Volume 34, Number 1, 2009, pp. 67–79

GENERALIZED PROJECTION METHOD FOR NON-LIPSCHITZ MULTIVALUED MONOTONE VARIATIONAL INEQUALITIES

PHAM NGOC ANH, LE DUNG MUU AND JEAN-JACQUES STRODIOT

Dedicated to Nguyen Van Hien on the occasion of his sixty-fifth birthday

Abstract. We generalize the projection method for solving strongly monotone multivalued variational inequalities when the cost operator is not necessarily Lipschitz. At each iteration at most one projection onto the constrained set is needed. When the convex constrained set is not polyhedral, we embed the proposed method in a polyhedral outer approximation procedure. This allows us to obtain the projections by solving strongly convex quadratic programs with linear constraints. We also discuss how to use the proposed method to implement inexact proximal point methods.

1. Introduction

Let K be a nonempty closed convex subset in $\mathbb{R}^n$ and $F : K \to 2^{\mathbb{R}^n}$ be a multivalued mapping such that $K \subset \mathrm{dom}F \equiv \{x : F(x) \neq \emptyset\}$. We consider the multivalued variational inequality problem given as

$$\text{find } x^* \in K : \exists\, w^* \in F(x^*) : \ \langle w^*, x - x^* \rangle \geq 0 \ \ \forall x \in K. \qquad \mathrm{VIP}(K, F)$$

As usual, we will refer to K as the constrained (or feasible) set and to F as the cost operator (or mapping) of $\mathrm{VIP}(K, F)$. This problem has important applications in different fields (see e.g. [8, 16]). In the case F is singlevalued, there exist many algorithms for solving $\mathrm{VIP}(K, F)$. The interested readers are referred to the comprehensive monographs [8, 16, 26] as well as to the papers [2, 6, 7, 10, 12, 17, 18, 21, 23, 25, 29] and the references quoted therein. Rather few algorithms have been developed for solving multivalued variational inequalities (see e.g. [3, 5, 13, 23, 28]). Most of these algorithms require that the cost mapping F is Lipschitz with respect to the Hausdorff distance. However, in general, the Lipschitz constant associated with F is not easy to compute.

Received April 7, 2008; in revised form November 11, 2008.

2000 Mathematics Subject Classification. 65K10, 90C25.

Key words and phrases. Multivalued variational inequalities, generalized projection method, polyhedral approximation, inexact proximal point method.

The work presented here was completed while the second author was on leave at the University of Namur, Belgium. He wishes to thank the University of Namur for financial support in the form of a FUNDP Research Scholarship.

In a recent paper [3], we have proposed a fixed point method for solving multivalued variational inequalities involving monotone cost operators. The assumptions were that the cost operator F is Lipschitz with respect to the Hausdorff distance. So, at each iteration k, we can find x^{k+1} and $t^{k+1} \in F(x^k)$ such that the inequality $||t^k - t^{k+1}|| \leq L||x^k - x^{k+1}||$ is satisfied.

In this paper, we consider a multivalued variational inequality VIP(K, F) whose cost operator F is not assumed to be Lipschitz. First, we generalize the projection method to the case of strongly monotone (not necessarily Lipschitz) variational inequalities. The main features of this new method are that

(i) the cost operator is not required to be Lipschitz.

(ii) at each iteration, at most one projection onto the feasible domain is needed.

(iii) the search direction can be determined from any point in the image of the current iterate.

Next, we embed the proposed algorithm in a polyhedral approximation procedure. This allows us to obtain the projections by solving strongly convex quadratic programs with linear constraints. From a computational view point, this is helpful, since the projection onto a closed convex set may be difficult to determine when the convex set is not simple. Finally, we discuss an application of the proposed method in the framework of the proximal point method. Namely, we give an approximation rule and show how to use it in the proposed algorithm to implement inexact proximal point methods for (not necessarily strongly) monotone multivalued variational inequalities whose cost operator is not assumed to be Lipschitz.

The paper is organized as follows. In the next section, we consider the variational inequality problem VIP(K, F) with F a strongly monotone operator. Then we describe a generalized projection method for solving this problem. Section 3 is devoted to show how to embed the new method in a polyhedral outer approximation procedure. In Section 4, we discuss how to apply our method to obtain an implementation of the inexact proximal point method with the aim of solving a variational inequality problem involving a monotone operator.

2. A generalized projection method

First, let us recall the well known concepts of monotonicity that will be used in the sequel.

Definition 2.1. The operator $F : K \to 2^{\mathbb{R}^n}$ is said to be:
a) strongly monotone on K with modulus $\beta > 0$ if

$$\langle w_x - w_y, x - y\rangle \geq \beta||x - y||^2 \quad \forall x, y \in K, w_x \in F(x), w_y \in F(y).$$

b) monotone on K if

$$\langle w_x - w_y, x - y\rangle \geq 0 \quad \forall x, y \in K, w_x \in F(x), w_y \in F(y).$$

Note that if F is strongly monotone on K, then there exists a compact set D such that

$$\forall x \in K \setminus D, \exists y \in K \cap D : \langle v, y - x \rangle < 0 \ \forall v \in F(x). \tag{CO}$$

Indeed, assume that this coercivity property does not hold, then for every closed ball B_r centered at zero with radius r, there exists $x^r \in K \setminus B_r$ such that

$$\forall y \in K \cap B_r, \ \exists u^r \in F(x^r) : \langle u^r, x^r - y \rangle \leq 0.$$

In particular, let $r_0 > 0$ and $y^0 \in K \cap B_{r_0}$. Then

$$\forall r > r_0, \ \exists u^r \in F(x^r) : \langle u^r, x^r - y^0 \rangle \leq 0.$$

On the other hand, by the strong monotonicity of F, we have

$$\langle u^r, x^r - y^0 \rangle \geq \langle v^0, x^r - y^0 \rangle + \beta ||x^r - y^0||^2,$$

where $v^0 \in F(y^0)$. Since $||x^r|| > r$, this inequality implies that $\langle u^r, x^r - y^0 \rangle \to +\infty$ as $r \to \infty$, which contradicts $\langle u^r, x^r - y^0 \rangle \leq 0$.

