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CONVEX METRICS REVISITED

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Dedicated to Nguyen Van Hien on the occasion of his sixty-fifth birthday

Abstract. This short note gives a new proof and some extensions of the classical result of Witzgall that any convex weak metric is derived from a gauge.

1. Introduction

A gauge is a real-valued function γ defined on a real vector space V satisfying the following properties for any u, v (see [4]):

G1: $\gamma(u) \geq 0$,
G2: $\gamma(ru) = r.\gamma(u)$ for any $r \geq 0$,
G3: $\gamma(u + v) \leq \gamma(u) + \gamma(v)$.

Any gauge γ defines a distance measure d_γ by

$$d_\gamma(x, y) = \gamma(y - x)$$

which is easily seen to be a weak metric, i.e. satisfies the following properties:

D1: $d(x, y) \geq 0$, (nonnegativity)
D2: $d(x, x) = 0$, (identity)
D3: $d(x, z) \leq d(x, y) + d(y, z)$. (triangle inequality)

Furthermore, any gauge γ is evidently a convex function, and it follows that the distance d_γ derived from it is a convex function $V \times V \to \mathbb{R}^+$, and also for any $x \in V$ each of the functions $d_\gamma(x, .)$ and $d_\gamma(., x)$ is always a convex function $V \to \mathbb{R}^+$.

Witzgall ([5, 6]) proved that the converse also holds: any weak metric defined on a finite dimensional real vector space, for which distance up to (and from) any fixed point is convex, is necessarily derived from a gauge.

In this note, we give another proof of this result, using geometric arguments which applies also in case of infinite dimension and, for dimension 1, may be restricted to a convex subset. The proof consists of two steps. First we show that each distance up to (and from) a fixed point is derived from a gauge. Secondly, using some small technical lemmas, we prove that all these gauges are the same for all fixed points.

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2. Another proof of Witzgall's theorem and some generalizations

In the next theorem we show that each distance induced by a weak metric up to (and from) any fixed point is derived from a gauge. The result holds for weak metrics on a convex subset in a general vector space.

Theorem 1. *Let C be a convex subset of V and $d : C \times C \to \mathbb{R}^+$ a weak metric on C. If for all $x \in C$ both derived functions*

$$d^x : C - x \to \mathbb{R}^+ : u \mapsto d(x, x + u)$$

$$d_x : C - x \to \mathbb{R}^+ : u \mapsto d(x + u, x)$$

are convex, then each of these functions d^x and d_x is a gauge restricted to its respective domain.

Proof. We detail the proof for d^x, where $x \in C$ is arbitrary, the case of d_x being fully similar. We prove that d^x satisfies properties G1,G2,G3, restricted to $C - x$. First note that since $x \in C$, we always have $0 \in C - x$, and $C - x$ is always convex as a translate of the convex C.

By D1 it is evident that d^x satisfies G1.

To show G2, consider first any $\lambda \in [0, 1]$ and $u \in C - x$. Then $\lambda u = (1 - \lambda)0 + \lambda u \in C - x$, and by convexity of d^x and $d^x(0) = d(x, x) = 0$ (using D2), we obtain already

$$d^x(\lambda u) \leq \lambda d^x(u).$$

In order to obtain also the inverse inequality, the triangle inequality D3 shows that

$$d(x, x + u) \leq d(x, x + \lambda u) + d(x + \lambda u, x + u) \tag{2.1}$$

since $x + \lambda u \in C$. A similar reasoning as above, using $-u \in C - x - u$ and $1 - \lambda$ and the convexity of d_{x+u} on $C - x - u$, yields

$$d_{x+u}(-(1 - \lambda)u) \leq (1 - \lambda)d_{x+u}(-u)$$

or $d(x+\lambda u, x+u) \leq (1-\lambda)d(x, x+u)$. Combined with (2.1) we therefore obtain

$$d(x, x + u) \leq d(x, x + \lambda u) + (1 - \lambda)d(x, x + u)$$

or $\lambda d(x, x + u) \leq d(x, x + \lambda u)$ as sought to conclude $d^x(\lambda u) = \lambda d^x(u)$.

