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ACCELERATING CONVERGENCE OF A CLASS OF SPLITTING ALGORITHMS WITH ITERATIVE FOLDINGS

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Dedicated to Nguyen Van Hien on the occasion of his sixty-fifth birthday

Abstract. We analyze here the asymptotic convergence properties of a class of splitting algorithms for finding a zero of the sum of two maximal monotone operators, not necessarily differentiable. We use the concept of proto- differentiability to obtain some new bounds on the global error of the sequence of iterates and explain the bad spiraling effect already observed in practice. After linking our model with the Lawrence-Spingarn's folding operators, we show how to accelerate convergence by skipping the averaging steps in a particular way. Numerical results on medium-term stochastic planning problems confirm the nice behavior of the modified algorithm.

1. Introduction

This work originates in the numerical resolution of the following large-scale optimization problem:

$$\text{Minimize} \quad f(y) = \sum_{i=1}^{p} F_i(y_i) + C(y_1, \ldots, y_p), \tag{1}$$

where F_i's and C are closed proper convex functions which can take the value $+\infty$ to model local constraints. This special structure models a system made of p smaller subsystems, each of them generates a cost $F_i(y_i)$ depending on a local decision y_i. The optimal decision for the whole system consists in a compromise between these local costs and a global coupling cost C. In many applications, C stands for the indicator function $\delta_{\mathcal{A}}$ of a linear or affine subspace $\mathcal{A}$ which defines coupling constraints between the subsystems. For instance, in stochastic programming applications [26, 23], F_i represents the cost of scenario i and $\mathcal{A}$ the space of non-anticipative decisions; in network routing problems F_i can be the cost of traffic congestion on arc i of the network while $\mathcal{A}$ is the space of flows respecting the conservation law at every node [8, 21]; in unit commitment

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problems F_i usually stands for the production cost of unit i and $\mathcal{A}$ models the constraints of supply satisfaction.

Introducing the subdifferential operators for F and C leads to an inclusion equivalent problem which consists in finding a zero of the sum of two maximal monotone operators, a well studied model for which various splitting methods have been explored. We are interested in the class of splitting methods derived from the Douglas-Rachford and Peaceman-Rachford methods where iterations based on both resolvent operators alternate (see for instance the seminal paper by Lions and Mercier [16]). When applied to (1), it corresponds to the so-called Alternate Direction Method of Multiplier and its numerical behavior has been deeply analyzed by Eckstein, who observed the spiraling effect on the sequence of iterates which tends to penalize the performance of the method [6].

In the more special framework where C is the indicator function of a linear subspace, it corresponds to the method of partial inverses [27, 28] or the proximal decomposition method [18]. Related models with inexact subproblem computations have been studied too in [20].

Quantitative convergence results about splitting methods generally rely on strong monotonicity assumptions on operators ∂F, ∂C, or on their inverses [16, 29, 18, 9, 3] but more can be obtained by examining the properties of the global operator which drives the sequence of iterates. Surprisingly, the relations between this operator and the problem data has received to our knowledge only a little attention in the literature.

In the following, we analyze the asymptotic convergence properties of the class of splitting algorithm in the general case where the subdifferential operators are only proto-differentiable. New bounds on the convergence rate are derived and a geometric interpretation of the algorithm is given with connections with the folding operators introduced by Lawrence and Spingarn [10]. Finally, we show in the last section how to accelerate the convergence and break the spiraling effect by carefully balancing the averaging and the folding steps of the algorithm. Numerical results on medium-term stochastic planning problems confirm the nice behavior of the resulting algorithm.

2. Technical background

Hereafter, we denote $F(y) = \sum_{i=1}^{p} F_i(y_i)$ and we suppose that $\partial F + \partial C = \partial[F + C]$. Under this assumption, a solution $\bar{y}$ is an optimal solution for (1) if and only if there exists $\bar{u}$ such that

$$\bar{u} \in \partial F(\bar{y}) \tag{2a}$$

$$-\bar{u} \in \partial C(\bar{y}). \tag{2b}$$

We denote $P_F = (I + \partial F)^{-1}$ the resolvent operator of ∂F. This is the proximal operator [19] associated to F, i.e. the operator which maps to any $s \in \mathbb{R}^n$ the

unique solution of the following optimization problem:

$$P_F(s) = \arg\min_z F(z) + \frac{1}{2}\|z - s\|^2 .$$

We also consider $Q_F = (I + \partial F^{-1})^{-1}$ the resolvent operator associated to ∂F^{-1}. Q_F is the Moreau-Yosida regularization operator of ∂F; it has the same set of zeros as ∂F but it is in addition firmly non-expansive (i.e. such that, for any x, x' and $y \in Q_F(x), y' \in Q_F(x')$, we have $\langle x - x', y - y'\rangle \geqslant \|x - x'\|^2$). It is also the gradient operator of $\breve{F}$ the Moreau-Yosida regularization function of F:

$$\breve{F}(s) = \min_z F(z) + \frac{1}{2}\|z - s\|^2 .$$

P_F and Q_F satisfy

$$P_F + Q_F = I_n, \tag{3}$$

where I_n is the n-dimensional identity operator and the mapping $s \mapsto (P_F(s), Q_F(s))$ is a Lipschitz parameterization of the graph of ∂F:

$$gr(\partial F) = \{(P_F(s), Q_F(s)), s \in \mathbb{R}^n\} .$$

The operator of main interest in this study is the difference:

$$N_F = P_F - Q_F. \tag{4}$$

N_F is non-expansive and generalizes the notion of reflection across a linear subspace. Indeed, if F is the indicator function $\delta_{\mathcal{A}}$ of the subspace $\mathcal{A}$, then $P_{\delta_{\mathcal{A}}} = \Pi_{\mathcal{A}}$ is the orthogonal projector onto $\mathcal{A}$; $Q_{\delta_{\mathcal{A}}} = \Pi_{\mathcal{A}^\perp}$ is the orthogonal projector onto $\mathcal{A}^\perp$ and $N_{\delta_{\mathcal{A}}} = R_{\mathcal{A}}$ is the orthogonal reflector across $\mathcal{A}$.