It is well known (see e.g. [8, 16]) that if F is upper semicontinuous with compact, convex values on K and the coercive condition (CO) is satisfied, then the variational inequality VIP(K, F) admits a solution, and if in addition, F is strongly monotone, then the solution is unique.

For each $x \in K$ and $v \in F(x)$, we define the following strongly convex programming problem

$$\min_{y \in K} \{ \langle v, y - x \rangle + \frac{1}{2\rho} \|y - x\|^2 \}, \tag{$P(x,w)$}$$

where $\rho > 0$ is a regularization parameter. Since K is a nonempty closed convex subset and the objective function is a strongly convex quadratic function, this problem admits a unique solution denoted by $s(x, v)$. Then we define a multivalued mapping $S : K \to 2^K$, by

$$S(x) \equiv \{ s(x, v) \in K : v \in F(x) \} \quad \text{with} \quad \mathrm{dom}\, S \subset K \subset \mathrm{dom} F.$$

It is well known (see e.g. [2, 16]) that x is a solution of VIP(K, F) if and only if $x \in S(x)$. This fact suggests solving VIP(K, F) by the iterative procedure $x^{j+1} \in S(x^j)$. It has been shown in [3] that if F is strongly monotone and Lipschitz continuous with respect to the Hausdorff distance on K, then the regularization parameters can be chosen depending on the strongly monotone modulus and the Lipschitz constant in such a way that the sequence $\{x^k\}$ linearly converges to the unique solution to VIP(K, F). However, from the numerical point of view, computing the Lipschitz constant can be costly and even sometimes impossible. Our aim now is to construct iteratively a sequence converging to a solution to VIP(K, F) without assuming the Lipschitz continuity of F. Given an iterate $x^j \in K$, we attempt to find a direction w^j on which the next iterate x^{j+1} is lying. By using an optimality condition for convex programming, we can see that $x^{j+1} \in S(x^j)$ if and only if

$$\langle v^j + \frac{1}{\rho}(x^{j+1} - x^j), y - x^{j+1} \rangle \geq 0 \quad \forall y \in K$$

holds true for some $v^j \in F(x^j)$. With $y = x^j \in K$, the last inequality becomes

$$\langle v^j, x^j - x^{j+1}\rangle - \frac{1}{\rho}\langle x^{j+1} - x^j, x^{j+1} - x^j\rangle \geq 0.$$

By taking $w^j = \frac{1}{\rho}(x^{j+1} - x^j)$, we obtain $x^{j+1} = x^j + \rho w^j$. It may happen that $x^j + \rho w^j$ does not belong to K for any $\rho > 0$. In this case, it is natural to take $x^{j+1} = P_K(x^j + \rho w^j)$, where P_K stands for the Euclidean projection onto K. Of course, we can also take $\rho = \rho_j$ depending on each iteration. In that case, the regularization parameter ρ_j plays the role of a stepsize at iteration j. The method can now be described in detail as follows:

ALGORITHM 2.1

Step 0. Choose a sequence $\{\rho_j\}$ such that

$$0 < \rho_j < 1 \ \ \forall j, \quad \sum_{j=0}^{\infty} \rho_j = +\infty, \quad \sum_{j=0}^{\infty} \rho_j^2 < +\infty.$$

Let $x^0 \in K$ and set $j := 0$.

Step 1. Take $v^j \in F(x^j)$.

If $v^j = 0$, then terminate: x^j solves VIP(K, F).

If $v^j \neq 0$, then find w^j, such that

$$\langle v^j, y - x^j\rangle + \langle w^j, y - x^j\rangle \geq 0 \ \ \forall y \in K. \tag{2.1}$$

If $w^j = 0$, terminate: x^j is a solution.

Otherwise, go to Step 2.

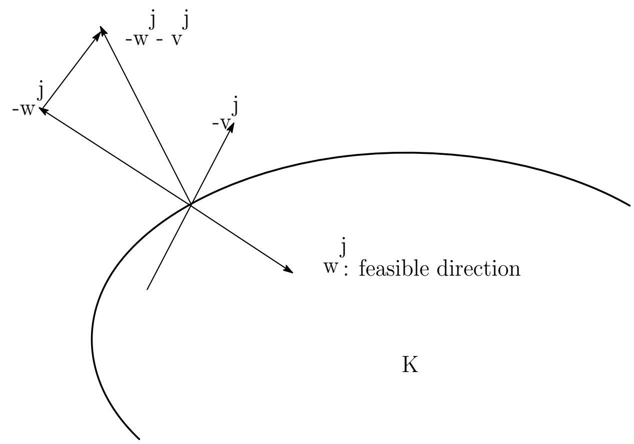
Step 2. Set

$$z^{j+1} := x^j + \rho_j w^j, \ x^{j+1} = P_K(z^{j+1}).$$

Let $j \leftarrow j + 1$ and go back to Step 1.

