ACTA MATHEMATICA VIETNAMICA
Volume 34, Number 1, 2009, pp. 81–90

SECOND-ORDER OPTIMALITY CONDITIONS IN SET-VALUED OPTIMIZATION BY A NEW TANGENTIAL DERIVATIVE

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Dedicated to Nguyen Van Hien on the occasion of his sixty-fifth birthday

Abstract. This paper gives new second-order necessary and sufficient optimality conditions in set-valued optimization. We define second-order tangential derivative/epiderivative of set-valued maps by taking contingent derivative of the first-order contingent derivative. The resulting derivatives/epiderivatives have strikingly simple structure and nice properties. The proposed derivatives are then employed to give new second-order optimality conditions for weak-minimality in set-valued optimization.

1. Introduction

Let X and Y be real normed spaces, let Q be a nonempty subset of X, let $C \subset Y$ be a proper pointed closed convex cone with nonempty interior, and let $F : X \rightrightarrows Y$ be a set-valued map. In this work we focus on the following set-valued minimization problem

$$(P) \qquad \operatorname{WMin} F(x) \quad \text{subject to} \quad x \in Q,$$

where we are interested in a weak-minimizer. A *weak-minimizer* of (P) is an element $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in F(\bar{x})$ and $(\cup_{x\in Q} F(x)) \cap (\{\bar{y} - \operatorname{int}(C)\}) = \emptyset$, where $\operatorname{int}(C)$ stands for the interior of C. We recall that given the cone C, the set of weakly minimal points of any nonempty set $A \subset Y$, henceforth denoted by $\operatorname{WMin}(A, C)$, is defined by $\operatorname{WMin}(A, C) = \{x \in A|\ A \cap (\{x\} - \operatorname{int}(C)) = \emptyset\}$. Therefore, for a solution $(\bar{x}, \bar{y})$, we have $\bar{y} \in F(\bar{x}) \cap \operatorname{WMin}(F(Q), C)$ where $F(Q) := \cup_{x\in Q} F(x)$.

In this short paper, we present new second-order optimality conditions for weak-optimality in set-valued optimization. Set-valued optimization presents an important generalization and unification of the scalar and the vector optimization problems. Moreover, there are many research areas that directly lead to optimization problems with set-valued objective, and/or set-valued constraints.

Received April 8, 2008.
2000 Mathematics Subject Classification. 90C26, 90C29, 90C30.
Key words and phrases. Set-valued optimization, second-order generalized contingent derivatives, second-order tangential derivatives, optimality conditions, contingent cones.

For instance, the duality principles in vector optimization, the gap functions for vector variational inequalities, inverse problems, fuzzy optimization, image processing, etc. all lead to optimization problems that can be conveniently framed as set-valued optimization problems. Furthermore, since the set-valued maps appear in many branches of pure and applied mathematics, set-valued optimization has the evident potential to remain as an important and active research topic in the near future. In set-valued optimization, there are two kinds of the optimality conditions, namely, by employing the derivatives of the involved set-valued maps and by using the alternative-type theorems. In the present work, we focus on the use of derivatives of the involved set-valued maps.

In 1988, Corley [7] employed contingent and circatangent derivatives to give general optimality conditions in set-valued optimization. Corley's results were subsequently refined by Luc-Malivert [24], and others. In these works the derivative notion revolves around the graphs of the set-valued maps. Another useful approach which is based on using the epigraphs of set-valued maps was proposed by Jahn-Rauh [18] and Bednarczuk and Song [3], and further pursued in [9, 15, 16, 23], among others. See also [8, 11, 25], and the references therein.

Although the field of first-order optimality conditions in set-valued optimization is still in making, recent developments in non-smooth scalar and vector optimization have shown a tremendous increase in interest towards the development of higher-order optimality conditions [12, 19]. Motivated by this, in [17], the second-order contingent epiderivatives were introduced and used to give second-order optimality conditions in set-valued optimization. These results were further refined in [20] where the second order asymptotic derivatives were used.

In this contribution, we present new second-order optimality conditions in set-valued optimization problems. The motivation behind this work is the desire to give second-order optimality conditions when the underlying second-order contingent sets are empty, and consequently the second-order contingent derivatives, and second-order epiderivatives are undefined. For this, we define a new second-order tangential derivative by taking *contingent derivative of a contingent derivative*. This new tangential derivative, is then used to obtain new optimality conditions in set-valued optimization. One major advantage of this approach is that the second-order derivative has a simpler structure. We will show that the optimality conditions given in this work are more general and subsumes the results obtained by using the second-order contingent derivatives.

