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AN EXISTENCE RESULT FOR MINTY VARIATIONAL INEQUALITIES

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Dedicated to Nguyen Van Hien on the occasion of his sixty-fifth birthday

Abstract. In this paper, we establish an existence result of solutions to Minty Variational Inequality under a generalized properly quasimonotone assumption that we introduce. This condition is weaker than those given in the literature for this problem. We illustrate it by an example. We also use this condition to extend the result given by Kassay, Kolumbán and Páles for Minty variational inequality systems.

1. Introduction

Let X and Y be two real Hausdorff topological vector spaces and let C be a nonempty set of Y. Let $\langle \cdot, \cdot \rangle : X \times Y \to R$ be a continuous bilinear function and $F : C \to \mathcal{P}(X)$ be a set-valued mapping with nonempty values. We consider the following Minty variational problem:

$$(1) \qquad M(F;C) \begin{cases} \text{Find } \; x \in C, \quad \text{such that:} \\ \langle v, y-x \rangle \geq 0 \quad \forall \; y \in C \; \forall \, v \in F(y). \end{cases}$$

In Konnov [13] this problem was called the *dual* formulation of the following variational inequality problem:

$$(2) \qquad VIP(F;C) \begin{cases} \text{Find } \; x \in C, \quad \text{such that } \exists \; u \in F(x): \\ \langle u, y-x \rangle \geq 0 \quad \forall \; y \in C. \end{cases}$$

It is well known in the literature that the existence of a solution to problem $M(F;C)$ implies, under suitable continuity conditions on F that $VIP(F;C)$ is also solvable (see, for example, [13, 14, 15, 16] and references therein). It is important to observe that the existence of solutions to $M(F;C)$ usually requires some monotonicity condition. In [15] it was assumed that F is a single monotone operator while in [14] this condition was required for multivalued monotone operators. In [16] the existence result was shown for pseudomonotone operators.

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In [13] was obtained the solvability of the Minty variational inequality under the semistrictly quasimonotone condition on F. In [12] it was studied the existence of solutions to $M(F;C)$ and to Minty variational inequality systems by using that F is a properly quasimonotone mapping.
The aim of this paper is to present existence results for these two problems under a weak condition on F, that we call the generalized properly quasimonotone condition.
The outline of this work is as follows. In Section 2 we give definitions and properties that will be useful. We also fix some notations. In Section 3 we present a generalization of the KKM principle and properties that we will use to prove the main existence result. We also consider an example to illustrate our requirements. Finally, we present in Section 4 an existence theorem of solutions to Minty variational inequality systems.

2. Preliminaries

We recall some concepts we will use. Throughout this section let X and Y be two real Hausdorff topological vector spaces. Let A be a nonempty subset of X and B a nonempty convex subset of Y. The following notions and properties can be seen, for example, with slight differences in [1, 5, 12, 17].

Definition 2.1. A multivalued mapping $F : A \to \mathcal{P}(X)$ is said to be a *KKM* mapping if, for any finite subset $\{x_1, \cdots, x_n\} \subset A$, it holds

$$\mathrm{co}\{x_1, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$$

where $\mathrm{co}\{x_1, \cdots, x_n\}$ denotes the convex hull of the set $\{x_1, \cdots, x_n\}$.

Definition 2.2. A multivalued mapping $F : A \to \mathcal{P}(B)$ is said to be a *generalized KKM* mapping if, for any finite subset $\{x_1, \cdots, x_n\} \subset A$, there is a finite subset $\{y_1, \cdots, y_n\} \subset B$ such that, for any subset $\{y_{i_1}, \cdots, y_{i_k}\} \subseteq \{y_1, \cdots, y_n\}$, it holds

$$\mathrm{co}\{y_{i_1}, \cdots, y_{i_k}\} \subseteq \bigcup_{j=1}^{k} F(x_{i_j}).$$

It is easy to see that every KKM mapping is a generalized KKM mapping. In [5] there is a counterexemple to illustrate that the converse does not hold.
We will generalize the following concept that we can find in [8].

Definition 2.3. A multivalued mapping $F : A \to \mathcal{P}(X)$ is said to be *properly quasimonotone* if for any finite subset $\{y_1, \cdots, y_m\} \subset A$ and for any $y \in \mathrm{co}\{y_1, \cdots, y_m\}$ it holds

$$\max_{1 \le j \le m} \inf_{v \in F(y_j)} \langle v, y_j - y \rangle \ge 0.$$

Let us note that this notion has a strong relation with the concept of 0-diagonally quasiconvexity in y. In fact, F is a properly quasimonotone mapping if and only if the function $\varphi(x,y) = \inf_{v\in F(y)} \langle v, y-x\rangle$ is 0-diagonally quasiconvex in y (see for example [5]).

