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INTERSECTION NUMBERS ON THE COMPACT VARIETY OF RATIONAL RULED SURFACES

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Abstract. We describe a natural action on the Quot scheme, R_d compactifying the space of degree d maps from $\mathbb{P}^1$ to the Grassmannian of lines. We identify the fixed points components for this action and the weights of the normal bundle of these components. We compute the degree of this Quot scheme under the generalized Plücker embedding by applying Atiyah-Bott localization formula.

1. Introduction

The Quot scheme is a fine moduli space equipped with an universal element. It has been used many times as a smooth compactification of the space of morphisms of a fixed degree from a curve C to a Grassmannian [2]. It is known that when C is of genus 0, the Quot scheme is irreducible and smooth. Recalling the notation of [4], we denote by R_d the Quot scheme parametrizing quotients of rank 2 and degree d of a trivial vector bundle $\mathcal{O}^4_{\mathbb{P}^1}$, and by R^0_d the open set of morphisms. S. A. Strømme in [6] computes the Betti numbers of R_d and gives a description with generators and relations of its cohomology ring. In particular, he gives a basis for the Picard group $A^1(R_d)$.

Here we compute the degree of the generalized Plücker embedding of the Quot scheme R_d. It is called generalized Plücker embedding because in some sense can be considered a generalization of the Plücker embedding given by the hyperplane class of the corresponding Grassmannian. In our case, we are considering the Grassmannian of lines in $\mathbb{P}^3$, which we will denote as $G(2,4)$, but similar computations can be obtained for a general Grassmannian $G(k,n)$ of $k-$planes in $\mathbb{C}^n$. In particular for $n-k=1$, the Quot scheme is a projective space and therefore intersection theory here is well understood. The method that we use here, involves the geometry of the Quot scheme and the Bott residue formula.

The Bott residue formula expresses the degree of certain zero-cycles classes on a smooth complete variety with an action of an algebraic torus in terms of local contributions supported on the components of the fixed point set. We describe a natural action on the Quot scheme. We identify the fixed points components for

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this action and the weights of the normal bundle of these components. We study the equivariant codimension one cohomology group of R_d for this action.

Notation. We work over the field of complex numbers $\mathbb{C}$. Let X be a projective homogeneous variety, then A_dX and A^dX can be taken to be the Chow homology and cohomology groups of X. We identify A^dX with $A_{n-d}X$ by the Poincaré duality isomorphism. We use cup product $\cup$ for the product in A^*X. The moduli space $\overline{M}_{0,n}(X,\beta)$ parametrizes marked stable maps from genus 0 curves to X in the cohomology class $\beta \in H^2(X,\mathbb{Z})$. Let γ_i be cycles on X, if $\sum \mathrm{codim}(\gamma_i) = \dim(\overline{M}_{0,n}(X,d))$, *the Gromov-Witten invariant* $I_{0,n,d}(\gamma_1,\ldots,\gamma_n)$ is defined as the top degree class:

$$I_{0,n,d}(\gamma_1,\ldots,\gamma_n) = \int_{[\overline{M}_{0,n}(X,d)]^{vir}} \pi_1^*(\gamma_1)\cup\ldots\cup\pi_n^*(\gamma_n), \tag{1}$$

where $[\overline{M}_{0,n}(X,d)]^{vir}$ denotes the virtual fundamental class of $\overline{M}_{0,n}(X,d)$.

2. The degree of the generalized Plücker embedding and the Vafa-intriligator formula.

When we fix the degree, d, and the rank, 2, of a locally free sheaf E on $\mathbb{P}^1$, we are fixing its Hilbert polynomial,

$$P(t) = \chi(E(t)) = d + 2t + 2. \tag{2}$$

The moduli, $\mathrm{Quot}_d(\mathbb{P}^1, P(t))$, of quotients with fixed Hilbert polynomial $P(t)$, is a fine moduli space which we will denote as R_d. We observe the quotient $\mathcal{O}^4_{\mathbb{P}^1} \to E \to 0$, determines a point $q \in \mathrm{Quot}(\mathbb{P}^1, P(t))$ and a morphism $f_q : \mathbb{P}^1 \to G(2,4)$, by the universal property of the Grassmannian. By definition, there is an universal quotient

$$\mathcal{O}^4_{\mathrm{Quot}\times\mathbb{P}^1} \to \mathcal{E}_{\mathrm{Quot}\times\mathbb{P}^1}. \tag{3}$$

S. A. Strømme in [6] gives a basis for the Picard group $A^1(R_d)$, formed by the divisors:

$$\alpha = c_1(\pi_{1*}(\mathcal{E}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d)) - c_1(\pi_{1*}(\mathcal{E}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1))),$$
$$\beta = c_1(\pi_{1*}(\mathcal{E}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1)),$$

where $\mathcal{E}$ is the universal quotient over $R_d \times \mathbb{P}^1$, and π_1, π_2 are the projection maps over the first and second factors respectively. Let $e : R_d^0 \times \mathbb{P}^1 \to G(2,4)$ be the evaluation map and $T_1 \in H_6(G(2,4),\mathbb{Z})$, $T_a \in H_4(G(2,4),\mathbb{Z})$ be the classes of an hyperplane and $a-$plane respectively. The hyperplane class determines the Plücker embedding of the Grassmannian $G(2,4)$ as a quadric in $\mathbb{P}^5$ which corresponds to a variety of lines in $\mathbb{P}^3$. The $a-$plane T_a corresponds to lines in $\mathbb{P}^3$ contained in a given plane. The following set of morphisms define Weil divisors on R_d^0:

$$D_i := \{\varphi \in R_d^0 |\ \ e(t,\varphi)\cap T_{a_i} \neq \emptyset\},$$

$$Y := \{\varphi \in R_d^0 |\ e(t_i,\varphi) \in T_1 \text{ for a fixed } \ t_i \in \mathbb{P}^1\}.$$

These divisors extend to divisors on R_d.

Let P_d be the degree of R_d by the morphism induced by the polarization given by the divisor α, by Lemma 3.2 of [5], P_d is the degree of the top codimensional cohomology class given by the autointersection

$$P_d = \int_{[R_d]} [Y]^{4d+4} \cap [R_d] = \int_{[R_d]} \alpha^{4d+4} \cap [R_d]. \tag{4}$$

This intersection number is computed in [7] via Quantum Cohomology. Note that in this case, the intersection is transversal in the Quot scheme compactification, and therefore the intersection number is the same than the given by integrating over the space of stable maps, that is the Gromov-Witten invariant $I_{0,4d+4,d}(T_1, \overset{4d+4}{\dots}, T_1)$. It can be obtained too by means of the formulas of Vafa and Intriligator, [2]. This does not happen when considering intersection numbers containing D_i.