2. Derivatives and epiderivatives

We begin with by recalling the definitions of some tangential cones and sets (see [1, 26] for details).

Definition 2.1. Let Z be a real normed space, let $S \subset Z$ be nonempty and let $w \in Z$.

1. The second order contingent set $T^2(S, \bar{z}, w)$ of S at $\bar{z} \in \mathrm{cl}(S)$ (closure of S) in the direction $w \in Z$ is the set of all $z \in Z$ such that there are a sequence

$(z_n) \subset Z$ with $z_n \to z$ and a sequence $(\lambda_n) \subset P := \{t \in R |\ t > 0\}$ with $\lambda_n \downarrow 0$ so that $\bar{z} + \lambda_n w + (\lambda_n^2/2) z_n \in S$.

2. The contingent cone $T(S, \bar{z})$ of S at $\bar{z} \in cl(S)$ is the set of all $z \in Z$ such that there are a sequence $(z_n) \subset Z$ with $z_n \to z$ and a sequence $(\lambda_n) \subset P$ with $\lambda_n \downarrow 0$ so that $\bar{z} + \lambda_n z_n \in S$.
3. The interiorly contingent cone $IT(S, \bar{z})$ of S at $\bar{z}$ is the set of all $v \in Z$ such that for any sequences $(\lambda_n) \subset \mathbb{P}$ and $(v_n) \subset Z$ with $\lambda_n \downarrow 0$ and $v_n \to v$, there exists an integer $m \in \mathbb{N}$ such that $\bar{z} + \lambda_n v_n \in S$ for all $n \geq m$.

Remark 2.1. It is known that the contingent cone $T(S, \bar{z})$ is a nonempty closed cone (cf. [1]). However, $T^2(S, \bar{z}, w)$ is only a closed set (possibly empty), non-connected in general, and it may be nonempty only if $w \in T(S, \bar{z})$. On the other hand the interiorly contingent cone $IT(S, \bar{z})$ is an open cone. As concern the relationship between $T(S, \bar{z})$ and $IT(S, \bar{z})$, we have $IT(S, \bar{z}) = Z \backslash T(Z \backslash S, \bar{z})$. For any $S \subset Z$, the identities $T(S, \bar{z}) = T(\mathrm{cl}(S), \bar{z})$ and $IT(S, \bar{z}) = IT(\mathrm{int}(S), \bar{z})$ hold. Moreover, for a convex solid set S, we have $\mathrm{cl}(IT(S, \bar{z})) = T(S, \bar{z})$ and $\mathrm{int}(T(S, \bar{z})) = IT(S, \bar{z})$. Some details and examples of these cone are given in [1, 4, 19, 26, 27].

Let X and Y be real normed spaces and let $F : X \rightrightarrows Y$ be a set-valued map. The effective domain and the graph of F are given by $\mathrm{dom}(F) := \{x \in X |\quad F(x) \neq \emptyset\}$ and $\mathrm{gph}(F) := \{(x, y) \in X \times Y |\quad y \in F(x)\}$, respectively. Given a proper convex pointed cone $C \subset Y$, the profile map $F_+ : X \rightrightarrows Y$ is defined by: $F_+(x) := (F + C)(x) = F(x) + C$, for every $x \in \mathrm{dom}(F)$. Then the epigraph of F is just the graph of F_+, that is, $\mathrm{epi}(F) = \mathrm{gph}(F_+)$.

We recall that given normed spaces X, Y and a set-valued map $F : X \rightrightarrows Y$, the *contingent derivative* of F at $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ is the set-valued map $D_c F(\bar{x}, \bar{y}) : X \rightrightarrows Y$ defined by

$$\mathrm{gph}(D_c F(\bar{x}, \bar{y})) = T(\mathrm{gph}(F), (\bar{x}, \bar{y})). \tag{1}$$

Using the above notion, the *generalized contingent epiderivative* of F at $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ is the set-valued map $D_g F(\bar{x}, \bar{y}) : X \rightrightarrows Y$ given by

$$D_g F(\bar{x}, \bar{y})(x) := \mathrm{Min}(D_c(F + C)(x), C) \quad x \in \mathrm{dom}(D_c(F + C)(\bar{x}, \bar{y})). \tag{2}$$

Here $\mathrm{Min}(A, C) = \{x \in A |\ A \cap (\{x\} - C) = \{x\}\}$ is the set of all minimal point of any nonempty set A with respect to C.