Rachford class splitting methods [16, 6, 7] for solving (1) essentially use recursive applications of operators N_F and N_C. Actually, solutions of system (2) can be deduced from fixed points of $N_C \circ N_F$ with the following theorem:

Theorem 2.1. *The following propositions are equivalent:*

i) $\bar{s} = N_C \circ N_F(\bar{s})$,
ii) $\bar{s} = N_C(\bar{d})$; $\bar{d} = N_F(\bar{s})$,
iii) $(\bar{y}, \bar{u}) = (P_F(\bar{s}), Q_F(\bar{s})) = (P_C(\bar{d}), -Q_C(\bar{d}))$,
iv) $(\bar{y}, \bar{u})$ *is a solution of* (2).

Proof. A proof can be found for instance in [7]. □

Given a relaxation parameter $0 < \alpha \leqslant 1$, the Rachford class methods consist in applying the following fixed point averaged iterations from an arbitrary starting point s^0:

$$s^{k+1} = ((1 - \alpha)I_n + \alpha N_C \circ N_F)(s^k). \tag{5}$$

The cases $\alpha = 1$ and $\alpha = \frac{1}{2}$ are respectively known to correspond to the Peaceman-Rachford and the Douglas-Rachford methods, developed in [16] for any maximal monotone operators, i.e. not necessarily subdifferentials of convex

functions. When $\alpha = \frac{1}{2}$, the algorithm corresponds [5] to the alternating direction method of multipliers (algorithm 1).

Algorithm 1

Require: $\epsilon > 0, y^{k=0}, u^{k=0} \in \mathbb{R}^n$

repeat
  $s^k = y^k + u^k$
  for all $i = 1, \dots, p$ **do**
    $y_i^{k+\frac{1}{2}} = \arg\min_{y_i} F_i(y_i) + \frac{1}{2} \left\| y_i - y_i^k \right\|^2 - \left\langle u_i^k, y_i \right\rangle$
  end for
  $y^{k+1} = \arg\min_y C(y) + \frac{1}{2} \left\| y - y^{k+\frac{1}{2}} \right\|^2 + \left\langle u^k, y \right\rangle$
  $u^{k+1} \leftarrow u^k - (y^{k+1} - y^{k+\frac{1}{2}})$
until $\left\| y^{k+\frac{1}{2}} - y^k \right\| < \epsilon$

The sequence $\left\{(y^k, u^k)\right\}_k$ generated by this method always converges to a primal-dual solution of (2) from any starting point.

3. Differentiability properties of generalized reflections

By identity (3) and definition (4), we have

$$N_F = I_n - 2Q_F. \tag{6}$$

Therefore, N_F has the same first order differentiability properties as $Q_F = \nabla \breve{F}$ which actually correspond to second-order differentiability properties of $\breve{F}$. The question of twice-differentiability of $\breve{F}$ has received much interest in the study of the proximal point method [2] as well as inexact versions [14] so as to develop quasi-Newton like methods. It has been studied in the convex case by [17], Lemaréchal and Sagastizabál [11, 12] and in a more general context by Poliquin and Rockafellar [22]. The first result (Proposition 3.1) needs ∂F to be differentiable. Before stating it, we first recall this notion for multi-valued operators.

Definition 3.1. ∂F is differentiable at $\bar{y}$ if $\partial F(\bar{y}) = \{\bar{u}\}$ and if there is a linear mapping S such that

$$\partial F(y) \subset \bar{u} + S(y - \bar{y}) + o(\|y - \bar{y}\|)B,$$

where B is the unit ball of $\mathbb{R}^n$.

Proposition 3.1. *If ∂F is differentiable at $\bar{y}$ with $D[\partial F](\bar{y}) = S$ and $\partial F(\bar{y}) = \{\bar{u}\}$, then $\breve{F}$ is twice differentiable at $\bar{s} = \bar{y} + \bar{u}$ and*

$$\nabla Q_F(\bar{s}) = \nabla^2 \breve{F}(\bar{s}) = I_n - (I_n + S)^{-1}.$$

A proof can be found in [12] and references therein. The differentiability of N_F follows from (6).

Corollary 3.1. *If ∂F is differentiable at $\bar{y}$ with $D[\partial F](\bar{y}) = S$ and $\partial F(\bar{y}) = \{\bar{u}\}$, then N_F is differentiable at $\bar{s} = \bar{y} + \bar{u}$ and*

$$\nabla N_F(\bar{s}) = 2(I_n + S)^{-1} - I_n.$$

This result is generalized in [17] and [22] with the help of *proto-differentiability* of ∂F. This notion was introduced in [25] as a way to generalize second order differentiability of convex functions. It corresponds to the convergence of the graphs of the family of operators indexed by $\tau > 0$:

$$\Delta_\tau(\bar{y}|\bar{u}) : h \mapsto \frac{\partial F(\bar{y} + \tau h) - \bar{u}}{\tau}.$$

Definition 3.2. ∂F is proto-differentiable at $\bar{y}$ for $\bar{u} \in \partial F(\bar{y})$ if $\{\Delta_\tau(\bar{y}|\bar{u})\}_\tau$ graphically converges as τ positively goes to zero. The limit operator $D[\partial F](\bar{y}|\bar{u})$ is the proto-derivative mapping.

The differentiability of ∂F actually corresponds to the proto-derivative being a linear mapping. It was highlighted in [17] and [22] that the relevant property to obtain twice differentiability of $\breve{F}$ is not linearity of $D[\partial F](\bar{y}|\bar{u})$ but rather the linearity of its graph, i.e. the generalized linearity of $D[\partial F](\bar{y}|\bar{u})$.