Remark 2.1. (i) The main subproblem in this algorithm is to find $w^j \neq 0$ satisfying (2.1). Clearly (2.1) holds if $w^j = -v^j$. In fact the inequality (2.1) means that $v^j + w^j \in -N_K(x^j)$, where $N_K(x^j)$ denotes the (outward) normal cone of K at x^j. In the case K is given by $K \equiv \{x : g(x) \leq 0\}$ with g a subdifferentiable convex function, one can take $w^j = -v^j$ when $g(x^j) < 0$ and w^j such that $-(v^j + w^j) \in \partial g(x^j)$ when $g(x^j) = 0$. In both cases $v^j + w^j \in -N_K(x^j)$. In the particular case where F is the subdifferential of a convex function f, it is suggested in [22] to determine $-w^j$ as the vector of the smallest norm in $F(x^j) + N_K(x^j)$. This vector is a descent direction for the objective function f at the current iterate. In [22], it is also shown how to compute this descent direction when K is a polyhedral convex set given by a finite number of affine inequalities.

(ii) The direction w^j defined by $v^j + w^j \in -N_K(x^j)$ with $v^j \in F(x^j)$ takes into account not only the cost operator F, but also the constrained set K. This is helpful in certain cases, for example for avoiding the projection onto K. Indeed it may happen that $(-F(x^j)) \cap K = \emptyset$, but that $(-F(x^j)) \cap \big(N_K(x^j) + w^j\big) \neq \emptyset$ for some $w^j \neq 0$ such that $x^j + \rho w^j \in K$ for some $\rho > 0$ (see the figure below).



If at each iteration j, we take $w^j = -v^j$, then Algorithm 2.1 becomes the following usual projection method

Step 0. Choose a sequence $\{\rho_j\}$ such that

$$0 < \rho_j < 1 \ \ \forall j, \quad \sum_{j=0}^{\infty} \rho_j = +\infty, \quad \sum_{j=0}^{\infty} \rho_j^2 < +\infty.$$

Let $x^0 \in K$ and set $j := 0$.

Step 1. Take $v^j \in F(x^j)$.

If $v^j = 0$, then terminate: x^j solves VIP(K, F).

If $v^j \neq 0$, take $x^{j+1} = P_K(x^j - \rho_j v^j)$. Let $j \leftarrow j+1$ and go back to Step 1.

Convergence of Algorithm 2.1 is ensured by the following theorem.

Theorem 2.1. *Suppose that the mapping F is strongly monotone on K with modulus $\beta > 0$. Then the sequence $\{x^j\}$ constructed by Algorithm 2.1 satisfies*

$$||x^{j+1} - x^*||^2 \leq (1 - 2\beta\rho_j)||x^j - x^*||^2 + \rho_j^2 ||w^j||^2 \qquad \forall j,$$

where x^* is the unique solution of VIP(K, F). Moreover, if $0 < \rho_j < \dfrac{1}{2\beta}$ for every j, and the sequence $\{w^j\}$ is bounded, then $x^j \to x^*$.

We need the following lemma for the proof of the theorem.

Lemma 2.1. *Let $\{\alpha_j\}$ be a sequence of nonnegative numbers such that*

$$\alpha_{j+1} \leq (1 - \lambda_j)\alpha_j + \epsilon_j \quad \forall j,$$

where

$$\lambda_j \in (0,1) \quad \forall j, \quad \sum_{j=0}^{\infty} \lambda_j = +\infty, \;\; \epsilon_j > 0 \quad \forall j, \quad \sum_{j=0}^{\infty} \epsilon_j < +\infty.$$

Then $\lim_{j\to\infty} \alpha_j = 0$.

Proof. Applying iteratively the inequality $\alpha_{j+1} \le (1-\lambda_j)\alpha_j + \epsilon_j$ for j, $j-1$, ..., 0, we obtain

$$\alpha_{j+1} \le \alpha_0 \prod_{k=1}^{j}(1-\lambda_k) + \sum_{i=1}^{j} \epsilon_{i-1} \prod_{k=i}^{j}(1-\lambda_k) + \epsilon_j \quad \forall j. \tag{2.2}$$

Since $\sum_{j=0}^{\infty} \epsilon_j < +\infty$, $\sum_{j=0}^{\infty} \lambda_j = +\infty$ and

$$\prod_{i=k}^{j}(1-\lambda_i) \le \exp\big(-\sum_{i=k}^{j} \lambda_i\big) \to 0 \;\text{ as } j \to \infty,$$

for all k, letting $j \to \infty$, we see from (2.2) that $\lim_{j\to\infty} \alpha_j = 0$. □

Proof of the theorem. Let x^* be the solution of $\mathrm{VIP}(K, F)$. Since $x^{j+1} = P_K(z^{j+1})$ and $x^* = P_K(x^*)$, we have, by the nonexpansivity of the Euclidean projection mapping, that

$$||x^{j+1} - x^*||^2 = ||P_K(z^{j+1}) - P_K(x^*)||^2 \le ||z^{j+1} - x^*||^2. \tag{2.3}$$

Replacing z^{j+1} by $x^j + \rho_j w^j$, we obtain

$$||z^{j+1} - x^*||^2 = ||x^j + \rho_j w^j - x^*||^2 = ||x^j - x^*||^2 + 2\rho_j \langle w^j, x^j - x^*\rangle + \rho_j^2 ||w^j||^2.$$