We say that epiderivative $D_g F(\bar{x}, \bar{y})$ *dominates* at $x \in \mathrm{dom}(D_c F_+(\bar{x}, \bar{y}))$, if $D_c F_+(\bar{x}, \bar{y})(x) \subseteq D_g F(\bar{x}, \bar{y})(x) + C$. It is known that if $D_g F(\bar{x}, \bar{y})$ dominates at all $x \in \mathrm{dom}(D_c F_+(\bar{x}, \bar{y}))$ then

$$\mathrm{epi}(D_g F(\bar{x}, \bar{y})) = T(\mathrm{epi}(F), (\bar{x}, \bar{y})). \tag{3}$$

On the other hand, given a set-valued map $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, the contingent epiderivative $DF(\bar{x}, \bar{y}) : X \to Y$ is a *single-valued map* satisfying (3).

All the above derivatives and epiderivatives have their natural second-order analogues. For instance, the second-order contingent derivative of $F : X \rightrightarrows$

Y, at $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ in the direction $(\bar{u},\bar{v}) \in X \times Y$ is a set-valued map $D_c^2F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \rightrightarrows Y$ given by

$$(4) \qquad D_c^2F(\bar{x},\bar{y},\bar{u},\bar{v})(x) := \left\{ y \in Y \mid \quad (x,y) \in T^2(\mathrm{gph}(F),(\bar{x},\bar{y}),(\bar{u},\bar{v})) \right\}.$$

In recent years the above derivative has been used to give very general optimality condition in set-valued optimization (see [17]). Notice that if $(\bar{u},\bar{v}) = (0_X, 0_Y)$ in the above definition, where 0_X and 0_Y are the zero elements in X and Y, we recover the contingent derivative $D_cF(\bar{x},\bar{y})$ of F at $(\bar{x},\bar{y})$. In particular, if $F : X \to Y$ is a single valued map which is twice continuously Fréchet differentiable around $\bar{x} \in \Omega \subset X$, then the second order contingent derivative of the restriction F_Ω of F to Ω at $\bar{x}$ in a direction $\bar{u}$ is given by the formula (see [1, p. 215]):

$$(5) \quad D^2F_\Omega(\bar{x},F(\bar{x}),\bar{u},F'(\bar{x})(\bar{u}))(x) = F'(\bar{x})(x) + F''(\bar{x})(\bar{u},\bar{u}) \text{ for } x \in T^2(\Omega,\bar{x},\bar{u}).$$

It is empty when $x \notin T^2(\Omega,\bar{x},\bar{u})$.

We now introduce new second-order tangential derivatives and epiderivatives.

Definition 2.2. Let X and Y be real normed spaces, and let $C \subset Y$ be a pointed closed convex cone. Let $F : X \rightrightarrows Y$ be a set-valued map, let $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$, and let $(\bar{u},\bar{v}) \in X \times Y$.

(1) A set-valued map $D^2F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \rightrightarrows Y$ defined by
$$D^2F(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \{y \in Y \mid \quad (x,y) \in T(T(\mathrm{gph}(F),(\bar{x},\bar{y})),(\bar{u},\bar{v}))\}$$
is called second-order tangential derivative of F at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$.

(2) A single-valued map $D_e^2F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \to Y$ defined by
$$\mathrm{epi}(D_e^2F(\bar{x},\bar{y},\bar{u},\bar{v})) = T(T(\mathrm{epi}(F),(\bar{x},\bar{y})),(\bar{u},\bar{v}))$$
is called second-order tangential epiderivative of F at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$.

(3) A set-valued map $D_g^2F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \rightrightarrows Y$ defined by
$$D_g^2F(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \mathrm{Min}(D^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v})(x),C)$$
$$x \in \mathrm{dom}(D^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v}))$$
is called generalized second-order tangential epiderivative of F at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$.

The above concepts are defined in analogy to the notions of contingent derivatives and (generalized) contingent epiderivatives. If $(\bar{u},\bar{v}) = (0_X, 0_Y)$ then the above notions recover the contingent derivative, the contingent epiderivative $DF(\bar{x},\bar{y})(\cdot)$ and the generalized contingent epiderivative $D_gF(\bar{x},\bar{y})(\cdot)$ of F at $(\bar{x},\bar{y})$, respectively.