In proving our main results we need the following two theorems.

Theorem 2.4. [3, Theorem 6] *A topological space X is compact if and only if the finite intersection axiom is verified, that is, if $\{F_i : i \in I\}$ is a family of closed sets in X for which any finite intersection is nonempty, then $\bigcap_{i\in I} F_i \neq \emptyset$.*

Theorem 2.5. *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Let $F : K \to \mathcal{P}(X)$ be a multivalued mapping such that:*

(a) *for each $x \in K$, $F(x)$ is a nonempty convex subset of K;*
(b) *for each $y \in K$, the set $F^{-1}(y) = \{x \in K | y \in F(x)\}$ contains an open subset O_y of K (that may be empty);*
(c) $\bigcup_{y\in K} O_y = K$.

Then, there exists a point $x_0 \in K$ such that $x_0 \in F(x_0)$.

3. Results for minty variational inequality

Let C be a nonempty subset of a Hausdorff topological vector space Y and let $F : C \to \mathcal{P}(X)$. Let $M_F : C \to \mathcal{P}(C)$ be the multivalued mapping given in [12] and defined by

$$M_F(y) = \{x \in C \mid \langle v, y-x\rangle \geq 0 \ \forall v \in F(y)\}. \tag{3}$$

Let us observe that x is a solution of problem $M(F;C)$ if and only if

$$x \in \bigcap_{y\in C} M_F(y).$$

The following result will be useful.

Lemma 3.1. *Let C be a nonempty convex subset of a Hausdorff topological vector space Y and let $F : C \to \mathcal{P}(X)$. Then, the multivalued mapping $M_F(y)$ has closed values related to C:*

$$\mathrm{cl}_C M_F(y) = M_F(y) \ \forall y \in C, \tag{4}$$

where $\mathrm{cl}_C M_F$ denotes the closure of M_F in relation to the set C.

Proof. It is sufficient to show that $\mathrm{cl}_C M_F(y) \subseteq M_F(y)$ for each $y \in C$ since the other inclusion is always valid. Given $y \in C$ consider $x \in C$ such that $x \notin M_F(y)$. Therefore there is $v_y \in F(y)$ verifing $\langle v_y, y-x\rangle < 0$. Then, by the continuity of the function $\langle v_y, y-\cdot\rangle$ there is a neighborhood $U(x)$ of x such that $\langle v_y, y-u\rangle < 0$ for all $u \in U(x)$. Thus, this relation holds for all $u \in U(x)\cap C$. Hence, $x \notin \mathrm{cl}_C M_F(y)$. Then, we conclude that $\mathrm{cl}_C M_F(y) \subseteq M_F(y)$ and the proof is complete. □

Now we present a generalization of the KKM principle (Fan [9]). We follow a similar argument to that in [11, Theorem 2.1.1]. Let us recall that a family of sets $\{S_i : i \in I\}$ has the finite intersection property if any finite intersection of elements of this family is nonempty.

Theorem 3.2. *Let C be a nonempty convex subset of a Hausdorff topological vector space Y and let $F : C \to \mathcal{P}(X)$. Then, the multivalued mapping $M_F : C \to \mathcal{P}(C)$ verifies the generalized KKM property if and only if the family of sets $\{M_F(y) : y \in C\}$ has the finite intersection property.*

Proof. Let M_F be a generalized KKM mapping. Assume to the contrary that there is a finite set $D = \{y_1, \cdots, y_n\} \subseteq C$ such that:

$$\bigcap_{i=1}^{n} M_F(y_i) \;=\; \emptyset \; . \tag{5}$$

So, there is a finite subset $\{x_1, \cdots, x_n\}$ of C such that:

$$\mathrm{co}\{x_{i_1}, \cdots, x_{i_k}\} \subseteq \bigcup_{j=1}^{k} M_F(y_{i_j}) \tag{6}$$

for any subset $\{x_{i_1}, \cdots, x_{i_k}\}$ of $\{x_1, \cdots, x_n\}$. We take $K = \mathrm{co}\{x_1, \cdots, x_n\}$. Therefore, K is a nonempty compact convex subset of C. We define $T : K \to \mathcal{P}(K)$ by

$$T(x) = \mathrm{co}\{x_i | x \notin M_F(y_i)\} \; . \tag{7}$$

Now we show that T verifies the conditions of Theorem 2.5:

(a) From (5) and (7) we deduce that $T(x)$ is a nonempty convex subset of K for every $x \in K$.