Then, by (2.3), we have

$$||x^{j+1} - x^*||^2 \le ||x^j - x^*||^2 + 2\rho_j \langle w^j, x^j - x^*\rangle + \rho_j^2 ||w^j||^2. \tag{2.4}$$

On the other hand, applying inequality (2.1) with $y = x^*$, we obtain

$$\langle v^j, x^* - x^j\rangle + \langle w^j, x^* - x^j\rangle \ge 0,$$

which implies

$$\langle v^j, x^* - x^j\rangle \ge \langle w^j, x^j - x^*\rangle. \tag{2.5}$$

Since F is strongly monotone on K with modulus $\beta > 0$, and $x^* \in K$, $v^j \in F(x^j)$, we have

$$\langle v^j - w, x^j - x^*\rangle \ge \beta ||x^j - x^*||^2 \qquad \forall w \in F(x^*).$$

Hence

$$\langle v^j, x^* - x^j\rangle \le -\beta ||x^j - x^*||^2 - \langle w, x^j - x^*\rangle \qquad \forall w \in F(x^*). \tag{2.6}$$

Since $x^* \in K$ is the solution of $\mathrm{VIP}(K, F)$, there exists $w^* \in F(x^*)$ satisfying

$$\langle w^*, x^j - x^*\rangle \ge 0. \tag{2.7}$$

Substituting (2.7) into (2.6) with $w = w^*$, we obtain

$$\langle v^j, x^* - x^j \rangle \leq -\beta ||x^j - x^*||^2.$$

Thus, by (2.5),

$$\langle w^j, x^j - x^* \rangle \leq -\beta ||x^j - x^*||^2. \tag{2.8}$$

Combining (2.4) and (2.8), we can deduce successively that

$$\begin{aligned} ||x^{j+1} - x^*||^2 &\leq ||x^j - x^*||^2 - 2\rho_j \beta ||x^j - x^*||^2 + \rho_j^2 ||w^j||^2 \\ &= (1 - 2\rho_j \beta) ||x^j - x^*||^2 + \rho_j^2 ||w^j||^2. \end{aligned}$$

To prove that $\lim_{j\to\infty} x^j = x^*$, we apply Lemma 2.1 with $\lambda_j := 2\rho_j\beta$, $\alpha_j := ||x^j - x^*||^2$, and $\epsilon_j = \rho_j^2 ||w^j||^2$. Since $0 < \rho_j < \frac{1}{2\beta}$ for all j and $\sum_{j=0}^{\infty} \rho_j = \infty$, we have $0 < \lambda_j < 1$ for all j and $\sum_{j=0}^{\infty} \lambda_j = \infty$. Furthermore, the sequence $\{||w^j||\}$ being bounded, and $\sum_{j=0}^{\infty} \rho_j^2 < \infty$, we have $\sum_{j=0}^{\infty} \epsilon_j < \infty$. Consequently, by Lemma 2.1, $\lim_{j\to\infty} ||x^j - x^*|| = 0$, which means that $x^j \to x^*$ as $j \to \infty$. □

Remark 2.2. In order to ensure the convergence of Algorithm 2.1, we have assumed that the sequence $\{w^j\}$ is bounded. By using an additional parameter τ_j, we can guarantee that the sequence $\{w^j\}$ is automatically bounded. It is the case if we take τ_j such that

$$0 < \tau_j < \min\left\{\frac{1}{2\beta\rho_j}, \frac{1}{||v^j||}\right\} \quad \text{for all } j \quad \text{and} \quad \sum_{j=0}^{\infty} \rho_j \tau_j = +\infty, \tag{2.9}$$

and if in Algorithm 2.1 we require, instead of (2.1), that

$$\langle \tau_j v^j, y - x^j \rangle + \langle w^j, y - x^j \rangle \geq 0 \quad \forall y \in K.$$

Indeed, if we take $u^j \in N_K(x^j) \equiv \{v : \langle v, x - x^j \rangle \leq 0 \ \forall x \in K\}$ such that $||u^j|| \leq c$ and $-\tau v^j - w^j = u^j$, then

$$||w^j|| = || - \tau_j v^j - u^j|| \leq \tau_j ||v^j|| + ||u^j|| \leq 1 + c,$$

where the second inequality follows from (2.9). By using the same arguments as in the proof of Theorem 2.1, we obtain

$$\begin{aligned} \|x^{j+1} - x^*\|^2 &\leq \|x^j - x^*\|^2 + 2\rho_j\tau_j \langle v^j, x^* - x^j \rangle + \rho_j^2 \|w^j\|^2 \\ &\leq \|x^j - x^*\|^2 - 2\rho_j\beta\tau_j \|x^j - x^*\|^2 + \rho_j^2 \|w^j\|^2 \\ &= (1 - 2\rho_j\beta\tau_j)\|x^j - x^*\|^2 + \rho_j^2 \|w^j\|^2. \end{aligned}$$

Again by (2.9), $2\rho_j\beta\tau_j < 1$ for all j. Then we can apply Lemma 2.1 with $\lambda_j = 2\rho_j\beta\tau_j < 1$, $\alpha_j = ||x^j - x^*||^2$, and $\epsilon_j = \rho_j^2 ||w^j||^2$ to show that $||x^j - x^*|| \to 0$ as $j \to \infty$.