Vafa and Intriligator's Formula. Let ζ be a primitive nth root of $(-1)^k$ and assume that $0 \leqq a_i \leqq k$ and $a_1 + \ldots + a_N = \dim(\overline{M}_{0,4d+4}(G(k,n),d))$. Then

$$I_{0,4d+4,d}(\gamma_{a_1}, \overset{4d+4}{\dots}, \gamma_{a_N}) = (-1)^{\binom{k}{2}} n^{-k} \sum_{i_1 > \ldots > i_k} \sigma_{a_1}(\zeta^I) \ldots \sigma_{a_N}(\zeta^I) \frac{\prod_{j \neq l}(\zeta^{i_j} - \zeta^{i_l})}{\prod_{j=1}^{k} \zeta^{(n-1)i_j}},$$

where $\zeta^I = (\zeta^{i_1, \ldots, \zeta^{i_k}})$ and σ_{a_i} are the elementary symmetric polynomials in k variables.

In [4] it is proved that the top-codimensional classes involving the divisors D_i have not enumerative meaning on the Quot scheme because there is an excess component of intersection contained in the boundary. Then the intersection is carried out in the Kontsevich compactification of stable maps, $\overline{M}_{0,3}(G(2,4),d)$.

The tool we are going to use for computing these degrees, is the Bott residue formula. For this purpose, we consider the equivariant action of a one dimensional torus $T = \mathbb{C}^*$ on the variety R_d. First we study the varieties of fixed points under this action and we compute the equivariant Chern classes in the Chow equivariant rings of the normal bundles restricted to the varieties of fixed points and the line bundles corresponding to the divisors we intersect.

Bott's residues formula, (see [1]).
Let X be a smooth, complete variety of dimension n and let E be a $T-$equivariant vector bundle over X. Then we have

$$\int_X (p(E) \cap [X]) = \sum_{F \subset R^T} \pi_{F*} \left(\frac{p^T(E_{|F}) \cap [F]_T}{c_{d_F}(\mathcal{N}_{F/X})} \right). \tag{5}$$

Here d_F denotes the codimension of the component F of fixed points in X and $p^T(E)$ is the polynomial of degree n in the line bundles corresponding to the cycles expressing the product in the equivariant Chow ring of F. The numerator will be a product of polynomials of degree 1 corresponding to the line bundles

restricted to F. The denominator will be a polynomial with the dimension of the normal bundle as degree.
In our case, we will apply the formula to compute the intersection

$$\int_{[R_d]} \alpha^{4d+4} \cap [R_d].$$

3. Varieties of fixed points under the $\mathbb{C}^*-$ action

We consider the diagonal action of the one-dimensional torus acting on the variety R_d.
A point in R_d is given by a quotient: $0 \to N \to \mathcal{O}^4 \to E \to 0$ in $\mathbb{P}^1$, where $\chi(E(t)) = 2t+2+d$. Let $\overline{w} = (w_0, w_1, w_2, w_3)$ be a quadruple of integral weights with $w_0 < w_1 < w_2 < w_3$. $\mathbb{C}^*$ acts on each point:

$$\forall t \in \mathbb{C}^*, \quad \begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}^4 & \to & \mathbb{C}^4, \\ t \cdot (x_0, x_1, x_2, x_3) = & & (t^{w_0}x_0, t^{w_1}x_1, t^{w_2}x_2, t^{w_3}x_3). \end{array}$$

Let $0 \to \mathcal{N} \to \mathcal{O}^4_{R_d \times \mathbb{P}^1} \to \mathcal{E} \to 0$ be the universal exact sequence in $R_d \times \mathbb{P}^1$. This action induces an isomorphism $\mathcal{O}^4 \to \mathcal{O}^4$, such that the weight corresponding to the trivial sheaf $\mathcal{O}_{R_d}$ is $w_0 + w_1 + w_2 + w_3$, since $\pi_* \wedge^4 \mathcal{O}^4_{R_d} \cong \mathcal{O}_{R_d}$, where $\pi : R_d \times \mathbb{P}^1 \to R_d$.

Let E be of rank 2 with a nonzero torsion of degree d. We suppose,

$$E \cong \mathcal{O}_{Z_d} \oplus 0 \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}. \tag{6}$$

The scheme $Hilb^d_{\mathbb{P}^1}$ will denote the Hilbert scheme of d points in $\mathbb{P}^1$ which is isomorphic to $\mathbb{P}^d$ and $\mathcal{Z}_d \subset \mathbb{P}^d \times \mathbb{P}^1$ will denote the incidence variety. The scheme $Hilb^d_{\mathbb{P}^1}$ parametrizes points $\mathcal{O}^4_{\mathbb{P}^1} \to E \to 0$, with E as in (6).

Definition 3.1. *Let $\underline{P} = (P_i)_{1 \leqslant i \leqslant r}$ be a family of polynomials with rational coefficients. $\underline{P}$ is said to be a good partition of the polynomial $\underline{P}$, if $Hilb^{P_i}_{\mathbb{P}^1} \neq \emptyset$ and $\underline{P} = \sum_{1 \leqslant i \leqslant r} P_i$.*

Examples of good partitions of the Hilbert polynomial of E..

(1) $P_1(t) = d,\ P_2(t) = 0,\ P_3(t) = t+1,\ P_4(t) = t+1$, is said to be a *good partition* of the polynomial $\underline{P}(t) = 2t+2+d$.
(2) Another good partition for d odd would be:

$$P_{\frac{d-1}{2}, \frac{d+1}{2}}(t) = \underbrace{\frac{d+1}{2}} + \underbrace{\frac{d-1}{2}} + \underbrace{t+1} + \underbrace{t+1} \tag{7}$$

and for d even:

$$P_{\frac{d}{2}, \frac{d}{2}}(t) = \underbrace{t+1} + \underbrace{t+1} + \underbrace{\frac{d}{2}} + \underbrace{\frac{d}{2}} \tag{8}$$

$\mathcal{O}^4_{\mathbb{P}^1} \to \mathcal{O}_{Z_{d/2}} \oplus \mathcal{O}_{Z_{d/2}} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to 0$, is a boundary point corresponding to the partition (8).

$$Hilb^{t+1}_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \times Hilb^{d/2}_{\mathbb{P}^1} \times Hilb^{d/2}_{\mathbb{P}^1}$$

is the subscheme of the scheme $Quot(\mathcal{O}^4_{\mathbb{P}^1}, P(t))$ that we will denote by $(t+1, t+1, \frac{d}{2}, \frac{d}{2})$ in order to simplify notation. The fixed point set of R_d under the action of the torus T is the union of the possible subschemes $Hilb^{\underline{P}}_{\mathbb{P}^1}$ [3].

3.1. **Fixed points in R_d under the $\mathbb{C}^*$-action.**

Proposition 3.2. *The varieties of fixed points in R_d under the $\mathbb{C}^*-$action are parametrized by,*

$$\begin{gathered}\mathbb{P}^d,\\ \mathbb{P}^{d-1}\times\mathbb{P}^1,\\ \mathbb{P}^{d-2}\times\mathbb{P}^2,\\ \vdots\\ \begin{cases}\mathbb{P}^{\frac{d+1}{2}}\times\mathbb{P}^{\frac{d-1}{2}} & \textit{if } d \textit{ odd},\\ \mathbb{P}^{\frac{d}{2}}\times\mathbb{P}^{\frac{d}{2}} & \textit{if } d \textit{ even}.\end{cases}\end{gathered}$$

There are 12 components of each type.