G2 then also follows for $\lambda > 1$ as soon as $\lambda u \in C - x$ since $\frac{1}{\lambda} \in [0, 1]$, so $d^x(u) = d^x(\frac{1}{\lambda}(\lambda u)) = \frac{1}{\lambda} d^x(\lambda u)$.

G3 now becomes a direct consequence of convexity : for any $u, v \in C - x$ with $u + v \in C - x$ we have

$$d^x(u+v) = d^x(2(\frac{1}{2}u+\frac{1}{2}v)) = 2d^x(\frac{1}{2}u+\frac{1}{2}v) \leq 2(\frac{1}{2}d^x(u)+\frac{1}{2}d^x(v)) = d^x(u)+d^x(v).$$

□

Remark. It may be noted that Witzgall [6] gives the following example of a weak metric d on $\mathbb{R}$ in which only the d^x are convex, but not all positively homogeneous, showing all assumptions above are needed:

$$d(x,y) = \begin{cases} 2(y-x) & \text{if } x < y \\ 2(x-y) & \text{if } 0 \le y \le x \\ 2x - y & \text{if } y < 0 \le x \\ x - y & \text{if } y \le x < 0 \end{cases}$$

The following three technical lemmas will allow us to prove that a weak metric on an arbitrary vector space is derived from a gauge, if each distance from any fixed point is a gauge.

Lemma 2. *Let $\gamma : V \to \mathbb{R}^+$ be a gauge on a real vector space V then for all $u, v \in V$,*

$$|\gamma(u) - \gamma(v)| \le M = \max\{\gamma(u-v), \gamma(v-u)\}.$$

Proof. For $u, v \in V$ we have

$$\gamma(u) = \gamma(u - v + v) \le \gamma(u - v) + \gamma(v)$$

and so $\gamma(u) - \gamma(v) \le \gamma(u - v) \le M$.
Analogously we find $\gamma(v) - \gamma(u) \le \gamma(u - v) \le M$. □

Lemma 3. *Let $\gamma : V \to \mathbb{R}^+$ be a gauge on a real vector space V then for all $u', v \in V$, $f : \mathbb{R} \to \mathbb{R}, \mu \to \gamma(u' + \mu v) - \gamma(\mu v)$ has the Lipschitz property with Lipschitz constant equal to $2\max\{\gamma(v), \gamma(-v)\}$.*

Proof. Let $\mu_1, \mu_2 \in \mathbb{R}$. Then

$$\begin{aligned} &|f(\mu_2) - f(\mu_1)| \\ &= |\gamma(u' + \mu_2 v) - \gamma(\mu_2 v) - \gamma(u' + \mu_1 v) + \gamma(\mu_1 v)|) \\ &\le |\gamma(u' + \mu_2 v) - \gamma(u' + \mu_1 v)| + |\gamma(\mu_1 v) - \gamma(\mu_2 v)|) \\ &\le \max\{\gamma((\mu_2 - \mu_1)v), \gamma((\mu_1 - \mu_2)v)\} + \max\{\gamma((\mu_1 - \mu_2)v), \gamma((\mu_2 - \mu_1)v)\} \\ &= 2|\mu_2 - \mu_1| \max\{\gamma(v), \gamma(-v)\}. \end{aligned}$$

□

Lemma 4. *Let γ, γ' be gauges on a real vector space V with $\gamma \not\le \gamma'$. Then for all $v \in V$ there exists $u \in V$ such that $\gamma(v + u) > \gamma(v) + \gamma'(u)$.*

Proof. Since $\gamma \not\le \gamma'$ there exists $u' \ne 0$ such that $\gamma(u') > \gamma'(u')$. We define $f : \mathbb{R} \to \mathbb{R}, \mu \mapsto \gamma(u' + \mu v) - \gamma(\mu v)$. The function f is Lipschitz and therefore continuous at 0 (see [1]). Since $f(0) = \gamma(u') > \gamma'(u') \ge 0$ there exists $\mu > 0$ with $f(\mu) > \gamma'(u')$. This implies $\gamma(u' + \mu v) - \gamma(\mu v) > \gamma'(u')$. Let $u = \frac{u'}{\mu}$. Then

$$\begin{aligned} \gamma(v + u) &= \gamma(v + \frac{u'}{\mu}) = \frac{1}{\mu}\gamma(\mu v + u') \\ &> \frac{1}{\mu}(\gamma'(u') + \gamma(\mu v)) = \gamma'(\frac{u'}{\mu}) + \gamma(v) = \gamma'(u) + \gamma(v). \end{aligned}$$