Definition 3.3. $D[\partial F](\bar{y}|\bar{u})$ is generalized linear if $D[\partial F](\bar{y}|\bar{u}) = S + \mathbf{N}_{\mathcal{H}}$ where $\mathcal{H}$ is a linear subspace, S is symmetric positive semi-definite.

Generalized linearity for multi-valued mappings is usually defined as the linearity of the graph. This formulation for maximal monotone subdifferential operators appears in [24, Proposition 4.1]. The following characterization of second order differentiability of $\breve{F}$ as well as the explicit formula for the Hessian is given in [17]. $A^\dagger$ denotes the Penrose-Moore pseudo-inverse of A; $im(A)$ and $ker(A)$ respectively denote the image and the kernel of A.

Proposition 3.2 (Lucet [17])**.** *The following propositions are equivalent:*

i) *∂F is proto-differentiable at $\bar{y}$ for $\bar{u}$ with a generalized linear proto-derivative $D[\partial F](\bar{y}|\bar{u}) = S + \mathbf{N}_{\mathcal{H}}$.*
ii) *$\breve{F}$ is twice differentiable at $\bar{s} = \bar{y} + \bar{u}$ and*

$$\nabla^2 \breve{F}(\bar{s}) = \nabla Q_F(\bar{s}) = \left[\Pi_{im(S)+\mathcal{H}^\perp} \circ \left[I_n + (\Pi_{\mathcal{H}} S \Pi_{\mathcal{H}})^\dagger\right] \circ \Pi_{im(S)+\mathcal{H}^\perp}\right]^\dagger.$$

A generalized linear subdifferential mapping is the subdifferential of a *generalized quadratic* or *partially quadratic* function. This class of functions, as noted in [22], can be viewed as a quadratic function with $+\infty$ as a possible eigenvalue. To simplify the notations, we will use a decomposition of the space in three subspaces, each of them corresponds respectively to eigenvalues in $\{0\}$, $\mathbb{R}^*$ and $\{+\infty\}$.

Proposition 3.3. *In Definition 3.3, S can be chosen such that $im(S) \subset \mathcal{H}$.*

Proof. $D[\partial F](\bar{y}|\bar{u})$ has the form:

$$Sy + \mathbf{N}_{\mathcal{H}}(y) \begin{cases} Sy + \mathcal{H}^{\perp} & \text{if } y \in \mathcal{H} \\ \emptyset & \text{else.} \end{cases}$$

In the first case, $y = \Pi_{\mathcal{H}} y$ and $Sy + \mathcal{H}^{\perp} = \Pi_{\mathcal{H}}(Sy) + \mathcal{H}^{\perp}$ so S can be replaced by $\Pi_{\mathcal{H}} S \Pi_{\mathcal{H}}$ which satisfies $im(\Pi_{\mathcal{H}} S \Pi_{\mathcal{H}}) \subset \mathcal{H}$. □

Hereafter, $\mathbb{R}^n$ will be decomposed into the direct sum:

$$\begin{aligned} \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 &= \mathbb{R}^n \\ \mathcal{H}_1 &= \mathcal{H} \cap im(S) = im(S) \\ \mathcal{H}_2 &= \mathcal{H} \cap ker(S) \\ \mathcal{H}_3 &= \mathcal{H}^{\perp}. \end{aligned}$$

Their respective dimensions are denoted by n_1, n_2, n_3 with $n_1 + n_2 + n_3 = n$. It is quite clear that these spaces are mutually orthogonal since $ker(S) = im(S)^{\perp}$. The fact that the sum is direct comes from the inclusion $im(S) \subset \mathcal{H}$.

Remark 3.1. This decomposition is closely related to the $\mathcal{U}$-$\mathcal{V}$ decomposition of [13]. More precisely, $\mathcal{U} = \mathcal{H}$ is the space along which F is smooth and $\mathcal{V} = \mathcal{H}^{\perp}$ is the subspace where F is not differentiable. Here, we additionally require F to have a Hessian along $\mathcal{U}$ and we consider again the subspaces where F is linear and where it is strongly convex.

Now, we choose an orthogonal basis B which fits this block decomposition so that S writes in this new basis:

$$B^{-1} S B = \begin{pmatrix} L & & \\ & 0 & \\ & & 0 \end{pmatrix}, \tag{7}$$

where L is a $n_1 \times n_1$ invertible submatrix. In this setting, we have the following proposition:

Proposition 3.4. *The following statements are equivalent:*

i) *∂F is proto-differentiable at $\bar{y}$ for $\bar{u}$ with a generalized linear proto-derivative $D[\partial F](\bar{y}|\bar{u}) = S + \mathbf{N}_{\mathcal{H}_1 \oplus \mathcal{H}_2}$; $im(S) = \mathcal{H}_1$,*

ii) *$\breve{F}$ is twice differentiable at $\bar{s} = \bar{y} + \bar{u}$ and*

$$\nabla^2 \breve{F}(\bar{s}) = \nabla Q_F(\bar{s}) = B \begin{pmatrix} (I_{n_1} + L^{-1})^{-1} & & \\ & 0 & \\ & & I_{n_3} \end{pmatrix} B^{-1},$$

iii) *N_F is differentiable at $\bar{s} = \bar{y} + \bar{u}$ and*

$$\nabla N_F(\bar{s}) = B \begin{pmatrix} 2(I_{n_1} + L)^{-1} - I_{n_1} & & \\ & I_{n_2} & \\ & & -I_{n_3} \end{pmatrix} B^{-1}. \tag{8}$$