Proof. Following the work of Bifet [3], we see that to study the components of fixed points under the $\mathbb{C}^*-$action, is equivalent to study good partitions for the Hilbert polynomial $\underline{P}(t) = 2t + 2 + d$.

(1) Corresponding to the partition,

$$P_{d,0}(t) = d + 0 + \underbrace{t+1} + \underbrace{t+1},$$

we have,

$$Hilb^d_{\mathbb{P}^1} \times Hilb^0_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \cong \mathbb{P}^d.$$

There are 12 of this kind.

(2) Corresponding to the partition,

$$P_{b,a}(t) = b + a + \underbrace{t+1} + \underbrace{t+1}, \quad b \geqslant a > 0,\ b + a = d,$$

$$Hilb^b_{\mathbb{P}^1} \times Hilb^a_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \cong \mathbb{P}^b \times \mathbb{P}^a$$

There are $\frac{d}{2} - 1$ different components if d even, and $\frac{d+1}{2} - 1$ if d odd. These are parametrized by:

$$\begin{gathered}\mathbb{P}^{d-1}\times\mathbb{P}^1,\\ \mathbb{P}^{d-2}\times\mathbb{P}^2,\\ \vdots\\ \begin{cases}\mathbb{P}^{\frac{d+1}{2}}\times\mathbb{P}^{\frac{d-1}{2}} & \text{if } d \text{ odd},\\ \mathbb{P}^{\frac{d}{2}}\times\mathbb{P}^{\frac{d}{2}} & \text{if } d \text{ even}.\end{cases}\end{gathered}$$

For each one there are also $\binom{4}{2}$ components.

There are $12 \cdot \frac{d}{2} = 6d$ fixed point components if d even, and $12 \cdot \frac{d+1}{2} = 6 \cdot (d+1)$ if d odd. □

The Euler characteristic of R_d is given by the formula (see Corollary 1.4 of [6]),

$$\chi(R_d) = \mathrm{rk}_{\mathbb{Z}}\ A\,(R_d) = \binom{4}{2}\binom{d+4-1}{d}.$$

Since the Chow ring of $\mathbb{P}^d$ is $\mathbb{Z}[h]/\langle h^d\rangle$ the contribution to the Euler characteristic of the component of fixed points of the first kind is $d+1$ cycles. The contribution of the components of the second kind to the Euler characteristic is $a \cdot b$, since $A^*(\mathbb{P}^b \times \mathbb{P}^a) = \mathbb{Z}(H,h)/\langle H^b, h^a\rangle$.

Example 1. (Fixed points in R_3 under the $\mathbb{C}^*$-action) The Quot scheme R_3 parametrizes quotients with Hilbert polynomial,

$$P_3(t) = 2t + 2 + 3.$$

There are two kinds of fixed points varieties in R_3 corresponding to the two partitions of the polynomial $P_3(t)$.

(1)

$$P_{3,0}(t) = 3 + 0 + \underbrace{t+1} + \underbrace{t+1},$$

$$Hilb^3_{\mathbb{P}^1} \times Hilb^0_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \cong \mathbb{P}^3.$$

There are $\binom{4}{2} \times 2 = 12$ of this kind, each one contributes 4 cycles, since the Chow ring of $\mathbb{P}^3$ is $A^*(\mathbb{P}^3) \cong \mathbb{Z}(h)/\langle h^4\rangle$, here h is the class of a hyperplane in $\mathbb{P}^3$.

(2)

$$P_{2,1}(t) = 2 + 1 + \underbrace{t+1} + \underbrace{t+1}$$

$$Hilb^2_{\mathbb{P}^1} \times Hilb^1_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \cong \mathbb{P}^2 \times \mathbb{P}^1.$$

There are $\binom{4}{2} \times 2 = 12$ of this kind, and each one contributes 6 cycles in $A^*(\mathbb{P}^2 \times \mathbb{P}^1) \cong \mathbb{Z}(H,h)/\langle H^3, h^2\rangle$, where H is the class of a hyperplane in $\mathbb{P}^2$ and h in $\mathbb{P}^1$.

This number coincides with the Euler characteristic, since

$$120 = \chi\,(R_3) = \binom{4}{2}\binom{3+2(4-2)-1}{3}.$$

There are 24 components of fixed points.

We have seen that the position of the trivial sheaves and the torsion sheaves is important. It determines different components up to isomorphism. It also determines the weights that act.

3.2. **The $\mathbb{C}^*$-equivariant Chern classes.** We shall associate to the component $Hilb^d_{\mathbb{P}^1} \times Hilb^0_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \times Hilb^{t+1}_{\mathbb{P}^1} \cong \mathbb{P}^d$ the quadruple $(d, 0, t+1, t+1)$ to simplify notation, and $c_1^T(\alpha|_{(d,0,t+1,t+1)})$, $c_1^T(\beta|_{(d,0,t+1,t+1)})$ will be the corresponding first $\mathbb{C}^*$-equivariant Chern classes in the equivariant Chow ring $A^T_*(\mathbb{P}^d)$.

Theorem 3.3. *The first equivariant Chern classes of the fixed points components are*