□

Theorem 5. *Let $d : V \times V \to \mathbb{R}^+$ be a weak metric on a real vector space V. If $\forall v \in V$ $d^v : V \to \mathbb{R}, u \mapsto d(v, v+u)$ is a gauge then all d^v are equal, i.e. d is derived from a gauge.*

Proof. Let $v_1, v_2 \in V$ be such that $d^{v_1} \neq d^{v_2}$. Then either $d^{v_1} \nleq d^{v_2}$ or $d^{v_2} \nleq d^{v_1}$. Suppose that $d^{v_1} \nleq d^{v_2}$. By Lemma 4 there exists $u \in V$ such that

$$d^{v_1}(v_2 - v_1 + u) > d^{v_1}(v_2 - v_1) + d^{v_2}(u).$$

Let $z = v_2 + u$, then

$$\begin{aligned} d(v_1, z) &= d(v_1, v_1 + z - v_1) = d^{v_1}(z - v_1) = d^{v_1}(v_2 + u - v_1), \\ d(v_1, v_2) &= d(v_1, v_1 + v_2 - v_1) = d^{v_1}(v_2 - v_1), \\ d(v_2, z) &= d(v_2, v_2 + u) = d^{v_2}(u). \end{aligned}$$

It follows that $d(v_1, z) > d(v_1, v_2) + d(v_2, z)$, which contradicts the triangle inequality. □

3. The one-dimensional case

The following theorem specifies necessary and sufficient conditions for a weak metric d on a convex subset C in $\mathbb{R}$ to be derived from a gauge when $V = \mathbb{R}$.

Theorem 6. *Let $d : C \times C \to \mathbb{R}$ be a weak metric on a convex subset $C \subset \mathbb{R}$ (case $n = 1$). Then the following statements are equivalent*

(1) *d is a convex function,*
(2) *all functions d^x and d_x ($x \in C$) as defined in Theorem 1 are convex,*
(3) *d is derived from a gauge.*

Proof. The only nontrivial part is the implication (2) to (3). By Theorem 1, (2) implies that for each $x \in C$ both d^x and d_x are positively homogeneous. In particular this means that for all $x \in C$ some nonnegative numbers $d^x(1)$ and $d_x(-1)$ exist such that for all $u > 0$, $u \in C - x$ we have $d^x(u) = d^x(1)u$ and $d_x(-u) = d_x(-1)u$. Observe that $d^x(1)$ (respectively $d_x(-1)$) are uniquely defined, except when $x = \max C$ (respectively $x = \min C$), if this exists, in which case the value is arbitrary. (Note also that the notations $d^x(1)$ and $d_x(-1)$ are not meant to imply that $1 \in C - x$ or $-1 \in C - x$.)

Now for any pair $x < y \in C$ we have

$$\begin{aligned} d(x, y) &= d(x, x + (y - x)) = d^x(y - x) = d^x(1)(y - x) \\ d(x, y) &= d(y - (y - x), y) = d_y(-(y - x)) = d_y(-1)(y - x) \end{aligned}$$

and hence $d^x(1) = d_y(-1)$ as soon as $x < y$.

Thus, for any $x' \in C$ with $x < x'$, either $x' = \max C$ and we may choose $d^{x'}(1) = d^x(1)$, or some $y \in C$ exists with $y > x'$, and we have $d^x(1) = d_y(-1) = d^{x'}(1)$. Also, either $x = \min C$ and we may choose $d_x(-1) = d_{x'}(-1)$ or some $z \in C$ exists with $z < x$, and we have $d_x(-1) = d^z(1) = d_{x'}(-1)$. In each case we have, $d^x(1) = d_x(-1) =: \alpha$ for all $x \in C$. Similarly, using $d(y,x)$ above, we obtain $d_x(1) = d^x(-1) =: \beta$ for all $x \in C$. Then, defining

$$\gamma(u) =: \begin{cases} \alpha u & (u \geq 0) \\ \beta(-u) & (u < 0) \end{cases}$$

we obtain, for all $x, y \in C$, $d(x,y) = \gamma(y - x)$. $\square$

Remark. We give an example of a weak metric defined on a convex subset of $\mathbb{R}$ for which all d^x are gauges but not necessarily equal. This shows in particular that the second condition above may not be weakened to consider the functions d^x only.