The following result clarifies the relationship between the second-order tangential derivative and the second-order contingent derivative.

Proposition 2.1. *Assume that $F : X \rightrightarrows Y$ is C-convex. Then the second-order tangential derivative $D^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v})$ of the map $(F+C)$ coincides with the* *second-order contingent derivative* $D_c^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v})$ *of* $(F+C)$ *provided that* $0_Y \in D_c^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v})(0_X)$.

Proof. The proof follows from the known fact that for a convex subset S of a normed space B, the equality $T^2(S,x,y) = T(T(S,x),y)$ holds provided that $0 \in T^2(S,x,y)$. □

The following examples will further clarify the differences and the similarities between the two notions.

Example 2.1. Let $\{F_1, F_2\} : \mathbb{R} \rightrightarrows \mathbb{R}$ be two set-valued maps given by

$$F_1(x) \ := \ \{y \in \mathbb{R} \mid y \geq x^4\} \quad \text{for all } x \in \mathbb{R},$$
$$F_2(x) \ := \ \{y \in \mathbb{R} \mid y \geq x^{3/2}\} \quad \text{for all } x \in \mathbb{R}.$$

Let $(\bar{x},\bar{y}) = (0,0)$ and let $(\bar{u},\bar{v}) = (1,0)$. Define $R_+ = \{r \in R | \ r \geq 0\}$. Then $D_c^2(F_1+R_+)(0,0,1,0)(x) = D^2(F_1+R_+)(0,0,1,0)(x) = R_+$ for every $x \in \mathbb{R}$.

On the other hand

$$D^2(F_1+R_+)(0,0,1,0)(x) = R_+ \quad \text{for every} \quad x \in \mathbb{R},$$

whereas the second-order contingent derivative is not defined.

Remark 2.2. Although the above example is given in a finite-dimensional setting, it clearly highlights the advantages of using the second-order tangential derivatives. A similar situation persists in infinite dimensional setting. For example, several examples of empty as well as nonempty second-order contingent sets in the space of continuous functions defined over bounded intervals can be found in the interesting work of Kawasaki [22]. On the other hand, the contingent cone of the contingent cone is always nonempty. It should be mentioned that the computation of the second-order contingent sets of positive cones in some function spaces is of great importance in many applications such as sensitivity analysis and inverse problems. The interested reader is referred to the important contributions of Cominetti and Penot [6] and Bednarczuk et al. [2] for more details. (see also [22] and [21].)

3. Optimality conditions

Let X, Y, Z be normed spaces and let the spaces Y and Z be partially ordered by nontrivial pointed closed convex cones $C \subset Y$ and $D \subset Z$. We assume that C and D have nonempty interiors, that is, $\text{int}(C) \neq \emptyset$ and $\text{int}(D) \neq \emptyset$. Let $Q_0 \subset X$ be nonempty. Let $F : Q_0 \rightrightarrows Y$ and $G : Q_0 \rightrightarrows Z$ be given set-valued maps.

We are concerned with the following set-valued optimization problems:

(P_0) WMin $F(x)$ subject to $x \in Q_0$.

(P_1) WMin $F(x)$ subject to $x \in Q_1 := \{x \in Q_0 | \ G(x) \cap -D \neq \emptyset\}$.

In (P_0) and (P_1), we seek for a weak-minimizer. Clearly, $(\bar{x},\bar{y}) \in \text{gph}(F)$ is a weak-minimizer of (P_1) if $\bar{y} \in \text{WMin}(F(Q_1), C)$, where $F(Q_1) := \cup_{x \in Q_1} F(x)$.

Observe that (P_1) reduces to (P_0), if $G(x) = 0_Z$. In this case the set of constraints Q_0 is not explicitly specified. If additionally we have $Q_0 = X$, then (P_1) is an unconstrained set-valued optimization problem. The optimality notion given in the above definition is a global one, that is, the whole set $F(Q_1)$ has been taken into account. Its local versions is defined as follows: The point $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ is said to be a *local weak-minimizer,* if there exists a neighborhood U of $\bar{x}$ such that $\bar{y} \in \mathrm{WMin}(F(Q_1 \cap U), C)$.