$$
\begin{aligned}
&c_1^T(\alpha|_{(t+1,t+1,d,0)}) = h + w_0 + w_1, \quad c_1^T(\beta|_{(t+1,t+1,d,0)}) = d\,w_0 + d\,w_1 + d\,w_2,\\
&c_1^T(\alpha|_{(t+1,d,t+1,0)}) = h + w_0 + w_2, \quad c_1^T(\beta|_{(t+1,d,t+1,0)}) = d\,w_0 + d\,w_1 + d\,w_2,\\
&c_1^T(\alpha|_{(d,0,t+1,t+1)}) = h + w_2 + w_3, \quad c_1^T(\beta|_{(d,0,t+1,t+1)}) = d\,w_0 + d\,w_2 + d\,w_3,\\
&c_1^T(\alpha|_{(t+1,t+1,0,d)}) = h + w_0 + w_1, \quad c_1^T(\beta|_{(t+1,t+1,0,d)}) = d\,w_0 + d\,w_1 + d\,w_3,\\
&c_1^T(\alpha|_{(d,t+1,0,t+1)}) = h + w_1 + w_3, \quad c_1^T(\beta|_{(d,t+1,0,t+1)}) = d\,w_0 + d\,w_1 + d\,w_3,\\
&c_1^T(\alpha|_{(d,t+1,t+1,0)}) = h + w_1 + w_2, \quad c_1^T(\beta|_{(d,t+1,t+1,0)}) = d\,w_0 + d\,d\,w_1 + d\,w_2,\\
&c_1^T(\alpha|_{(t+1,t+1,0,d)}) = h + w_0 + w_1, \quad c_1^T(\beta|_{(t+1,t+1,0,d)}) = d\,w_0 + d\,w_1 + d\,w_3,\\
&c_1^T(\alpha|_{(t+1,0,t+1,d)}) = h + w_0 + w_2, \quad c_1^T(\beta|_{(t+1,0,t+1,d)}) = d\,w_0 + d\,w_2 + d\,w_3,\\
&c_1^T(\alpha|_{(0,d,t+1,t+1)}) = h + w_2 + w_3, \quad c_1^T(\beta|_{(0,d,t+1,t+1)}) = d\,w_1 + d\,w_2 + d\,w_3,\\
&c_1^T(\alpha|_{(t+1,0,d,t+1)}) = h + w_0 + w_3, \quad c_1^T(\beta|_{(t+1,0,d,t+1)}) = d\,w_0 + d\,w_2 + d\,w_3,\\
&c_1^T(\alpha|_{(0,t+1,d,t+1)}) = h + w_1 + w_3, \quad c_1^T(\beta|_{(0,t+1,d,t+1)}) = d\,w_1 + d\,w_2 + d\,w_3,\\
&c_1^T(\alpha|_{(0,t+1,t+1,d)}) = h + w_1 + w_2, \quad c_1^T(\beta|_{(0,t+1,t+1,d)}) = d\,w_1 + d\,w_2 + d\,w_3,\\
&c_1^T(\alpha|_{(t+1,t+1,a,b)}) = H + h + w_0 + w_1,\\
&c_1^T(\beta|_{(t+1,t+1,a,b)}) = aH + bh + dw_0 + dw_1 + aw_2 + bw_3,\\
&c_1^T(\alpha|_{(t+1,a,t+1,b)}) = H + h + w_0 + w_2,\\
&c_1^T(\beta|_{(t+1,a,t+1,b)}) = aH + bh + dw_0 + dw_1 + aw_1 + bw_3,\\
&c_1^T(\alpha|_{(a,b,t+1,t+1)}) = H + h + w_2 + w_3,\\
&c_1^T(\beta|_{(t+1,a,t+1,b)}) = aH + bh + dw_0 + aw_1 + dw_1 + bw_3,\\
&c_1^T(\alpha|_{(t+1,a,b,t+1)}) = H + h + w_0 + w_3,\\
&c_1^T(\beta|_{(t+1,a,b,t+1)}) = aH + bh + dw_0 + aw_1 + bw_1 + dw_3,\\
&c_1^T(\alpha|_{(a,t+1,b,t+1)}) = H + h + w_0 + w_3,\\
&c_1^T(\beta|_{(a,t+1,b,t+1)}) = aH + bh + aw_0 + dw_1 + bw_2 + dw_3,\\
&c_1^T(\alpha|_{(a,t+1,t+1,b)}) = H + h + w_1 + w_2,\\
&c_1^T(\beta|_{(a,t+1,t+1,b)}) = aH + bh + aw_0 + dw_1 + dw_2 + bw_3.\\
&c_1^T(\alpha|_{(t+1,t+1,b,a)}) = H + h + w_0 + w_1,\\
&c_1^T(\beta|_{(t+1,t+1,b,a)}) = aH + bh + dw_0 + dw_1 + bw_2 + aw_3.\\
&c_1^T(\alpha|_{(t+1,a,t+1,b)}) = H + h + w_0 + w_1,\\
&c_1^T(\beta|_{(t+1,a,t+1,b)}) = aH + bh + dw_0 + aw_1 + dw_2 + bw_3.\\
&c_1^T(\alpha|_{(a,b,t+1,t+1)}) = H + h + w_1 + w_2,\\
&c_1^T(\beta|_{(a,b,t+1,t+1)}) = aH + bh + aw_0 + bw_1 + dw_2 + dw_3.\\
&c_1^T(\alpha|_{(t+1,a,b,t+1)}) = H + h + w_0 + w_3,\\
&c_1^T(\beta|_{(t+1,a,b,t+1)}) = aH + bh + dw_0 + aw_1 + bw_2 + dw_3.\\
&c_1^T(\alpha|_{(a,t+1,b,t+1)}) = H + h + w_1 + w_3,\\
&c_1^T(\beta|_{(a,t+1,b,t+1)}) = aH + bh + aw_0 + dw_1 + bw_2 + dw_3.\\
&c_1^T(\alpha|_{(a,t+1,t+1,b)}) = H + h + w_1 + w_2,\\
&c_1^T(\beta|_{(a,t+1,t+1,b)}) = aH + bh + aw_0 + dw_1 + dw_2 + bw_3.
\end{aligned}
$$

Proof. We consider the universal quotient in $R_d \times \mathbb{P}^1$ restricted to the fixed point component $(d, 0, t+1, t+1)$. It is enough to consider one component of fixed points by the symmetry of the computations.

$$0 \to \mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,-d)\oplus\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}\to\mathcal{O}^4_{\mathbb{P}^d\times\mathbb{P}^1}\to\mathcal{O}_{\mathcal{Z}_d}\oplus 0\oplus\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}\to 0.$$

$$\begin{tikzcd}
\mathbb{P}^3\times\mathbb{P}^1 \supset \mathcal{Z}_d \arrow[dr] \arrow[d, "\pi_1"] & \\
\mathbb{P}^d &
\end{tikzcd}$$

Let h denote the positive generator of the projective space $\mathbb{P}^d$.
Since $\mathcal{Z}_d$ is π_1-flat, the restriction of the coherent sheaf B_{d-1} to the fixed point component $(d,0,t+1,t+1)$ is well defined by,

$$\begin{aligned}B_{d-1}|_{(d,0,t+1,t+1)} &= \pi_{1*}[(\mathcal{O}_{\mathcal{Z}_d}\oplus 0\oplus\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1})\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1)]\\ &= \pi_1^*\mathcal{O}_{\mathcal{Z}_d}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1)\oplus\pi_{1*}\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1)\oplus\pi_{1*}\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1),\end{aligned}$$

where $\pi_{1*}\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1)=\mathcal{O}^d_{\mathbb{P}^d}$.
We consider the exact sequence defining the sheaf $\mathcal{O}_{\mathcal{Z}_d}$:

$$0\to\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,-d)\to\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}\to\mathcal{O}_{\mathcal{Z}_d}\to 0, \tag{9}$$

and we tensorize with the line bundle $\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1)$ and take the long exact sequence associated to the pushforward π_{1*}:

$$\begin{aligned}0\to\pi_{1*}(\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,-d)\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1))\to\pi_{1*}(\mathcal{O}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1))\to\\ \to\pi_{1*}(\mathcal{O}_{\mathcal{Z}_d}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1))\to 0,\end{aligned} \tag{10}$$