3. Polyhedral approximation

In the algorithm described in the previous section, at each iteration j, the iterate x^{j+1} is defined as the projection on K of the preference point z^{j+1}. This task leads to the problem of minimizing the quadratic function $||x - z^{j+1}||^2$ over the closed convex set K. In the case K is a polyhedron, this problem can be solved efficiently by using available softwares. When K is not polyhedral, we suggest approximating K by polyhedral convex sets. Polyhedral outer approximations of a convex set are based upon the fact that any nonempty closed convex set can be approximated by polyhedral convex sets. This technique has been widely used in convex programming and variational inequality problems (see e.g. [10, 15]). In this section, we embed Algorithm 2.1 in a polyhedral outer approximation procedure in order to solve problem VIP(K, F). We also suppose that the closed convex set K is given as

$$K := \{x \in \mathbb{R}^n \;:\; g_j(x) \leq 0, j \in J\},$$

where J is a finite index set and the functions g_j $(j \in J)$ are convex and subdifferentiable on $\mathbb{R}^n$. By taking $g(x) := \max_{j\in J} g_j(x)$, we can write $K = \{x \in \mathbb{R}^n |\; g(x) \leq 0\}$. Suppose now that Slater's condition is satisfied, i.e., that there exists v^0 such that $g(v^0) < 0$.

For getting the convergence of the polyhedral approximation algorithm described below, we need the following result.

Theorem 3.1. ([21], [30, Theorem 6.1, p.180]) *Let $\{x^k\} \subset \mathbb{R}^n \setminus K$ be a bounded sequence, let $v^0 \in \operatorname{int} K$, $y^j \in [v^0, x^k] \setminus \operatorname{int} K$, $p^k \in \partial g(y^k)$ and $0 \leq \alpha_k \leq g(y^k)$ such that $\alpha_k - g(y^k) \to 0$ as $k \to +\infty$. If, for every k, the affine functions $l_k(x) := \langle p^k, x - y^k\rangle + \alpha_k$ satisfy*

$$l_k(x^k) > 0, \quad l_k(x^{k+1}) \leq 0, \quad l_k(x) \leq 0 \qquad \forall x \in K,$$

then every accumulation point of the sequence $\{x^k\}$ belongs to K.

Now we are in a position to describe the polyhedral approximation algorithm.

ALGORITHM 3.1

Initialization. Choose a sequence $\{\delta_k\}$ of positive numbers such that $\delta_k \searrow 0$, and take a polyhedral convex set T_0 containing K. For example

$$T_0 := \{x : g(x^0) + \langle v^0, x - u^0\rangle \leq 0\},$$

where $u^0 \in \mathbb{R}^n$ and $v^0 \in \partial g(x^0)$.

Iteration k $(k = 0, 1, ...)$. Find $x^k \in T_k$ such that

$$\exists\, v^k \in F(x^k) : \; \langle v^k, x - x^k\rangle \geq -\delta_k \;\forall x \in T_k \supset K. \tag{3.1}$$

a) If $x^k \in K$, take $T_{k+1} = T_k$.

b) If $x^k \notin K$, then construct a hyperplane l_k such that

$$l_k(x) \leq 0 \ \forall x \in K, \ l_k(x^k) > 0,$$

and take

$$T_{k+1} := \{x \in T_k : l_k(x) \leq 0\}.$$

Increase k by 1 and go to iteration k.

Theorem 3.2. *Assume that*

(i) *F is upper semicontinuous, with compact values, and strongly monotone on T_0.*

(ii) *The cutting hyperplane l_k used in the algorithm is constructed as in Theorem 3.1.*

Then the sequence $\{x^k\}$ generated by the algorithm converges to the solution to VIP(F, K).

Proof. Let x^* be the exact solution to VIP(F, K). Applying (3.1) with $x = x^*$, we obtain

$$\langle v^k, x^* - x^k \rangle \geq -\delta_k \quad \forall k. \tag{3.2}$$

If F is strongly monotone with modulus β, then

$$\langle v^k - v^*, x^k - x^* \rangle \geq \beta ||x^k - x^*||^2 \quad \forall v^k \in F(x^k), v^* \in F(x^*),$$

which implies

$$\langle v^k, x^k - x^* \rangle \geq \langle v^*, x^k - x^* \rangle + \beta ||x^k - x^*||^2.$$

Then by (3.2), it follows that

$$\beta ||x^k - x^*||^2 + \langle v^*, x^k - x^* \rangle \leq \delta_k.$$

Thus

$$\beta ||x^k - x^*||^2 \leq \delta_k + ||v^*|| ||x^k - x^*|| \quad \forall k.$$

Since $\{\delta_k\}$ is bounded, it follows directly from this inequality that $\{x^k\}$ is also bounded. Then there is a cluster point $\bar{x}$ of the sequence $\{x^k\}$ with $\bar{x} \in K$. Indeed, if $x^k \notin K$ for every k large enough, then this is guaranteed by Theorem 3.1. On the other hand, if $x^k \in K$ for infinitely many k, then it is a consequence of the closedness of K. Now, we show that $\bar{x}$ solves VIP(F, K). Let x be any point in K and let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ such that $x^{k_j} \to \bar{x}$ as $j \to \infty$. Since $K \subset T_k$ for all k, again by (3.1), we have

$$\langle v^{k_j}, x - x^{k_j} \rangle \geq -\delta_{k_j}. \tag{3.3}$$

Since $\{x^{k_j}\}$ is bounded, by upper semicontinuity of F, we see that the sequence $\{v^{k_j}\}$ is bounded too (see e.g. [4, Proposition 11, p.112]), and therefore we may assume, taking a subsequence if necessary, that $v^{k_j} \to \overline{v}$ as $j \to +\infty$. Since F is closed at $\overline{x}$, we have $\overline{v} \in F(\overline{x})$. Now letting j in (3.3) tend to $+\infty$, as $\delta_{k_j} \to 0$, we obtain