Proof. i) and ii) are simple reformulations of Proposition 3.2 i) and ii) while iii) comes from (6). We give the detail of the formula of $\nabla Q_F(\bar{s})$:

$$\begin{aligned}
\nabla Q_F(\bar{s}) &= \nabla^2 \breve{F}(\bar{s}) \\
&= \left[\Pi_{\mathcal{H}_1\oplus\mathcal{H}_3} \circ \left[I_n + (\Pi_{\mathcal{H}_1\oplus\mathcal{H}_2} \circ S \circ \Pi_{\mathcal{H}_1\oplus\mathcal{H}_2})^\dagger\right] \circ \Pi_{\mathcal{H}_1\oplus\mathcal{H}_3}\right]^\dagger \\
&= \left[\Pi_{\mathcal{H}_1\oplus\mathcal{H}_3} \circ \left[I_n + S^\dagger\right] \circ \Pi_{\mathcal{H}_1\oplus\mathcal{H}_3}\right]^\dagger \\
&= \left[\Pi_{\mathcal{H}_1\oplus\mathcal{H}_3} + S^\dagger\right]^\dagger \\
&= B \begin{pmatrix} (I_{n_1} + L^{-1})^{-1} & & \\ & 0 & \\ & & I_{n_3} \end{pmatrix} B^{-1}.
\end{aligned}$$

□

The case $\mathcal{H}_3 = \{0\}$ is treated at Corollary 3.1. We now consider another extreme case when $\mathcal{H}_1 = \{0\}$.

Case $\mathcal{H}_1 = \{0\}$. When $\mathcal{H}_1 = \{0\}$, the proto-derivative $D[\partial F](\bar{u}|\bar{y})$ reduces to $\mathbf{N}_\mathcal{H}$. It means that $\mathcal{H}_2 = \mathcal{H}$, $\mathcal{H}_3 = \mathcal{H}^\perp$ and from Proposition 3.4, N_F is differentiable with

$$\nabla N_F(\bar{s}) = \begin{pmatrix} I_{n_2} & \\ & -I_{n_3} \end{pmatrix} = R_\mathcal{H}. \tag{9}$$

Notably, if ∂F is the subdifferential of a polyhedral function, then it is proto-differentiable everywhere and generalized linear with $\mathcal{H} = 0$ almost everywhere. In addition, for ϵ small enough, N_F has an expansion which holds exactly

$$N_F(s) = N_F(\bar{s}) + \nabla N_F(\bar{s})(s - \bar{s}) \qquad \forall s / \|s - \bar{s}\| \leqslant \epsilon. \tag{10}$$

As a remark, some other functions which are not polyhedral can however present a subdifferential with generalized linear proto-derivative having $\mathcal{H}_1 = \{0\}$. This includes for example:

$$F : y \mapsto \frac{1}{3}|y|^3 \qquad\qquad \nabla F : y \mapsto \begin{cases} y^2 & \text{if } y \geqslant 0 \\ -y^2 & \text{if } y < 0 \end{cases}$$

or its conjugate, proto-derivatives of which at 0 for 0 are respectively $y \mapsto 0$ and $\mathbf{N}_{\{0\}}$. In the next section we will however concentrate on polyhedral functions because of the additional structure (10) it provides.

4. Asymptotic behavior in the polyhedral case

Let Algorithm 1 converge to $(\bar{y}, \bar{u})$, $\bar{s} = \bar{y} + \bar{u}$, $\bar{d} = \bar{y} - \bar{u}$. If N_F and N_C are respectively differentiable at $\bar{s}$ and $\bar{d}$, then, from recursion (5), it comes:

$$e^{k+1} = \left[(1-\alpha)I + \alpha \nabla N_C(\bar{d}) \nabla N_F(\bar{s})\right] e^k + o(\left\| e^k \right\|),$$

where $e^k = s^k - \bar{s}$ denotes the error at each iteration. We now specialize these results when F and C are polyhedral functions. From (9) and (10), we have for ϵ small enough and as soon as $\left\|e^k\right\| < \epsilon$:

$$e^{k+1} = \left[(1-\alpha)I + \alpha \nabla N_C(\bar{d}) \nabla N_F(\bar{s})\right] e^k. \tag{11}$$

To simplify, we will suppose that $\left\|e^0\right\| < \epsilon$. According to (9), $\nabla N_C(\bar{d}) \nabla N_F(\bar{s})$ is the product of two reflections hence it is isometric. It consequently writes in some basis D:

$$\nabla N_C(\bar{d}) \nabla N_F(\bar{s}) = D \begin{pmatrix} I_p & & & & \\ & -I_q & & & \\ & & E_1 & & \\ & & & \ddots & \\ & & & & E_r \end{pmatrix} D^{-1},$$

where $p + q + 2r = n$ and $E_i = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{pmatrix}$, $0 < \theta_1 \leqslant \ldots \leqslant \theta_r < \pi$. The operator $J_\alpha = (1-\alpha)I + \alpha \nabla N_C(\bar{d}) \nabla N_F(\bar{s})$ writes in this basis:

$$J_\alpha = D \begin{pmatrix} I_p & & & & \\ & (1-2\alpha)I_q & & & \\ & & (1-\alpha)I_2 + \alpha E_1 & & \\ & & & \ddots & \\ & & & & (1-\alpha)I_2 + \alpha E_r \end{pmatrix} D^{-1}. \tag{12}$$

The error vector expressed in basis D can be decomposed in $2 + r$ subvectors, independent one from another. The first subvector (of size p) is constant, so it must be 0_p from some rank, i.e. the algorithm converges in a finite number of steps in this first subspace. The second subvector (of size q) converges linearly at a rate $|1 - 2\alpha|$ in a one-dimensional subspace of $\mathcal{E}_q$, in a finite number of steps if $\alpha = \frac{1}{2}$. The r remaining 2-dimensional subvectors are given from one iteration to another by an α-averaged rotation of angle θ_i.

Remark 4.1. The exactness of the expansion (10) is necessary to affirm that the first subvector converges in a finite number of steps. Otherwise, we only could have said that it converges sublinearly. This occurs for instance in the following counter-example:

$$F(y) = \frac{1}{3}\left|y\right|^3; C(y) = 0; s^0 > 0$$

where s^{k+1} is given by the recursion $s^{k+1} = (\frac{1}{4} + s^k)^{\frac{1}{2}} - (\frac{1}{4})^{\frac{1}{2}}$.