The vanishing of $R^1\pi_{1*}(\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,-1))$ implies that $\pi_{1*}(\mathcal{O}_{\mathcal{Z}_d}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1))=\mathcal{O}^d_{\mathbb{P}^d}$. Therefore the rank of $B_{d-1}|_{(d,0,t+1,t+1)}$ is $2d+d=3d$ and the restriction of β_d to the component $(d,0,t+1,t+1)$ is $\mathcal{O}_{\mathbb{P}^d}$. Since $\bigwedge^d\mathcal{O}^d_{\mathbb{P}^d}=\mathcal{O}_{\mathbb{P}^d}$ the corresponding weight is $dw_0+dw_2+dw_3$.
The first equivariant Chern class $\alpha_d|_{(d,0,t+1,t+1)}$ is defined by

$$\alpha_d|_{(d,0,t+1,t+1)}=c_1(B_d|_{(d,0,t+1,t+1)})-c_1(B_{d-1}|_{(d,0,t+1,t+1)}),$$

with

$$\begin{aligned}B_d|_{(d,0,t+1,t+1)} &= \pi_{1*}[(\mathcal{O}_{\mathcal{Z}_d}\otimes 0\oplus\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1})\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d)]\\ &= \pi_{1*}\mathcal{O}_{\mathcal{Z}_d}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d)\oplus\pi_{1*}\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d)\oplus\pi_{1*}\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d).\end{aligned}$$

We have the following exact sequence

$$0\to\pi_{1*}(\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,0))\to\pi_{1*}(\mathcal{O}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d))\to\pi_{1*}(\mathcal{O}_{\mathcal{Z}_d}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d))\to 0. \tag{11}$$

We have that $\pi_{1*}\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d)=\mathcal{O}^{d+1}_{\mathbb{P}^d}$ and $R^1\pi_{1*}(\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,0))=0$. We also have that $\pi_{1*}(\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,0))=\mathcal{O}_{\mathbb{P}^d}(-1)$, therefore,

$$\pi_{1*}(\mathcal{O}_{\mathcal{Z}_d}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(d))=\mathcal{O}_{\mathbb{P}^d}(1)\oplus\mathcal{O}^{d-1}_{\mathbb{P}^d}.$$

The rank of $B_d|_{(d,0,t+1,t+1)}$ is $2(d+1)+d = 3d+2$ and the weight $dw_0 + (d+1)w_2 + (d+1)w_3$, therefore the restriction of α_d to $(d,0,t+1,t+1)$ is

$$\boxed{h + w_2 + w_3}$$

We now compute the restrictions of the divisors α_d and β_d to the components of fixed points of the second kind. Consider the following incidence variety

$$\begin{tikzcd}
 & \mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1 \supset \mathcal{Z}_b \times \mathcal{Z}_a \arrow[dl, "\pi_3"'] \arrow[d, "\pi_{12}"] & \\
\mathbb{P}^1 & \mathbb{P}^b \times \mathbb{P}^a \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathbb{P}^b & & \mathbb{P}^a
\end{tikzcd}$$

and the restriction of the coherent sheaf B_{d-1} to the fixed point component $(b,a,t+1,t+1)$,

$$B_{d-1}|_{(b,a,t+1,t+1)} = \pi_{12*}[(\mathcal{O}_{\mathcal{Z}_b} \oplus \mathcal{O}_{\mathcal{Z}_a} \oplus \mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}) \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(d-1)]$$

where

$$b + a = d,\ b \geqslant a > 0,$$

and

$$\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,-1,-a) \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(d-1)) = \pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,-1,b-1).$$

Since $R^1\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,-1,b-1) = 0$, the following exact sequence stands:

$$0 \to \mathcal{O}(0,-1) \otimes \mathcal{O}^b \to \mathcal{O}^d \to \pi_{12*}(\mathcal{O}_{\mathcal{Z}_a} \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(d-1)) \to 0 \text{ over } \mathbb{P}^b \times \mathbb{P}^a.$$

Let h, H denote the positive generators of the projective spaces $\mathbb{P}^a$ and $\mathbb{P}^b$, respectively. The first Chern class of the bundle $\pi_{12*}(\mathcal{O}_{\mathcal{Z}_a} \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(d-1))$ is bh. For computing the first Chern class of the subbundle $\pi_{12*}(\mathcal{O}_{\mathcal{Z}_b} \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(d-1))$, we see that

$$\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,-1,-b) \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(d-1)) = \pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,0,a-1),$$

and

$$R^1\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,0,a-1) = 0,$$

therefore

$$0 \to \mathcal{O}(-1,0) \otimes \mathcal{O}^a \to \mathcal{O}^d \to \pi_{12*}(\mathcal{O}_{Z_b} \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(d-1)) \to 0 \text{ on } \mathbb{P}^b \times \mathbb{P}^a.$$

By symmetry with the previous case, the first Chern class of the bundle $\pi_{12*}(\mathcal{O}_{\mathcal{Z}_b} \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(d-1))$ is aH. It follows that $c_1(B_{d-1}|_{(b,a,t+1,t+1)}) = aH + bh$, and its weight is $bw_0 + aw_1 + dw_2 + dw_3$. Finally, the restriction of β_d to the component $(b,a,t+1,t+1)$, isomorphic to $\mathbb{P}^b \times \mathbb{P}^a$, is

$$\boxed{aH + bh + bw_0 + aw_1 + dw_2 + dw_3}$$

The restriction of B_d to $(b,a,t+1,t+1)$ is given by

$$B_d|_{(b,a,t+1,t+1)} = \pi_{12*}[(\mathcal{O}_{\mathcal{Z}_b} \oplus \mathcal{O}_{\mathcal{Z}_a} \oplus \mathcal{O} \oplus \mathcal{O}) \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(d)].$$

We see that

$$\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,-1,-a)\otimes\pi_3^*\mathcal{O}_{\mathbb{P}^1}(d))=\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,-1,b)$$

and

$$R^1\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,-1,b)=0.$$

Therefore we have the exact sequence:

$$0\to\mathcal{O}(0,-1)\otimes\mathcal{O}^{b+1}\to\mathcal{O}^{d+1}\to\pi_{12*}(\mathcal{O}_{\mathcal{Z}_a}\otimes\pi_3^*\mathcal{O}_{\mathbb{P}^1}(d))\to 0 \text{ over } \mathbb{P}^b\times\mathbb{P}^a.$$

Again from the fact that

$$\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,0,-b)\otimes\pi_3^*\mathcal{O}_{\mathbb{P}^1}(d))=\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,0,a),$$

and that

$$R^1\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,0,a)=0,$$

it follows that there is an exact sequence,

$$0\to\mathcal{O}(-1,0)\otimes\mathcal{O}^{a+1}\to\mathcal{O}^{d+1}\to\pi_{12*}(\mathcal{O}_{\mathcal{Z}_b}\otimes\pi_3^*\mathcal{O}_{\mathbb{P}^1}(d))\to 0 \text{ over } \mathbb{P}^b\times\mathbb{P}^a.$$