(b) We claim that for each $y \in K$ there is an open subset $\mathcal{O}_y$ of K such that $\mathcal{O}_y \subseteq T^{-1}(y)$. Indeed, if $T^{-1}(y) = \emptyset$, we take $\mathcal{O}_y = \emptyset$. Otherwise, let $x \in T^{-1}(y)$. We define

$$S_x = \bigcup_{i \in I_x} M_F(y_i) \text{ where } I_x = \{1 \le i \le n : x \notin M_F(y_i)\} \; , \tag{8}$$

and

$$O_x = K \backslash S_x . \tag{9}$$

By (5) we have that $I_x \neq \emptyset$ and by Lemma 3.1 we obtain that S_x is a nonempty closed subset in C. Therefore, $S_x \cap K$ is a closed subset of K. Indeed, we have $M_F(y_i) = C_i \cap C$ where C_i is the closure of $M_F(y_i)$, $C_i = \mathrm{cl} M_F(y_i)$ for each $i \in I_x$. Then, $S_x \cap K = (\bigcup_{i \in I_x} (C_i \cap C)) \cap K = (\bigcup_{i \in I_x} C_i) \cap C \cap K = (\bigcup_{i \in I_x} C_i) \cap K$. Hence, $O_x = K \backslash S_x = K \backslash (S_x \cap K)$ is open in K.

Now, we affirm that $O_x \subseteq T^{-1}(y)$. In fact, let $z \in O_x$. By (9) we have that $z \notin S_x$ and by (8) we obtain that $z \notin M_F(y_i)$ for every $i \in I_x$. Then, by the definition of T it results that $T(x) \subseteq T(z)$. Moreover, since $y \in T(x)$

we obtain that $z \in T^{-1}(y)$. Hence, we conclude that $O_x \subseteq T^{-1}(y)$. Finally, we take

$$\mathcal{O}_y = \bigcup_{x \in T^{-1}(y)} O_x \tag{10}$$

which is open in K and $\mathcal{O}_y \subseteq T^{-1}(y)$ for every $y \in K$. Thus, the assertion is valid.

(c) We show that $K = \bigcup_{y \in K} \mathcal{O}_y$. Indeed, let $z \in K$. So, item(a) implies that there is $v \in K$ such that $v \in T(x)$. By taking z and v in (7), (8), (9) and (10) instead of x and y respectively, we deduce that $z \in O_z \subseteq \mathcal{O}_v$. Hence, we deduce that $K \subseteq \bigcup_{y \in K} \mathcal{O}_y$. Since the other inclusion is obviously valid the result holds.

Thus, we apply Theorem 2.5 to conclude that there exists $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x})$. Therefore, by (6), (7) and (8) it holds

$$\bar{x} \in T(\bar{x}) = \text{co}\{x_i | i \in I_{\bar{x}}\} \subseteq \bigcup_{i \in I_{\bar{x}}} M_F(y_i). \tag{11}$$

At the same time, by the definition of $I_{\bar{x}}$ it results that

$$\bar{x} \notin \bigcup_{i \in I_{\bar{x}}} M_F(y_i). \tag{12}$$

So, we obtain a contradiction. Hence we deduce that $\{M_F(y) : y \in C\}$ has the finite intersection property.
Conversely we assume that the property above is satisfied. Hence, given a finite subset $D = \{y_1, \cdots, y_n\}$ we get that there is $x \in \bigcap_{i=1}^n M_F(y_i)$. Take $x_i = x$ for $i = 1, \cdots, n$. Therefore, for any subset $\{y_{i_1}, \cdots, y_{i_k}\}$ of D it results:

$$\text{co}\{x_{i_1}, \cdots, x_{i_k}\} = \{x\} \subseteq \bigcup_{j=1}^{k} M_F(y_{i_j}). \tag{13}$$

This implies that M_F is a generalized KKM mapping. The proof is complete. □

Let us observe that in the proof of Theorem 3.2 given in [5] is deduced a similar result under a strongly condition. In fact, they assume that M_F has closed values in Y, that is, $\text{cl}M_F = M_F$ while we have that F has only closed values related to $C \subseteq Y$, $\text{cl}_C\ M_F = M_F$.