We can derive an upper bound for the rate of convergence by considering the maximum between the norms of all the blocks of J_α but the first. We already mentioned that the norm of the second block is $|1 - 2\alpha|$; for the other ones the calculation gives:

Lemma 4.1. *The norm of the averaged rotation* $(1-\alpha)I_2 + \alpha E_i$ *is*

$$\|(1-\alpha)I_2 + \alpha E_i\|_2 = \left[(1-2\alpha)^2 \sin^2\left(\frac{\theta_i}{2}\right) + \cos^2\left(\frac{\theta_i}{2}\right)\right]^{\frac{1}{2}}.$$

Proof. Let us choose $v \in \mathbb{R}^2$ and compute the norm of $v' = (1-\alpha)v + \alpha E_i v$. We will use the fact that $\|E_i v\| = \|v\|$ and $\langle v, E_i v\rangle = \cos(\theta_i)$.

$$\begin{aligned}
\left\|v'\right\|^2 &= \|(1-\alpha)v + \alpha E_i v\|^2 \\
&= \left\|(1-2\alpha)\frac{1}{2}(v - E_i v) + \frac{1}{2}(v + E_i v)\right\|^2 \\
&= (1-2\alpha)^2\frac{1}{4}\|v - E_i v\|^2 + \frac{1}{4}\|v + E_i v\|^2 + \frac{1}{4}(1-2\alpha)\left(\underbrace{\|v\|^2 - \|E_i v\|^2}_{=0}\right) \\
&= (1-2\alpha)^2\frac{1}{4}\left(\|v\|^2 + \|E_i v\|^2 - 2\langle v, E_i v\rangle\right) + \frac{1}{4}\left(\|v\|^2 + \|E_i v\|^2 + 2\langle v, E_i v\rangle\right) \\
&= (1-2\alpha)^2\frac{1-\cos(\theta_i)}{2}\|v\|^2 + \frac{1+\cos(\theta_i)}{2}\|v\|^2 \\
&= (1-2\alpha)^2\sin^2\left(\frac{\theta_i}{2}\right)\|v\|^2 + \cos^2\left(\frac{\theta_i}{2}\right)\|v\|^2 \\
\left\|v'\right\| &= \left[(1-2\alpha)^2\sin^2\left(\frac{\theta_i}{2}\right) + \cos^2\left(\frac{\theta_i}{2}\right)\right]^{\frac{1}{2}}\|v\|.
\end{aligned}$$

□

We deduce from this formula the following bound for the convergence rate:

$$\limsup \frac{\left\|e^{k+1}\right\|}{\|e^k\|} \leqslant \max_{i=1}^{r}\left[(1-2\alpha)^2\sin^2\left(\frac{\theta_i}{2}\right) + \cos^2\left(\frac{\theta_i}{2}\right)\right]^{\frac{1}{2}}$$

which is minimized for the choice $\alpha = \frac{1}{2}$ and reduces to

$$\limsup \frac{\left\|e^{k+1}\right\|}{\|e^k\|} \leqslant \max_{i=1}^{r}\cos\left(\frac{\theta_i}{2}\right) = \cos\left(\frac{\theta_1}{2}\right).$$

Link with Foldings of Spingarn and Lawrence. We examine in this section the case when F and C are polyhedral but with proto-derivatives not necessarily generalized linear. N_F and N_C, as well as their composition $N_F \circ N_C$ belong to a special class of operators called *foldings* introduced in [10]. A folding is a non-expansive piecewise isometric mappings $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ having only a locally finite number of pieces called *folds*. These folds must be closed polyhedron with a non-empty disjoint interior. Actually, when F (equivalently C) is polyhedral, the folds of N_F are in one-to-one correspondence with the projection onto $\mathbb{R}^n \times \{0\}$ of the faces of $epi(F)$, the epigraph of F. Namely, if f is such a projected face then the polyhedron $U_f = f + \partial F(f)$ is a fold. See figure 1 for an example of such a construction. If we denote by d_f the projection of $\partial F(f)$ onto $span(f)$ (it

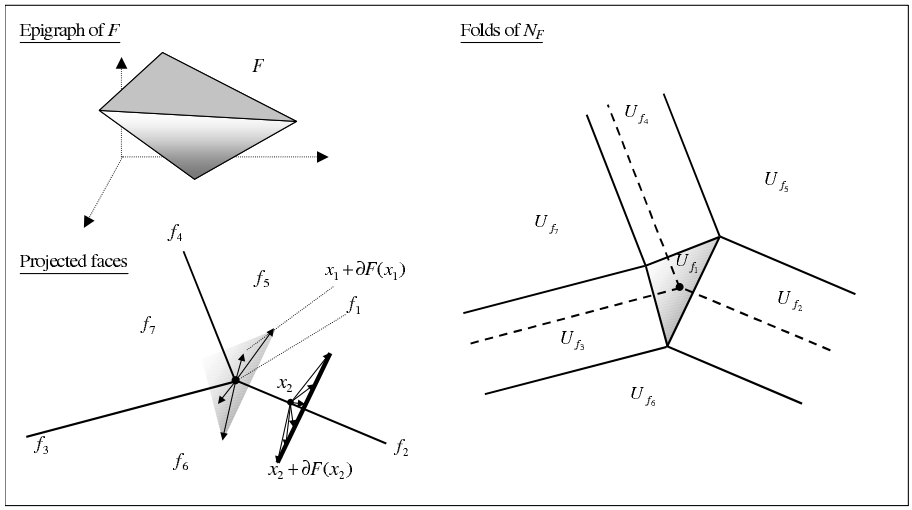

FIGURE 1. Folds U_{f_i} associated with projected faces f_i of a polyhedral convex function from $\mathbb{R}^2$ to $\mathbb{R}$.

is a singleton since $\partial F(f)$ is orthogonal to $span(f)$), then the restriction of N_F to U_f is $R_{aff(f)} - 2d_f$, i.e. the composition of the reflection through the affine subspace spanned by f and of the translation of $-2d_f$.