The first Chern class of $B_d|_{(b,a,t+1,t+1)}$ is $(a+1)H+(b+1)h$ and its weight $bw_0+aw_1+(d+1)w_2+(d+1)w_3$, therefore the restriction of α_d to $(b,a,t+1,t+1)$ is,

$$\boxed{H+h+w_2+w_3}$$

□

4. The normal bundle.

Theorem 4.1. *The Equivariant Chern classes of the Normal bundle in the equivariant Chow ring of the fixed points component are, for the first kind of components:*

$$\begin{aligned}c^T_{3d+4}(\mathcal{N}_{\mathbb{P}^d/R_d})=(h+(w_2-w_0))^{d+1}\cdot(h+(w_3-w_0))^{d+1}\cdot(w_0-w_1)\\ \cdot(h-(w_0-w_1))^{d-1}(w_2-w_1)\cdot(w_3-w_1),\end{aligned}$$

and for the second kind of components:

$$\begin{aligned}c^T_{3d+4}(\mathcal{N}_{\mathbb{P}^b\times\mathbb{P}^a}/R_d)=(H+w_2-w_0)^{b+1}\cdot(H+w_3-w_0)^{b+1}\cdot(h+w_2-w_0)^{a+1}\\ \cdot(-H+h+w_0-w_1)^{b-a-1}\cdot(h+w_0-w_1)^{a+1}\\ \cdot(H+w_1-w_0)^{b+1}\cdot(H-h+w_1-w_0)^{a-b-1},\end{aligned}$$

for $b-a\geqslant 0$.

Proof. We consider the normal bundle of the fixed points component in R_d.
We need to compute the weight of the normal bundle for each component of fixed points, and its equivariant Chern class. We first study the normal bundle for the components of the first kind.
We consider again the universal exact sequence

$$0\to\mathcal{N}\to\mathcal{O}^4_{R_d\times\mathbb{P}^1}\to\mathcal{E}\to 0.$$

The tangent space to the variety R_d is $\mathcal{T}_{R_d} \cong \pi_* \mathcal{H}om\,(\mathcal{N}, \mathcal{E})$ (see §7.7.1 of [6]), where $\pi : R_d \times \mathbb{P}^1 \to R_d$. If we restrict it to a component of fixed points of the first kind:

$$0 \to \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_3 \oplus \mathcal{Y}_4 \to \mathcal{O}^4_{\mathbb{P}^d \times \mathbb{P}^1} \to \mathcal{E}_{\mathbb{P}^d \times \mathbb{P}^1} \to 0,$$

where $\bigoplus_{i=1}^4 \mathcal{Y}_i$ is the kernel of the quotient map $\mathcal{O}^4_{\mathbb{P}^d \times \mathbb{P}^1} \to \mathcal{E}_{\mathbb{P}^d \times \mathbb{P}^1} \to 0$. The restriction of the normal bundle to the component of fixed points yields

$$0 \to \mathcal{T}_{\mathbb{P}^d} \to \mathcal{T}_R \to \mathcal{N}_{\mathbb{P}^d} \to 0,$$

$$\mathcal{N}_{\mathbb{P}^d/R_d} \cong \pi_* \oplus_{i \neq j} \mathcal{H}om(\mathcal{Y}_i, \mathcal{O}_{\mathbb{P}^1}/\mathcal{Y}_j).$$

Let us suppose that $\mathcal{E} \cong \mathcal{O}_{\mathcal{Z}_d} \oplus 0 \oplus \mathcal{O} \oplus \mathcal{O}$, it is enough to consider one component of fixed points by the symmetry of the computations, therefore,

$$0 \to \overbrace{\mathcal{O}(-1,-d)}^{w_0} \oplus \overbrace{\mathcal{O}}^{w_1} \oplus \overbrace{0}^{w_2} \oplus \overbrace{0}^{w_3} \to \mathcal{O}^4 \to \mathcal{O}_{\mathcal{Z}_d} \oplus 0 \oplus \mathcal{O} \oplus \mathcal{O} \to 0,$$

and

$$\mathcal{T}_{R_d} \cong \pi_* \mathcal{H}om(\mathcal{O}(-1,-d) \oplus \mathcal{O}, \mathcal{O}_{\mathcal{Z}_d} \oplus 0 \oplus \mathcal{O} \oplus \mathcal{O}).$$

Definitely, what we have is

$$\mathcal{N}_{\mathbb{P}^d/R_d} \cong \pi_* \mathcal{O}_{\mathcal{Z}_d} \otimes \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1}(1,d) \oplus (\pi_* \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1}(1,d))^2 \oplus (\pi_* \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1})^2. \tag{12}$$

Since $\pi_* \mathcal{O}_{\mathcal{Z}_d}$ is a bundle of rank d, the rank of the normal bundle is $3d+4$. Therefore we need to compute, $c^T_{3d+4}(\mathcal{N}_{\mathbb{P}^d/R_d})$ in the equivariant Chow ring of $\mathbb{P}^d$, (see §1.8 of [1]),

$$A^T_*(\mathbb{P}^d) = \mathbb{Z}(h,t) / \prod_{i=0}^{d} (h + w_i\, t).$$

It will be a polynomial of degree $3d+4$ in the variable h. The fiber of the normal bundle is isomorphic to

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{Z_d}) \oplus \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \oplus \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \oplus$$
$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}) \oplus \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}).$$

and the weights are by [3]:

$$\underbrace{\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{Z_d})}_{w_0 - w_1} \oplus \underbrace{\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1})}_{w_2 - w_1} \oplus \underbrace{\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1})}_{w_3 - w_1} \oplus$$
$$\underbrace{\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1})}_{w_2 - w_0} \oplus \underbrace{\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1})}_{w_3 - w_0}.$$

Given a $T-$equivariant vector bundle $E \to X$ we get a canonical decomposition $E = \bigoplus_{\chi \in \hat{T}} E^{\chi}$ where E^{χ} denotes the eigensubbundle consisting of vectors in E on which T acts with the character χ.

The $i-$esima equivariant Chern class of a $T-$equivariant bundle of rank r over $\mathbb{P}^d$ is such that the action of T over each fiber is given by the character. It is by §2.2.1, [1]:

$$c^T_i = \sum_{j=0}^{i} \binom{r-j}{i-j} c_j(E_{\chi_{i-j}}) \chi_{i-j}.$$

This formula relates the equivariant Chern classes of a bundle with the usual Chern classes. Thus, the problem is reduced to compute the usual Chern classes of the normal bundle and by using the Whitney formula,

$$\begin{aligned}&c^T_{3d+4}(\mathcal{N}_{\mathbb{P}^d/R_d})\\ &\qquad = c^T_d(\pi_*\mathcal{O}_{\mathcal{Z}_d})\cdot c^T_{d+1}(\pi_*\mathcal{O}(1,d))\cdot c^T_{d+1}(\pi_*\mathcal{O}(1,d))\cdot c^T_1(\pi_*\mathcal{O})\cdot c^T_1(\pi_*\mathcal{O}).\end{aligned}$$