Next we present two auxiliary results.

Lemma 3.3. *Let C be a nonempty convex subset of a Hausdorff topological vector space Y and let $F : C \to \mathcal{P}(X)$ be a multivalued mapping. Assume that the following condition is satisfied:*
(A1) *For any finite set $\{y_1, \cdots, y_m\} \subset C$ there is a finite subset $\{x_1, , \cdots, x_m\} \subset$*

C such that, for any subset $\{x_{i_1}, \cdots, x_{i_k}\} \subset \{x_1, \cdots, x_m\}$ and for any point $x \in \mathrm{co}\{x_{i_1}, \cdots, x_{i_k}\}$ it holds

$$\max_{1\le j\le k} \inf_{v\in F(y_{i_j})} \langle v, y_{i_j} - x\rangle \ \geq\ 0. \tag{14}$$

Then, M_F is a generalized KKM mapping.

Proof. Let $\{y_1, \cdots, y_m\} \subset C$. Then, by $(A1)$we have that there is $\{x_1, \cdots, x_m\} \subset C$ such that for any $x \in \mathrm{co}\{x_{i_1}, \cdots, x_{i_k}\}$ relation (14) is verified. Hence, for some $p \in \{i_1, \cdots, i_k\}$ it holds

$$\inf_{v\in F(y_p)} \langle v, y_p - x\rangle = \max_{1\le j\le k} \inf_{v\in F(y_{i_j})} \langle v, y_{i_j} - x\rangle \ \geq\ 0 \tag{15}$$

which implies that $x \in M_F(y_p) \subseteq \bigcup_{j=1}^{k} M_F(y_{i_j})$. Thus, the assertion follows. □

We say that $F : C \to \mathcal{P}(X)$ is a *generalized properly quasimonotone mapping* if it verfies condition $A1$, that is, if $\varphi(x,y) = \inf_{v\in F(y)} \langle v, y-x\rangle$ is 0- generalized quasiconvex in y.

Lemma 3.4. *Let C be a subset of a Hausdorff topological vector space Y and let $F : C \to \mathcal{P}(X)$ be a multivalued mapping. Assume that the following condition is satisfied:*
(A2) *There exist a nonempty compact subset B of C and a finite subset of C, $\{y_1, \cdots, y_l\}$, such that for each $x \in C\backslash B$ it holds:*

$$\min_{1\le j\le l} \ \inf_{v_j\in F(y_j)} \langle v_j, y_j - x\rangle \ < 0\ . \tag{16}$$

Then

$$A \ = \bigcap_{i=1}^{l} M_F(y_i) \tag{17}$$

is a compact subset of B.

Proof. First we claim that $A \subset B$. Indeed, if $A = \emptyset$, the assertion follows. Now, let $A \neq \emptyset$. For purpose of contradiction, we suppose that there is a point $x \in C$ such that

$$x \in \bigcap_{i=1}^{l} M_F(y_i) \text{ and } x \notin B. \tag{18}$$

Therefore $x \in C\backslash B$. By assumption $(A2)$, there is $j \in \{1, \cdots, l\}$ verifying:

$$\inf_{v_j\in F(y_j)} \langle v_j, y_j - x\rangle \ < 0\ . \tag{19}$$

Hence, $x \notin M_F(y_j)$ which is in contradiction with our assumption. Thus, we get that $A \subset B$. Since A is a closed subset related to C and B is closed we obtain that A is closed in B. As a closed subset of a compact set is compact, it follows that A is a compact subset of B, the proof is complete. □

We note that condition (A2) is also used in [10, 12]. For $l = 1$ it is considered for several authors, for example [1, 4, 5, 7], Now, we are able to give the following result.