In [10], Lawrence and Spingarn also study the asymptotic behavior of averaged iterations of foldings in the case $\alpha = \frac{1}{2}$:

$$s^{k+1} = \frac{1}{2}s^k + \frac{1}{2}\mathcal{F}(s^k).$$

They restrict themselves to the class of positively homogeneous foldings, by centering $\mathcal{F}$ around the convergence point and considering a neighborhood of 0. This actually corresponds to handle the semi-derivative $D\mathcal{F}(\bar{s})$ of $\mathcal{F}$ at $\bar{s}$ and to study the sequence of error vectors $\left\{e^k\right\}_k$ given by averaged iterations:

$$e^{k+1} = \frac{1}{2}e^k + \frac{1}{2}D\mathcal{F}(\bar{s})(e^k).$$

Their main result states that, from some rank the sequence $\left\{e^k\right\}_k$ stays in some subspace $V(e^0)$ (depending on the initial iterate e^0) and the restriction of $D\mathcal{F}(\bar{s})$ on $V(e^0)$ is isometric. More specially, consequences i) and ii) of Theorem 4.1 claims that the sequence $\left\{e^k\right\}_k$ spirals in $V(e^0)$.

Theorem 4.1 (Lawrence-Spingarn [10])**.** *Let $\mathcal{F}$ be a folding and $e^{k+1} = \frac{1}{2}(I + D\mathcal{F})e^k$ converge to 0. Then*

i) *there exist a subspace $V(e^0)$ and a rank K such that*

$$k \geqslant K \Rightarrow span(e^k, e^{k+1}, \ldots) = V(e^0) = cone(e^k, e^{k+1}, \ldots).$$

ii) *The restriction of $D\mathcal{F}$ to $V(e^0)$ is an isometry.*

This result is more general than our development because it does not necessitate $\mathcal{F}$ to be differentiable but simply semi-differentiable which is the case everywhere. It is worth noting that even in the differentiable case, conclusion i) does generally not extend to choices $\alpha < \frac{1}{2}$ because of the second block in (12) which may keep the associated component in a half-line. A second remark is that consequence iii) asserts that the restriction of $\frac{1}{2}(I + D\mathcal{F}(\bar{s}))$ to $V(e^0)$ has the same structure as $J_{\alpha=\frac{1}{2}}$ in (12). Consequently, the convergence rate also satisfies for some $0 < \theta_1 \leqslant \theta_2 \leqslant \ldots \leqslant \theta_{r'} < \pi$:

$$\limsup \frac{\left\|e^{k+1}\right\|}{\|e^k\|} \leqslant \max_{i=1}^{r'} \cos\left(\frac{\theta_i}{2}\right) = \cos\left(\frac{\theta_1}{2}\right).$$

This geometric interpretation of "spiraling" convergence will be used in the next section to propose possible acceleration of the method.

5. Accelerating the convergence

Theorem 4.1 states that the sequence of errors asymptotically describes a spiral around 0. We propose here a slight modification of recursion (5) and of algorithm 1 with a negligible computational cost. The main idea is to follow "chords" of the spiral instead of the spiral itself.

5.1. **Skipping average in Algorithm 1.**

We have shown that the asymptotic convergence rate depends on the largest angle θ_1. The closer this angle will be from π, the faster the convergence will be. Now, if we operate several applications of $\mathcal{F}$ before averaging, say L applications, then the set of angles $(\theta_1, \ldots, \theta_r)$ will be replaced by $(L\theta_1, \ldots, L\theta_r)$. Consequently, if some angles are small, it can be profitable to alternate the number of applications of $\mathcal{F}$ before averaging. Formally, we propose the following method. Let us select a set of $a+1$ integers $L_0, \ldots, L_a$ with $L_0 = 1$ that we will refer as the *averaging sequence*. We set $\sigma = \sum_{l=0}^{a} L_l$ and apply the recursion:

$$s^{(k+1)\sigma} = \frac{1}{2}\left(I + \mathcal{F}^{L_0}\right) \circ \ldots \circ \frac{1}{2}\left(I + \mathcal{F}^{L_a}\right) s^{k\sigma}. \tag{13}$$

The superscript of s gives the number of required computations of $\mathcal{F}$. Figure 2 graphically shows the acceleration for several choices of sequence. The number of computation of $\mathcal{F}$ is identical for the three choices.

The involved operator

$$\mathcal{R} = \frac{1}{2}\left(I + \mathcal{F}\right) \circ \frac{1}{2}\left(I + \mathcal{F}^{L_1}\right) \circ \ldots \circ \frac{1}{2}\left(I + \mathcal{F}^{L_a}\right)$$

is an averaged operator, so successive applications converge to an element of $fix(\mathcal{R}) = \bigcap_{l=0}^{a} fix(\mathcal{F}^{L_l}) = fix(\mathcal{F})$. Proofs of this fact can be found in [1, 4]. The modified iterations (13) gives Algorithm 2.

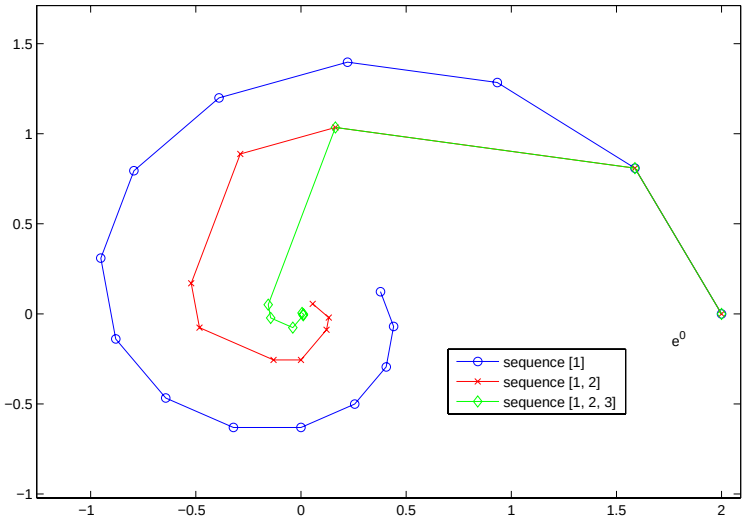

FIGURE 2. Convergence rate improvement for several choices of averaging sequence.