Let $C = [1,2]$ and define

$$d : C \times C \to \mathbb{R}^+ : (x,y) \mapsto \begin{cases} x(y-x) & \text{if } x \leq y \\ 0 & \text{if } x \geq y \end{cases}$$

Evidently d satisfies D1 and D2. For D3 take three points x, y, z in C. The inequality is trivial when $x \geq y$ or when two of the three points coincide. If $x < y$ we consider the following cases:

(1) if $x < z < y$ then $d(x,y) = x(y-x) = x(y-z+z-x) = x(y-z)+x(z-x) < z(y-z) + x(z-x) = d(z,y) + d(x,z)$,
(2) if $x < y < z$ then $d(x,y) = x(y-x) < x(z-x) = d(x,z) \leq d(x,z)+d(z,y)$,
(3) if $z < x < y$ then $y \leq 2 = 1+1 < x+z$ and so $(x-z)y < (x-z)(x+z) = x^2 - z^2$ which implies $x(y-x) < z(y-z)$ or $d(x,y) < d(x,z) + d(z,y)$.

For each $x \in C$ the map

$$d^x : C - x \to \mathbb{R}^+ : u \mapsto d(x, x+u) = \begin{cases} xu & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases}$$

is easily seen to be positively homogeneous. For G3 we take two points $u, v \in C-x$ such that $u + v \in C - x$. The inequality is trivial when $u + v \leq 0$ so we suppose $u + v > 0$ and consider the following cases:

(1) if u and v both are positive then $d^x(u+v) = x(u+v) = xu + xv = d^x(u) + d^x(v)$,
(2) if $u > 0$ and $v < 0$ then $d^x(u+v) = x(u+v) = xu + xv < xu = d^x(u) \leq d^x(u) + d^x(v)$.

Finally for $x = \frac{5}{4}$ and $x' = \frac{7}{4}$ we have $\frac{1}{4} \in (C-x) \cap (C-x')$ and $d^x(\frac{1}{4}) = \frac{5}{16} \neq \frac{7}{16} = d^{x'}(\frac{1}{4})$ and so the weak gauges d^x and $d^{x'}$ do not coincide on the intersection of their domains.

4. Some open questions

(1) The proof of Lemma 4 clearly shows that the less the two gauges differ, the smaller μ will have to be chosen, thus enlarging u quite rapidly. This means that in Theorem 5 the z, constructed in order to violate the triangle inequality, will have to be taken the farther away, the closer the unit balls for distance up to x and y are. Therefore our argument does not apply for convex $C \neq V$, and it is not clear how to adapt it to such cases.

One may note that a proof for $C \subset \mathbb{R}^2$ would suffice, since the triangle inequality involves three points, so any proof may always be restricted to the two-dimensional plane containing them. It should be noted that recently Guerrini [2] gave another proof of the result, valid when C is the positive cone of a Riesz space.

One may also try to construct counterexamples. In other words, the question is if one can define a family of gauges for all points from a bounded set, which only slowly change, in such a way that they define together a convex function d on this set, which would also satisfy the triangle inequality, because for its violation one would always need to use a z outside the bounds of the set.

It might also be that in dimension higher than one the full equivalences of Theorem 6 do not hold, in the sense that full convexity of d on $C \times C$ is required to prove it to be derived from a gauge.

(2) Does this theorem still hold when distance may be ∞, as happens for gauges with 0 on the unit ball's boundary?

(3) Can some similar result be derived for quasiconvex metrics? E.g. If a weak metric is quasiconvex in each parameter, might it be derived from a gauge composed with a concave function?

This looks much more dubious, although it might be related to the fact that the composition of a weak metric with any concave real functional which is nonnegative and vanishes at 0 is still a weak metric (see [3]).

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