The notion of weak-minimality requires that the ordering cone has a nonempty interior which is a quite stringent requirement. Nonetheless, many important cones have nonempty interior, as shown in the following example.

Example 3.1. (1) In the n-dimensional Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, the cone $\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) |\ x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0\}$ has a nonempty interior.
(2) Consider the space of continuous functions $C([a, b], \mathbb{R})$ with the norm $\|x\| = \sup\{|x(t)||\ t \in [a, b]\}$. Then the cone $K = \{x \in C([a, b], \mathbb{R}) |\ x(t) \geq 0$ for any $t \in [a, b]\}$ has a nonempty interior.
(3) Consider the space $\ell^2(\mathbb{N}, \mathbb{R})$ with the well-known structure of a Hilbert space. The convex cone $K = \{x = \{x_i\}_{i \geq 0} |\ x_0 \geq 0,\ \Sigma_{i=1}^n x_i^2 \leq x_0^2\}$ has a nonempty interior given by $\mathrm{int}(K) = \{x = \{x_i\}_{i \geq 0} |\ x_0 > 0,\ \Sigma_{i=1}^n x_i^2 < x_0^2\}$.
(4) Let ℓ^∞ be the space of bounded sequences of real numbers, equipped with the norm $\|x\| = \sup_{n \in N}\{|x_n|\}$. The convex cone $K = \{x = \{x_n\}_{n \in \mathbb{N}} |\ x_n \geq 0$, for any $n \in \mathbb{N}\}$ has a nonempty interior.
(5) Consider the space $C^1([a, b], \mathbb{R})$ of real continuously differentiable functions equipped with the norm $\|f\|_1 = \{\int_a^b (f(t))^2 dt + \int_a^b (f'(t))^2 dt\}^{1/2}$ for any $t \in C^1([a, b], \mathbb{R})$. It is known that the cone $K = \{f \in C^1([a, b], \mathbb{R}) |\ f(t) \geq 0$, for any $t \in [a, b]\}$ has a nonempty interior.
(6) Let $(X, \|\cdot\|)$ be a normed vector space and X^* be the topological dual of X^*. Let $f \in X^*$, and let $\epsilon > 0$. The convex cone $K_{f,\epsilon} = \{x \in X |\ f(x) \geq \epsilon\|x\|\}$ has a nonempty interior given by $\mathrm{int}(K_{f,\epsilon}) = \{x \in X |\ f(x) > \epsilon\|x\|\}$.

In the following necessary optimality conditions for (P_0), ∂C stands for the boundary of C.

Theorem 3.1. *Assume that* $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ *is a local weak-minimizer of* (P_0). *Then for every* $\bar{u} \in \mathrm{dom}(D(F + C)(\bar{x}, \bar{y}))$ *and for every* $\bar{v} \in D(F + C)(\bar{x}, \bar{y})(\bar{u}) \cap (-C)$, *we have*

$$D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \cap IT(-\mathrm{int}(C), \bar{v}) = \emptyset \tag{6}$$

for all $x \in \mathrm{dom}(D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$.

Proof. Assume that (6) does not hold, and assume that there exists $x \in \mathrm{dom}(D^2\ (F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$ such that

$$y \in D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \cap IT(-\mathrm{int}(C), \bar{v}).$$

Then $(x,y) \in T(T(\text{epi}(F),(\bar{x},\bar{y})),(\bar{u},\bar{v}))$. Consequently, there are sequences $(t_n) \subset \mathbb{P}$ and $((x_n,y_n)) \subset X \times Y$ such that $t_n \downarrow 0$ and $(x_n,y_n) \to (x,y)$ with

$$(\bar{u}+t_n x_n, \bar{v}+t_n y_n) \in T(\text{epi}(F),(\bar{x},\bar{y})) \quad \text{for every } n \in \mathbb{N}.$$

Since $t_n \downarrow 0$, $y_n \to y$ and $y \in IT(-\text{int}(C),\bar{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\bar{v}+t_n y_n \in -\text{int}(C) \quad \text{for all } n > n_1.$$