Theorem 3.5. *Let C be a convex subset of a Hausdorff topological vector space Y and let $F : C \to \mathcal{P}(C)$ be a multivalued mapping verifying conditions* (A1) *and* (A2)*. Then, there exists a solution of problem $M(F;C)$.*

Proof. By condition (A1) we can use Theorem 3.2 to deduce that the family of sets $\{M_F(y) : y \in C\}$ has the finite intersection property. Therefore, the set A defined in (17) is nonempty. Furthermore, the family of sets $\{M_F(y) \cap A : y \in C\}$ verifies the same property. In addition, for each $y \in C$ we have that $M_F(y) \cap A = \mathrm{cl}M_F(y) \cap C \cap A = \mathrm{cl}M_F(y) \cap A$ is closed in A, so it is compact in A. Therefore, by Theorem 2.4 we conclude that

$$\emptyset \neq \bigcap_{y \in C} (M_F(y) \cap A) = \bigcap_{y \in C} M_F(y). \tag{20}$$

The proof is complete. □

In [12] is assumed that F is a properly quasimonotone mapping which implies our condition (A1) by taking $x_i = y_i$. The converse is not valid in general. We illustrate this situation by considering a modification of Example 4.3 given in [8]. Let $X = R^2$, $y_1 = (0,1)$, $y_2 = (0,0)$, $y_3 = (1,0)$. Let $F : X \to X$ be defined by $F(y_1) = (1,0)$, $F(y_2) = (1,0)$, $F(y_3) = (0,1)$ and $F(y) = 0$ otherwise. This function F is not properly quasimonotone (take $y = \frac{1}{3}(y_1 + y_2 + y_3)$). It is easy to see that F verifies condition (A1) by taking, for example, $x_1 = (-1,1)$, $x_2 = (0,0)$, $x_3 = (1,-1)$ associated with y_1, y_2 and y_3 respectively and $x = (0,0)$ otherwise. In addition, if we define $C = \{(y_1,y_2) \in R^2 : \ y_2 \geq -y_1, |y_1| \leq 1, |y_2| < 2\}$, we have that condition (A2) is satisfied by taking $B = \{(y_1,y_2) \in R^2 : \ 0 \leq y_1 \leq 1, 0 \leq y_2 \leq \frac{1}{2}\}$. Hence, by Theorem 3.5 we conclude that there exists a solution to this problem. Actually, the solution is $x = (0,0)$.

Related results can be found in [9, 5, 1, 17].

4. Minty variational inequality systems

Let X_i and Y_i be real Hausdorff topological vector spaces, for each $i = 1, \cdots, n$. Let C_i be a nonempty subset of Y_i and let $\langle \cdot, \cdot \rangle : X_i \times Y_i \to R$ be a continuous bilinear function for each $i = 1, \cdots, n$. Let $C = C_1 \times \cdots \times C_n$ and $F_i : C \to \mathcal{P}(X_i)$ be a set-valued mapping with nonempty values, for each $i = 1, \cdots, n$. We consider the Minty variational problem $M(F_1, \cdots, F_n; C_1, \cdots, C_n)$ given by:

$$\begin{cases} \text{Find } \ (x_1, \cdots, x_n) \in C \ \text{ such that for each } i = 1, \cdots, n \\ \langle v, y - x_i \rangle \geq 0, \quad \forall \ y \in C_i \ \forall \ v \in F_i(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n). \end{cases} \tag{21}$$

As a consequence of Theorem 3.5 we establish the following existence result for this problem which extends a theorem given in [12].

Theorem 4.1. *Let us consider problem $M(F_1, \cdots, F_n; C_1, \cdots, C_n)$. Assume that there is a compact subset $B_i \subseteq C_i$, for each $i \in I = \{1, \cdots n\}$, such that the following conditions hold:*

(a) *Fixed $x_j \in B_j$ for each $j \in I \backslash \{i\}$ the multivalued mapping*

$$x \mapsto F_i(x_1, \cdots, x_{i-1}, x, x_{i+1}, \cdots, x_n)$$

verifies condition $(A1)$, for each $j \in I \backslash \{i\}$, $i = 1, \cdots, n$.

(b) *There exists $\{y_1, \cdots, y_l\}$such that for each $x \in C_i \backslash B_i$ we have*

$$\min_{1 \leq j \leq l} \inf_{v_j \in F_i(x_1, \cdots, x_{i-1}, y_j, x_{i+1}, \cdots, x_n)} \langle v_j, y_j - x \rangle < 0.$$

(c) *Fixed $y_i \in C_i$, the set-valued function*

$$(x_1, \cdots x_{i-1}, x_{i+1}, \cdots) \mapsto F_i(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)$$

is lower semicontinuous on $C_1 \times \cdots C_{i-1} \times C_{i+1} \cdots \times C_n$, for $i = 1, \cdots, n$.

Then, there exists a solution of problem $M(F_1, \cdots, F_n; C_1, \cdots, C_n)$.

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