Algorithm 2

Require: $\epsilon > 0; 0 < \alpha \leqslant 1; y^{k=0}, u^{k=0} \in \mathbb{R}^n$

repeat
  for all $l = 0, \ldots, a$ **do**
    SAVING REFERENCE ALLOCATIONS
    $(y^{ref}, u^{ref}) \leftarrow (y^k, u^k)$
    for all $j = 1, \ldots, L_l$ **do**
      SOLVING SUBPROBLEMS
      for all $i = 1, \ldots, p$ **do**
        $y_i^{k+\frac{1}{2}} = \arg\min_{y_i} F_i(y_i) + \frac{1}{2}\left\|y_i - y_i^k\right\|^2 - \left\langle u_i^k, y_i\right\rangle$
      end for
      COORDINATION
      $y^{k+1} = 2\left\{\arg\min_y C(y) + \frac{1}{2}\left\|y - y^{k+\frac{1}{2}}\right\|^2 + \left\langle u^k, y\right\rangle\right\} - y^k$
      $u^{k+1} \leftarrow u^k - (y^{k+1} - 2y^{k+\frac{1}{2}} + y^k)$
      $k \leftarrow k + 1$
    end for
    AVERAGING
    $(y^k, u^k) \leftarrow \alpha(y^{k-1}, u^{k-1}) + (1 - \alpha)(y^{ref}, u^{ref})$
  end for
until $\left\|y^{k+\frac{1}{2}} - y^k\right\| < \epsilon$

5.2. **Asymptotic behavior.** We have

$$\limsup \frac{\left\|e^{(q+1)\sigma}\right\|}{\left\|e^{q\sigma}\right\|} = \max_{i=1,\dots,r} \left\{ \left| \prod_{l=0}^{a} \cos\left(\frac{L_l \theta_i}{2}\right) \right| \right\}.$$

In order to reflect the number of computation of $\mathcal{F}$, we consider the mean convergence rate to be the σ-root

$$\tau(\theta) = \max_{i=1,\dots,r} \left\{ \left| \prod_{l=0}^{a} \cos\left(\frac{L_l \theta_i}{2}\right) \right| \right\}^{\frac{1}{\sigma}}.$$

The question of the acceleration of the convergence in comparison to the classical algorithm ($a = 0$) depends on the repartition of the θ_i's in $]0, \pi[$. If θ_1 is close to 0, then the convergence will be accelerated. One case when the classical implementation should not be changed is when all θ_i's are close to π. In this case, the classical method is very fast and making several applications of the folding before averaging will slow down the convergence. Some computations of $\mathcal{F}$ will be used to make small steps whereas they would have been used instead to perform large steps in the original method. We consider on figure 3 the case where $dim(V(e^0)) = 4$. We draw for $a = 1$ and $L_1 = 2$ the area of $]0, \pi[\times]0, \pi[$, $\theta_1 \leqslant \theta_2$ where the convergence rate is decreased, i.e. the area where $\tau(\theta) \leqslant \rho(\theta) = |\cos(\theta_1)|$. It seems difficult to deduce a priori the values of these angles from the problem data and even more so to determine an adequate averaging sequence. In the last section, we present some attempts of sequences and observe the results on a powerplant management problem.

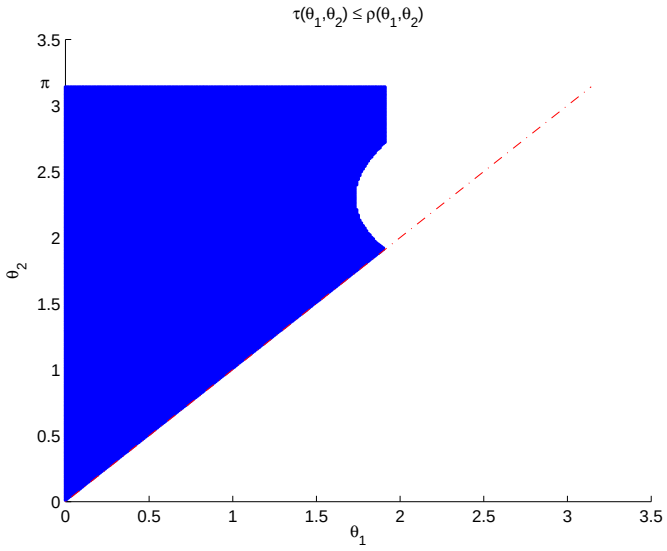

FIGURE 3. Domain of $]0, \pi[\times]0, \pi[$, $\theta_1 \leqslant \theta_2$ where the asymptotic convergence is improved for $a = 1$, $L_1 = 2$

6. Numerical experiments

We compare in this last section the classical algorithm ($a = 0$) with its modified version on a powerplant management problem. We dispose of p powerplants and we have to determine how much each of them must produce over a time period so as to satisfy a given demand d and minimize a production cost. y_i is the amount of production of plant i over the period. F_i models the linear cost of production of plant i as well as its local constraint (management of a reservoir, capacity of production, ...). C is the indicator function $\delta_{\mathcal{A}}$ of the affine subspace $\mathcal{A}$ of production satisfying the demand:

$$\mathcal{A} = \{(y_1, \ldots, y_p)/y_1 + \ldots + y_p = d\}\,.$$

Usually, a scaling factor λ is added so as to improve the problem conditioning. It consists in replacing $F(\cdot)$ and $C(\cdot)$ by $F(\lambda^{-1}\cdot)$ and $C(\lambda^{-1}\cdot)$. For $a = 1$ and $L_1 = 2$, we applied both algorithms for several values of this scaling parameter. The number of subproblem resolutions and CPU time savings are reported on figures 4 and 5. 100% coincide with no saving at all while 50% means that the modified algorithm is twice faster.