For any $n > n_1$, we fix an element $(u_n,v_n) = (\bar{u}+t_n x_n, \bar{v}+t_n y_n)$, and notice that

$$(u_n,v_n) \in T(\text{epi}(F),(\bar{x},\bar{y})).$$

In view of the definition of contingent cone, for (u_n,v_n), there are sequences $(t_m) \subset \mathbb{P}$ and $(x_m,y_m) \in X \times Y$ such that $t_m \downarrow 0$, $(x_m,y_m) \to (u_n,v_n)$ and

$$\bar{y}+t_m y_m \in F(\bar{x}+t_m x_m)+C.$$

Moreover, since $v_n \in -\text{int}(C)$ and $y_m \to v_n$, there exists $m_1 \in \mathbb{N}$ such that $y_m \in -\text{int}(C)$ for all $m > m_1$. This, in view of the fact that C is a cone, further implies that $t_m y_m \in -\text{int}(C)$. Now assume that $w_m \in F(\bar{x}+t_m x_m)$ is such that $\bar{y}+t_m y_m \in w_m + C$. Then

$$w_m \in \bar{y} - \text{int}(C).$$

Since $a_m := (\bar{x}+t_m x_m) \to \bar{x}$, there exists $m_2 > 0$ such that $a_m \in \mathcal{N}(\bar{x})$, where $\mathcal{N}(\bar{x})$ is a suitable neighborhood of $\bar{x}$. Therefore, we have shown that there exists a sequence (w_m) such that

$$w_m \in F(a_m) \cap (\bar{y} - \text{int}(C)) \quad \text{for all } m > \{m_1, m_2\}.$$

However, this a contradiction to the weak-optimality of $(\bar{x},\bar{y})$. The proof is complete. □

Setting $(\bar{x},\bar{y}) = (0_X, 0_Y)$, we get the following known first-order optimality condition (see [7]).

Corollary 3.1. *Assume that* $(\bar{x},\bar{y}) \in \text{gph}(F)$ *is a local weak-minimizer of* (P_0). *Then*

$$D(F+C)(\bar{x},\bar{y}) \cap -\text{int}(C) = \emptyset \quad \textit{for all } \ x \in \text{dom}(D(F+C)(\bar{x},\bar{y})). \tag{7}$$

The following result shows that (6) is in fact a sufficient optimality condition provided that a certain convexity hypothesis holds.

Theorem 3.2. *Assume that* $\text{gph}(F+C)$ *is convex, and assume that the following condition holds:*
For every $\bar{u} \in \text{dom}(D(F+C)(\bar{x},\bar{y}))$ *and for every* $\bar{v} \in D(F+C)(\bar{x},\bar{y})(\bar{u}) \cap \{-\partial C\}$, *we have*

$$D^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v})(x) \cap IT(-\text{int}(C),\bar{v}) = \emptyset \tag{8}$$

for all $x \in D_0 := \text{dom}(D^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v}))$.

Then $(\bar{x},\bar{y}) \in \text{gph}(F)$ *is a weak-minimizer of* (P_0).

Proof. By setting $(\bar{u},\bar{v}) = (0_X, 0_Y)$ we obtain (7) from (8). We will prove sufficiency by using (8). Notice that $(x-\bar{x}) \in \mathrm{dom}(D(F+C)(\bar{x},\bar{y}))$ for $x \in Q_0$, and under the convexity assumption, we have $y-\bar{y} \subset DF(\bar{x},\bar{y})(x-\bar{x})$. Therefore, we obtain

$$(y-\bar{y}) \cap -\mathrm{int}(C) = \emptyset.$$

The weak-minimality then follows. □

The following is a necessary optimality conditions for (P_1), where we use the notation $(F,G)(x)$ to represent $(F+C)(x) \times (G+D)(x)$.

Theorem 3.3. *Let $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ be a local weak-minimizer of (P_1), and let $\bar{z} \in G(\bar{x})$. Then for every $\bar{u} \in \mathrm{dom}(D(FG)(\bar{x},\bar{y},\bar{z}))$, for every $(\bar{v},\bar{w}) \in D(FG)(\bar{x},\bar{y},\bar{z})(\bar{u}) \cap \{-C \times -D\}$ and for every $x \in D_1 := \mathrm{dom}(D^2(FG)(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w})$, we have*

$$D^2(FG)(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w})(x) \cap IT(-C,\bar{v}) \times IT(IT(-D,\bar{z}),\bar{w}) = \emptyset. \tag{9}$$

Proof. Assume that the assertion is not true. Then there exists $x \in D_1$ such that

$$(y,z) \in D^2(F,G)(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w})(x) \cap IT(-C,\bar{v}) \times IT(IT(-\mathrm{int}(D),\bar{z}),\bar{w}),$$

which implies that $(x,y,z) \in T(T(\mathrm{gph}(F,G),(\bar{x},\bar{y},\bar{z})),(\bar{u},\bar{v},\bar{w}))$.