$$\langle \overline{v}, x - \overline{x} \rangle \geq 0,$$

which implies that $\overline{x}$ solves VIP(F, K). $\square$

4. Application to inexact proximal point methods

The proximal point method is a fundamental tool for solving the inclusion $0 \in T(x)$ with T a maximal monotone operator. For the variational inequality $\mathrm{VIP}(K, F)$, the mapping T is defined as

$$T(x) := \begin{cases} F(x) + N_K(x) & \text{if } x \in K, \\ \emptyset & \text{if } x \notin K, \end{cases}$$

where $N_K(x)$ is the normal cone to K at x. It is well known (see e.g. [11, p.381]) that, if F is upper semicontinuous and monotone with compact values on K, then T is maximal monotone. Then, for any $c_k > 0$, the proximal mapping $P_k := (I + c_k T)^{-1}$ is single valued and nonexpansive on the whole space. The proximal point method constructs a sequence $\{x^k\}$ by taking $x^{k+1} = P_k(x^k)$ for all k. In the case of a variational inequality, where $T := F + N_K$, computing $x^{k+1} = P_k(x^k)$ is reduced to the problem of finding the unique solution of the following strongly monotone variational inequality

$$\text{find } x^{k+1} \in K \text{ and } w^{k+1} \in F_k(x^{k+1}) : \langle w^{k+1}, x - x^{k+1}\rangle \geq 0 \quad \forall x \in K,$$

where $F_k(x) := c_k F(x) + x - x^k$.

It is well known [27] that starting from any $x^0 \in K$, the sequence of iterates $\{x^k\}$ defined by $x^{k+1} = P_k(x^k)$ converges to a solution of the initial variational inequality $\mathrm{VIP}(K, F)$ provided that the regularization parameters c_k are bounded away from 0, i.e., $c_k > c > 0$ for all k. Since F_k is strongly monotone on K, Algorithm 2.1 proposed in Section 2 can be applied. In practice, we can solve the subproblem only approximately. In [27], criteria for approximation are given that ensure that the global convergence remains true. Following this idea, starting from an arbitrary point $\tilde{x}^0 \in K$, we construct a sequence $\{\tilde{x}^k\}$ of approximate solutions to subvariational inequalities by taking, for each k,

$$\tilde{x}^{k+1} \in K \text{ and } \tilde{w}^{k+1} \in F_k(\tilde{x}^{k+1}) : \langle \tilde{w}^{k+1}, x - \tilde{x}^{k+1}\rangle \geq -\delta_k \quad \forall x \in K, \tag{4.1}$$

where $\delta_k > 0$ and $\sum_{k=1}^{\infty} \delta_k < \infty$. We call a point $\tilde{x}^{k+1}$ satisfying (4.1) a δ_k-*solution* to the variational inequality

$$\text{find } x^{k+1} \in K \text{ and } w^{k+1} \in F_k(x^{k+1}) : \langle w^{k+1}, x - x^{k+1}\rangle \geq 0 \; \forall x \in K. \quad \mathrm{VIP}(K, F_k)$$

Convergence of the sequence defined by (4.1) is ensured by the following theorem.

Theorem 4.1. *Suppose that F is monotone and upper semicontinuous with compact, convex values on K, and that $F_k(.) = c_k F(.) + \nabla h(.) - \nabla h(\tilde{x}^k)$ with h being a Lipschitz differentiable, strongly convex function and $c_k \geq c > 0$ for all k. Then the sequence $\{\tilde{x}^k\}$ generated by (4.1) with $\delta_k > 0$ and $\sum_{k=1}^{\infty} \delta_k < \infty$ converges to a solution of the variational inequality problem* VIP(K, F).

Proof. Let $x = x^*$ be a solution of $\mathrm{VIP}(K, F)$. Applying (4.1) with $x = x^*$, and substituting $\tilde{w}^{k+1} = c_k \tilde{t}^{k+1} + \nabla h(\tilde{x}^{k+1}) - \nabla h(\tilde{x}^k)$ with $\tilde{t}^{k+1} \in F(\tilde{x}^{k+1})$, we have

$$\langle c_k \tilde{t}^{k+1}, x^* - \tilde{x}^{k+1}\rangle + \langle \nabla h(\tilde{x}^{k+1}) - \nabla h(\tilde{x}^k), x^* - \tilde{x}^{k+1}\rangle \geq -\delta_k. \tag{4.2}$$

Since $\tilde{t}^{k+1} \in F(\tilde{x}^{k+1})$, we obtain, by monotonicity of F, that

$$\langle \tilde{t}^{k+1} - t^*, x^* - \tilde{x}^{k+1} \rangle \leq 0 \quad \forall t^* \in F(x^*).$$

Furthermore, since x^* is a solution of VIP(K, F), there exists $t^* \in F(x^*)$ such that

$$\langle t^*, \tilde{x}^{k+1} - x^* \rangle \geq 0.$$

Thus, again by monotonicity, we obtain

$$\langle \tilde{t}^{k+1}, x^* - \tilde{x}^{k+1} \rangle \leq 0.$$

Then, by (4.2)

$$\langle \nabla h(\tilde{x}^{k+1}) - \nabla h(\tilde{x}^k), x^* - \tilde{x}^{k+1} \rangle \geq -\delta_k. \tag{4.3}$$