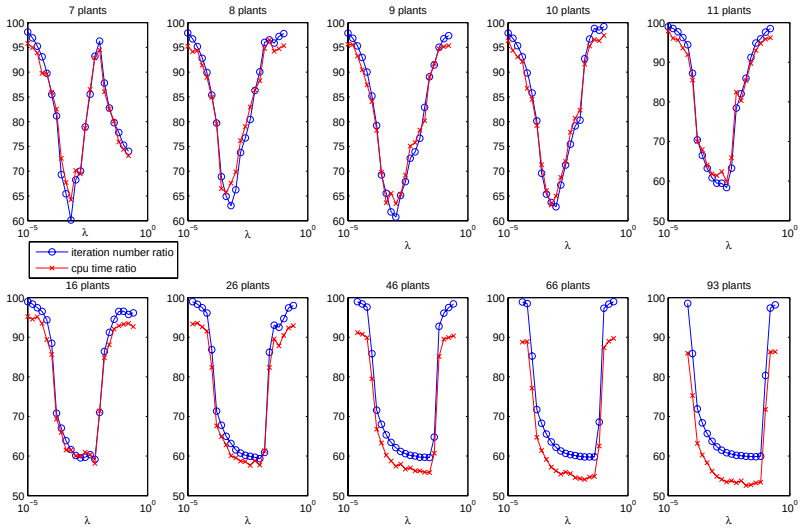

Figure 4. CPU and number of subproblem resolutions saving on a 50 periods problem with 6 reservoirs

In most cases, we observe between 20% and 40% of CPU savings. Interestingly, the maximum saving is provided for values of λ making the classical method the most efficient. We also compared several sequences, namely $[1]$; $[1, 2]$; $[1, 2, 3]$ and $[1, 2, 3, 4]$ on a larger problem. The quantity $\|y^{k+\frac{1}{2}} - y^k\|^2$ used in the stopping criterion is plotted with respect to CPU time on figure 6. Parameter ϵ is chosen

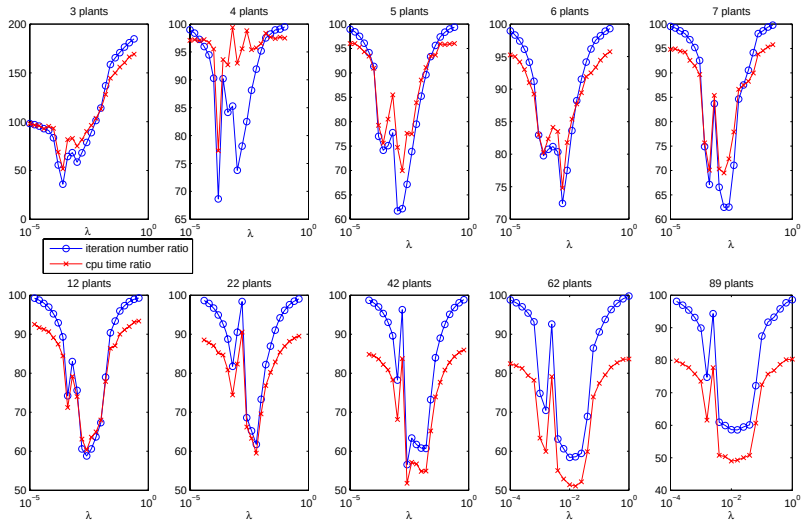

FIGURE 5. CPU and number of subproblem resolutions saving on a 350 period problem with 2 reservoirs

equal to 10^{-6}. On this instance, we succeeded in reducing the computation time by a factor 3 with the last two sequences.

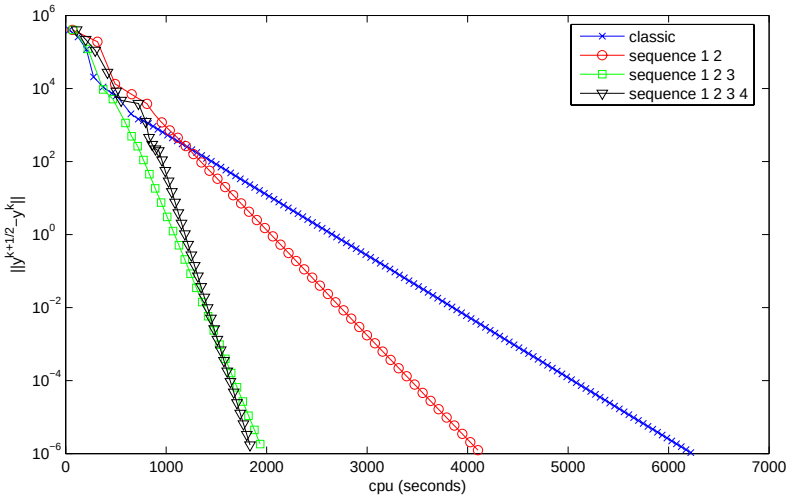

FIGURE 6. Speed of convergence for different choices of averaging sequence

7. Conclusion

This study focused on polyhedral separable optimization and revealed that the examined splitting method can theoretically be accelerated by skipping some of the averaging steps. Our preliminary numerical experiments have confirmed it was the case in practice, reducing effectively the linear rate of convergence. The theoretical setting employed in Sections 3 and 4 provides a unified view of the asymptotic behavior of the method: it covers the twice differentiable case already examined in [15] and allowed us to partially recover former results in the polyhedral case. We hope this approach will enable further developments on broader classes of functions. However, the polyhedral case still raises interesting theoretical and numerical questions like the asymptotic effect of multidimensional scaling or the development of automatic tuning strategies of the parameter α away from the solution.

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