Therefore, there are sequences $(t_n) \subset \mathbb{P}$, $((x_n,y_n,z_n)) \subset X \times Y \times Z$, such that $t_n \downarrow 0$, $(x_n,y_n,z_n) \to (x,y,z)$ and

$$(\bar{u}+t_nx_n, \bar{v}+t_ny_n, \bar{w}+t_nz_n) \in T(\mathrm{gph}(F,G),(\bar{x},\bar{y},\bar{z})). \tag{10}$$

Since $y \in IT(-\mathrm{int}(C),\bar{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\bar{v}+t_ny_n \in -\mathrm{int}(C) \quad \text{for all} \quad n > n_1.$$

Analogously, since $z \in IT(IT(-\mathrm{int}(D),\bar{z}),\bar{w})$, there exists $n_2 \in \mathbb{N}$ such that

$$\bar{w}+t_nz_n \in IT(-\mathrm{int}(D),\bar{z}) \quad \text{for all} \quad n > n_2.$$

For $n \geq \max\{n_1,n_2\}$ we fix elements $u_n := \bar{u}+t_nx_n$, $v_n := \bar{v}+t_ny_n$, and $w_n := \bar{w}+t_nz_n$. Notice that the following containments hold:

$$\begin{array}{rcl} (u_n,v_n,w_n) & \in & T(\mathrm{gph}(FG),(\bar{x},\bar{y},\bar{z})) \\ (v_n,w_n) & \in & -\mathrm{int}(C) \times IT(-\mathrm{int}(D),\bar{z}). \end{array}$$

In view of the definition of the contingent cone, there are sequences $(t_m) \subset \mathbb{P}$, $((x_m,y_m,z_m)) \subset X \times Y \times Z$, such that $t_m \downarrow 0$, $(x_m,y_m,z_m) \to (u_n,v_n,w_n)$ and

$$(\bar{y}+t_my_m, \bar{z}+t_mz_m) \in (F,G)(\bar{x}+t_mx_m)$$

which means that

$$\begin{array}{rcl} \bar{y}+t_my_m & \in & F(\bar{x}+t_mx_m)+C \\ \bar{z}+t_mz_m & \in & G(\bar{x}+t_mx_m)+D. \end{array}$$

Since $y_m \to v_n$ and $v_n \in -\mathrm{int}(C)$ there exists $m_1 > 0$ such that $t_m y_m \in -\mathrm{int}(C)$ for every $m > m_1$. Let $a_m \in F(\bar{x} + t_m x_m)$ be such that $\bar{y} - \mathrm{int}(C) \in a_m + C$, and consequently we have

$$a_m \in \bar{y} - \mathrm{int}(C). \tag{11}$$

Similarly, because $w_n \in IT(-\mathrm{int}(D), \bar{z})$ there exists $m_2 > 0$ such that $\bar{z} + t_m z_m \in -\mathrm{int}(D)$ for every $m > m_4$. Let $b_m \in G(\bar{x} + t_m x_m)$ be such that $\bar{z} + t_m z_m \in b_m + D$, and consequently we have

$$b_m \in -\mathrm{int}(D). \tag{12}$$

Therefore, we have shown that for sufficiently large m, we have $c_m := \bar{x} + t_m x_m \in \mathcal{N}(\bar{x})$, $G(c_m) \cap -D \neq \emptyset$ and $F(c_m) \cap (\bar{y} - \mathrm{int}(C)) \neq \emptyset$. This contradicts the optimality of $(\bar{x}, \bar{y})$. □

4. Concluding remarks

New second-order optimality conditions have been given by introducing new tangential derivatives of second-order. Several extensions of our results are possible. By using standard separation arguments the existence of Lagrange multiplier can be proved (see [18, 28]). It is of interest to give second-order optimality by using derivatives of F and G (not of (F, G)). It seems that Aubin's property can be used to reach this goal (see [13, 25]).

Acknowledgments

We would like to express our sincere appreciation to the referee for a careful reading and for the suggestions which brought substantial improvements to the manuscript.

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