For computing the Chern classes of the equivariant subbundle $\pi_*\mathcal{O}_{\mathcal{Z}_d}$ is required a little more work. For this purpose, we consider the exact sequence

$$0\to\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,-d)\to\mathcal{O}\to\mathcal{O}_{\mathcal{Z}_d}\to 0\quad\text{on } \mathbb{P}^{\mathrm{d}}\times\mathbb{P}^1,$$

$$0\to\pi_*\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,-d)\to\pi_*\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}\to\pi_*\mathcal{O}_{\mathcal{Z}_d}\to R^1\pi_*\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,-d)\to 0.$$

It follows $\pi_*\mathcal{O}_{\mathbb{P}^d\times\mathbb{P}^1}(-1,-d)=0$, since the fibers of this bundle are isomorphic to $H^0(\mathcal{O}_{\mathbb{P}^1}(-d))$ which are 0-dimensional vectorial spaces.

$$\begin{aligned}&R^1\pi_*\mathcal{O}(-1,-d)\\ &\qquad = R^1\pi_*(\pi_2^*\mathcal{O}_{\mathbb{P}^1}(-d)\otimes\pi^*\mathcal{O}_{\mathbb{P}^d}(-1))\\ &\qquad = R^1\pi_*\pi_2^*(\mathcal{O}_{\mathbb{P}^1}(-d))\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\end{aligned}$$

and $R^1\pi_*\pi_2^*(\mathcal{O}_{\mathbb{P}^1}(-d))\cong\mathcal{O}^{d-1}$ by Serre duality. Definitely, we have

$$0\to\mathcal{O}\to\pi_*\mathcal{O}_{\mathcal{Z}_d}\to\mathcal{O}^{d-1}\otimes\pi^*\mathcal{O}_{\mathbb{P}^d}(-1)\to 0\quad\text{on}\quad\mathbb{P}^{\mathrm{d}}$$

and therefore, by applying the Whitney formula, the total Chern class of the bundle is

$$\boxed{c_t(\pi_*\mathcal{O}_{\mathcal{Z}_d})=\prod_{i=0}^{d-1}(1-t)}$$

Now we can compute the Chern equivariant class of each equivariant subbundle of (12). Let $E_{w_i-w_j}$ denote the eigensubbundle consisting of vectors in $\mathcal{N}_{\mathbb{P}^d/R_d}$ on which $\mathbb{C}^*$ acts with weight w_i-w_j.

$$\begin{aligned}c^T_d(\pi_*\mathcal{O}_{\mathcal{Z}_d}) &= (w_0-w_1)\,(h-(w_0-w_1))^{d-1},\\ c^T_{d+1}(E_{w_2-w_0}) &= (h+w_2-w_0)^{d+1},\\ c^T_{d+1}(E_{w_3-w_0}) &= (h+w_3-w_0)^{d+1},\\ c^T_1(E_{w_2-w_1}) &= (w_2-w_1),\\ c^T_1(E_{w_3-w_1}) &= (w_3-w_1).\end{aligned}\tag{13}$$

We have,

$$\begin{aligned}c^T_{3d+4}(\mathcal{N}_{\mathbb{P}^d/R_d}) =&(h+(w_2-w_0))^{d+1}\cdot(h+(w_3-w_0))^{d+1}\cdot(w_0-w_1)\\ &\cdot(h-(w_0-w_1))^{d-1}\cdots(w_2-w_1)\cdot(w_3-w_1).\end{aligned}$$

We now study $c_{d_F}(\mathcal{N}_{F/R_d})$, for the components of fixed points of the second kind.

We consider the varieties of fixed points isomorphic to $\mathbb{P}^b\times\mathbb{P}^a$, with $b+a=d$, $b\geqslant a>0$.

Let be the component of fixed points defined by the universal quotient in $\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1$:

$$0 \to \mathcal{O}(-1,0,-b) \oplus \mathcal{O}(0,-1,-a) \oplus 0 \oplus 0 \to \mathcal{O}^4 \to \mathcal{O}_{\mathcal{Z}_b} \oplus \mathcal{O}_{\mathcal{Z}_a} \oplus \mathcal{O} \oplus \mathcal{O} \to 0$$

$$\begin{aligned}\mathcal{N}_{\mathbb{P}^b\times\mathbb{P}^a/R_d} =& \pi_{12*}\mathcal{H}om(\mathcal{O}(-1,0,-b), \mathcal{O}_{\mathcal{Z}_a}) \oplus \pi_{12*}\mathcal{H}om(\mathcal{O}(0,-1,-a), \mathcal{O}_{\mathcal{Z}_b}) \\ & \oplus \pi_{12*}\mathcal{H}om(\mathcal{O}(-1,0,-b), \mathcal{O})^2 \oplus \pi_{12*}\mathcal{H}om(\mathcal{O}(0,-1,-a), \mathcal{O})^2.\end{aligned}$$

We denote $E_{w_2-w_0}$ the subbundle $\pi_{12_*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,0,b))$ of $\mathcal{N}_{\mathbb{P}^b\times\mathbb{P}^a/R_d}$ on which $\mathbb{C}^*$ acts with weight $w_2 - w_0$. We compute its equivariant Chern classes:

$$c^T_{b+1}(\underbrace{\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,0,b))}_{E_{w_2-w_0}}) = (H + (w_2 - w_0))^{b+1},$$

$$c^T_{b+1}(\underbrace{\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,0,b))}_{E_{w_3-w_0}}) = (H + (w_3 - w_0))^{b+1},$$

$$c^T_{a+1}(\pi_{12*}(\underbrace{\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,1,a))}_{E_{w_2-w_0}}) = (h + (w_2 - w_0))^{a+1},$$

$$c^T_{a+1}(\pi_{12*}(\underbrace{\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,1,a))}_{E_{w_3-w_1}}) = (h + (w_3 - w_1))^{a+1},$$

$\mathbb{C}^*$ acts on $\pi_{12*}(\mathcal{O}_{\mathcal{Z}_b} \otimes \mathcal{O}(0,1,a))$ with weight $w_0 - w_1$. For computing its equivariant Chern class $c^T_b(E_{w_0-w_1})$ we consider the exact sequence,

$$\begin{gathered}0 \to \mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,0,-b) \otimes \mathcal{O}(0,1,a) \to \mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,1,a) \to \\ \to \mathcal{O}_{\mathcal{Z}_b} \otimes \mathcal{O}(0,1,a) \to 0\end{gathered}$$

$$\begin{gathered}0 \to \pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,1,-b+a)) \to \pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,1,a)) \to \\ \to \pi_{12*}(\mathcal{O}_{\mathcal{Z}_b} \otimes \mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,1,a)) \to R^1\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,1,-b+a)) \to 0\end{gathered} \tag{14}$$

$$\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,1,a)) \cong \mathcal{O}(0,1)^{a+1}$$