On the other hand, since h is strongly convex with modulus α, we have

$$L(x) := h(x^*) - h(x) - \langle \nabla h(x), x^* - x \rangle \geq \frac{\alpha}{2} ||x - x^*||^2 \quad \forall x \in K. \tag{4.4}$$

Using (4.4) with $\tilde{x}^k$ and $\tilde{x}^{k+1}$, we obtain that

(4.5)

$$\begin{aligned}
&L(\tilde{x}^k) - L(\tilde{x}^{k+1}) \\
&= h(\tilde{x}^{k+1}) - h(\tilde{x}^k) + \langle \nabla h(\tilde{x}^{k+1}), x^* - \tilde{x}^{k+1} \rangle - \langle \nabla h(\tilde{x}^k), x^* - \tilde{x}^k \rangle \\
&= h(\tilde{x}^{k+1}) - h(\tilde{x}^k) + \langle \nabla h(\tilde{x}^{k+1}), x^* - \tilde{x}^{k+1} \rangle - \langle \nabla h(\tilde{x}^k), x^* - \tilde{x}^k + \tilde{x}^{k+1} - \tilde{x}^{k+1} \rangle \\
&= h(\tilde{x}^{k+1}) - h(\tilde{x}^k) + \langle \nabla h(\tilde{x}^{k+1}) - \nabla h(\tilde{x}^k), x^* - \tilde{x}^{k+1} \rangle - \langle \nabla h(\tilde{x}^k), \tilde{x}^{k+1} - \tilde{x}^k \rangle \\
&\geq \frac{\alpha}{2} \|\tilde{x}^{k+1} - \tilde{x}^k\|^2 + \langle \nabla h(\tilde{x}^{k+1}) - \nabla h(\tilde{x}^k), x^* - \tilde{x}^{k+1} \rangle,
\end{aligned}$$

where the last inequality follows from the fact that h is strongly convex with modulus α. Then from (4.3) and (4.5), we deduce that

$$L(\tilde{x}^{k+1}) - L(\tilde{x}^k) \leq -\frac{\alpha}{2} ||\tilde{x}^{k+1} - \tilde{x}^k||^2 + \delta_k \leq \delta_k.$$

Since $\sum_{k=0}^{\infty} \delta_k < \infty$, it follows from the last inequality that the sequence $\{L(\tilde{x}^k)\}$ is convergent, and therefore

$$||\tilde{x}^{k+1} - \tilde{x}^k||^2 \to 0 \text{ as } k \to \infty.$$

Since $\{L(\tilde{x}^k)\}$ is convergent, using (4.4) with $x = \tilde{x}^k$, we can see that the sequence $\{\tilde{x}^k\}$ is bounded. Let $\tilde{x}$ be any cluster point of $\{\tilde{x}^k\}$. For simplicity of notation, we write $\tilde{x}^k \to \tilde{x}$. Let x be any point in K. From (4.1), it follows that

$$\langle c_k \tilde{t}^{k+1}, x - \tilde{x}^{k+1} \rangle + \langle \nabla h(\tilde{x}^{k+1}) - \nabla h(\tilde{x}^k), x - \tilde{x}^{k+1} \rangle \geq -\delta_k,$$

which, by Lipschitz continuity of ∇h, implies that

$$\begin{aligned}
\langle c_k \tilde{t}^{k+1}, \tilde{x}^{k+1} - x \rangle &\leq \langle \nabla h(\tilde{x}^{k+1}) - \nabla h(\tilde{x}^k), x - \tilde{x}^{k+1} \rangle + \delta_k \\
&\leq L||\tilde{x}^{k+1} - \tilde{x}^k|| ||\tilde{x}^{k+1} - x|| + \delta_k. \qquad (4.6)
\end{aligned}$$

Note that, since F is upper semicontinuous with compact values, $\{\tilde{x}^k\}$ is bounded and $\tilde{t}^{k+1} \in F(\tilde{x}^{k+1})$, we can conclude that $\{\tilde{t}^k\}$ is bounded ([4, Proposition 11,

p. 112]). Thus, taking a subsequence if necessary, we may assume that $\tilde{t}^k \to \tilde{t}$. Letting $k \to \infty$ in (4.6) and observing that the sequence $\{\tilde{x}^k\}$ is asymptotic regular, i.e., $||\tilde{x}^{k+1} - \tilde{x}^k|| \to 0$, and $c_k > c > 0$ for all k, we obtain

$$\langle \tilde{t}, x - \tilde{x} \rangle \geq 0.$$

Since x is arbitrary in K and $\tilde{t} \in F(\tilde{x})$, the last inequality shows that $\tilde{x}$ is a solution to VIP(K, F). □

In virtue of Theorem 4.1, we can use Algorithm 2.1 to implement the approximate proximal point algorithm with approximation criterion defined by (4.1).

Remark 4.1. The approximation defined by (4.1) allows us to verify whether the point $\tilde{x}^{k+1}$ is a δ_k-solution or not. Indeed, $\tilde{x}^{k+1}$ is a δ_k-solution to VIP(K, F_k) if and only if the optimal value of the convex program

$$\min \langle \tilde{t}^{k+1}, x \rangle \text{ subject to } x \in K$$

is greater or equal to $\langle \tilde{t}^{k+1}, \tilde{x}^{k+1} \rangle - \delta_k$.

References

[1] Y. I. Alber, Recurence relations and variational inequalities, *Soviet Mathematics Doklady*, **27** (1983), 511–517.

