{x}, x) + H(y, \bar{x}, x) \geq 0, \text{ for all } x \in S(\bar{x}), y \in T(\bar{x}, x),$$

(see the proof in [1]). □

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