[2] P. N. Anh, L. D. Muu, V. H. Nguyen, and J. J. Strodiot, On the Contraction and Nonexpensiveness Properties of the Marginal Mappings in Generalized Variational Inequalities Involving Co-coercive Operators. In: A. Eberhard, N. Hadjisavvas, and D. T. Luc (eds.), *Generalized Convexity and Monotonicity*, Springer-Verlag, New York, 2005, pp. 89–111.

[3] P. N. Anh, L. D. Muu, V. H. Nguyen and J. J. Strodiot, Using the Banach contraction principle to implement the proximal point method for multivalued monotone variational inequalities, *J. Optim. Theory Appl.* **124** (2005), 285–306.

[4] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley, New York, 1984.

[5] T. Q. Bao and P. Q. Khanh, A Projection-Type Algorithm for Pseudomonotone Nonlipschitzian Multivalued Variational Inequalities. In: A. Eberhard, N. Hadjisavvas, and D.T. Luc (eds.), *Generalized Convexity and Monotonicity*, Springer-Verlag, New York, 2005, pp. 113-129.

[6] G. Cohen, Auxiliary problem principle extended to variational inequalities, *J. Optim. Theory Appl.* **59** (1988), 325–333.

[7] N. El Farouq, Pseudomonotone variational inequalities: convergence of the auxiliary problem method, *J. Optim. Theory Appl.* **111** (2001), 306–325.

[8] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementary Problems*, Springer-Verlag, New York, 2003.

[9] S. C. Fang and E. L. Peterson, Generalized variational inequalities, *J. Optim. Theory Appl.* **38** (1982), 363–383.

[10] M. Fukushima, Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, *Math. Program.* **53** (1992), 99–110.

[11] E. G. Golshtein and N. V. Tretyakov, *Modified Lagrangians and Monotone Maps in Optimization*, Wiley, New York, 1996.

[12] T. T. Hue, J. J. Strodiot, and V. H. Nguyen, Convergence of the approximate auxiliary problem method for solving generalized variational inequalities, *J. Optim. Theory Appl.* **121** (2004), 119–145.

[13] A. N. Iusem and B. F. Svaiter, A variant of Korpelevich's method for variational inequalities with a new search strategy, *Optimization* **42** (1997), 309–321.

[14] A. N. Iusem and L. R. L. Pérez, An extragradient type algorithm for nonsmooth variational inequalities, *Optimization* **48** (2000), 309–332.

[15] J. E. Kelley, The cutting plane method for solving convex programs, *SIAM J. Appl. Math.* **8** (1960), 703-712.
[16] I. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer-Verlag, Berlin, 2001.
[17] I. Konnov I. and S. Kum, Descent methods for mixed variational inequalities in a Hilbert space, *Nonlin. Anal.: Theory, Methods Appl.* **47** (2001), 561–572.
[18] P. Marcotte, A new algorithm for solving variational inequalities, *Math. Program.* **33** (1995), 339–351.
[19] P. Marcotte and J. H. Wu, On the convergence of projection methods: application to the decomposition of affine variational inequalities, *J. Optim. Theory Appl.* **85** (1995), 347–362.
[20] L. D. Muu, An augmented penalty function method for solving a class of variational inequalities, *USSR Computational Mathematics and Mathematical Physics* **12** (1986), 1788–1796.
[21] L. D. Muu and W. Oettli, A combimed branch-and-bound and cutting plane method for solving a certain class of nonconvex optimization problems, *J. Glob. Optim.* **3** (1993), 377–391.
[22] V. H. Nguyen and J. J. Strodiot, A linearly constrained algorithm not requiring derivative continuity, *Eng. Struct.* **6** (1984), 7–11.
[23] M. A. Noor, General algorithm for variational inequalities, *J. Math. Japonica* **38** (1993), 47–53.
[24] M. A. Noor, Some predictor-corrector algorithms for multivalued variational inequalities, *J. Optim. Theory Appl.* **108** (2001), 659–671.
[25] M. Patriksson, Merit functions and descent algorithms for a class of variational inequality problems, *Optimization* **41** (1997), 37–55.
[26] M. Patriksson, *Nonlinear Programming and Variational Inequality Problems*, Kluwer Academic Publishers, Dordrecht, Holland, 1999.
[27] R.T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control* **14** (1976), 877–899.
[28] G. Salmon, J. J. Strodiot and V.H. Nguyen, A bundle method for solving variational inequalities, *SIAM J. Optim.* **14** (2004), 869–893.
[29] K. Taji and M. Fukushima, A new merit function and a successive quadratic programming algorithm for variational inequality problems, *SIAM J. Optim.* **6** (1996), 704–713.
[30] H. Tuy, *Convex Analysis and Global Optimization*, Kluwer Academic Publishers, Dordrecht, Holland, 1998.
[31] J. H. Wu, M. Florian, and P. Marcotte, A general descent framework for the monotone variational inequality problem, *Math. Program.* **53** (1992), 99–110.
[32] D. Zhu and P. Marcotte, An extended descent framework for variational inequalities, *J. Optim. Theory Appl.* **80** (1994), 349–366.

Posts and Telecommunications Institute of Technology, Hanoi, Vietnam
E-mail address: anhpn@ptit.edu.vn

Institute of mathematics, Hanoi, Vietnam
E-mail address: ldmuu@math.ac.vn

Department of Mathematics, University of Namur (FUNDP), Namur, Belgium
E-mail address: jean-jacques.strodiot@fundp.ac.be