We suppose $b > a$, thus we have

$$\begin{aligned}\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,1,-b+a) = 0, \\ R^1\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,1,-b+a) \neq 0,\end{aligned}$$

and

$$\begin{aligned}R^1\pi_{12*}&\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(-1,1,-b+a) \\ &= R^1\pi_{12*}(\pi^*_{12}\mathcal{O}(-1,1) \otimes \pi^*_3\mathcal{O}(-b+a)) \\ &= \mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a}(-1,1) \otimes R^1\pi_{12*}\pi^*_3\mathcal{O}(-b+a) \\ &\cong \mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a}(-1,1) \otimes \pi_{12*}\pi^*_3\mathcal{O}_{\mathbb{P}^1}(b-a-2) \\ &= \mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a}(-1,1) \otimes \mathcal{O}^{b-a-1}.\end{aligned}$$

In the case $a = b$, we have that

(15)
$$0 \to \mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a}(0,1) \to \pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,1,a) \to \pi_{12*}(\mathcal{O}_{\mathcal{Z}_b}\otimes\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,1,a)) \to 0.$$

In either case, by applying Whitney formula to (4), we see that,

$$c^T(E_{w_0-w_1}) = c_b^T(\pi_{12*}(\mathcal{O}_{\mathcal{Z}_b}\otimes\mathcal{O}(0,1,a))) = (-H+h+w_0-w_1)^{b-a-1}\cdot(h+w_0-w_1)^{a+1}.$$

□

$\mathbb{C}^*$ acts on $\pi_{12*}(\mathcal{O}_{\mathcal{Z}_a}\otimes\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,0,b))$ with weight $w_1 - w_0$. We now compute $c_a^T(\pi_{12*}(\mathcal{O}_{\mathcal{Z}_a}\otimes\mathcal{O}(1,0,b)))$. We consider also the case in which $a = b$. Consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(0,-1,-a)\otimes\mathcal{O}(1,0,b) \to \mathcal{O}(1,0,b) \to \mathcal{O}_{\mathcal{Z}_a}\otimes\mathcal{O}(1,0,b) \to 0,$$

$$\begin{aligned} &0 \to \pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,-1,b-a)) \to \pi_{12*}(\mathcal{O}(1,0,b)) \to \\ &\to \pi_{12*}(\mathcal{O}_{\mathcal{Z}_a}\otimes\mathcal{O}(1,0,b)) \to R^1\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,-1,b-a)) \to 0. \end{aligned} \tag{16}$$

We observe in this case

$$R^1\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,-1,b-a) \cong \mathcal{O}(1,-1)\otimes\pi_{12*}(\pi_3^*\mathcal{O}_{\mathbb{P}^1}(a-b-2)) = 0,$$

$$\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,-1,b-a)) \cong \mathcal{O}(1,-1)\otimes\mathcal{O}^{b-a+1}, \tag{17}$$

$$\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,0,b)) \cong \mathcal{O}(1,0)\otimes\mathcal{O}^{b+1}, \tag{18}$$

$\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,-1,b-a))$ is a bundle with total Chern class:

$$\begin{aligned} c_t(\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,-1,b-a)) &= ((H-h)t+1)^{b-a+1}, \\ c_t^T(\pi_{12*}\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,-1,b-a)) &= ((H-h)t+w_1-w_0)^{b-a+1}, \\ c_t(\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,0,b))) &= (Ht+1)^{b+1}, \\ c_t^T(\pi_{12*}(\mathcal{O}_{\mathbb{P}^b\times\mathbb{P}^a\times\mathbb{P}^1}(1,0,b))) &= (Ht+w_1-w_0)^{b+1}, \end{aligned}$$

By applying Whitney formula to (16), we get that

$$c_a(E_{w_1-w_0}) = \frac{(H+w_1-w_0)^{b+1}}{(H-h+w_1-w_0)^{b-a+1}},$$

therefore,

$$\begin{aligned} c_{3d+4}^T(\mathcal{N}_{\mathbb{P}^b\times\mathbb{P}^a}/R_d) = c_{b+1}^T(E_{w_2-w_0})\cdot c_{b+1}^T(E_{w_3-w_0})\cdot c_{a+1}^T(E_{w_2-w_0})\cdot \\ \cdot c_{a+1}^T(E_{w_3-w_1})\cdot\ c_b^T(E_{w_0-w_1})\cdot c_a^T(E_{w_1-w_0}), \end{aligned}$$

that is, for $b-a \geqslant 1$,

$$\begin{aligned} c_{3d+4}^T(\mathcal{N}_{\mathbb{P}^b\times\mathbb{P}^a}/R_d) = &(H+w_2-w_0)^{b+1}\cdot(H+w_3-w_0)^{b+1}\cdot \\ &\cdot(h+w_2-w_0)^{a+1}\cdot(-H+h+w_0-w_1)^{b-a-1}\cdot \\ &\cdot(h+w_0-w_1)^{a+1}\cdot(H+w_1-w_0)^{b+1}\cdot \\ &\cdot(H-h+w_1-w_0)^{a-b-1}, \end{aligned}$$

and for $a = b$,

$$\begin{aligned} c^T_{3d+4}(\mathcal{N}_{\mathbb{P}^a\times\mathbb{P}^a}/R_d) =&(H+w_2-w_0)^{a+1}\cdot(H+w_3-w_0)^{a+1}\cdot\\ &\cdot(h+w_2-w_0)^{a+1}\cdot(h+w_0-w_1)^{a+1}\cdot\\ &\cdot(H+w_1-w_0)^{a+1}\cdot(H-h+w_1-w_0)^{-1}. \end{aligned}$$

5. Appendix A: calculation of Plücker degree of R_3.

We want to compute the degree of R_3 by the morphism induced by the divisor α, that is, the generalized Plücker embbeding [7]. The intersection we compute, is

$$P_3 = \int_{R_3}(\alpha^{16}\cap[R_3]).$$

We apply Bott residue formula. We have 24 summands, one for each component of fixed points. We know what the denominator is (4), and the restrictions of α to each subvariety of fixed points. The 24 summands corresponding to the 24 components of fixed points are:

(1)

$$Hilb^0_{\mathbb{P}^1}\times Hilb^3_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}$$

$$\frac{(h+w_2+w_3)^{16}}{(h+w_2-w_1)^4(h+w_3-w_1)^4(w_1-w_0)(h-w_1+w_0)^2(w_2-w_0)(w_3-w_0)}$$

(2)

$$Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^3_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^0_{\mathbb{P}^1}$$

$$\frac{(h+w_0+w_2)^{16}}{(h+w_0-w_1)^4(h+w_2-w_1)^4(w_1-w_3)(h-w_1+w_3)^2(w_2-w_3)(w_0-w_3)}$$

(3)

$$Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^3_{\mathbb{P}^1}\times Hilb^0_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}$$

$$\frac{(h+w_0+w_3)^{16}}{(h+w_0-w_1)^4(h+w_3-w_1)^4(w_1-w_2)(h-w_1+w_2)^2(w_0-w_2)(w_3-w_2)}$$

(4)

$$Hilb^0_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^3_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}$$

$$\frac{(h+w_1+w_3)^{16}}{(h+w_1-w_2)^4(h+w_3-w_2)^4(w_2-w_0)(h-w_2+w_0)^2(w_1-w_0)(w_3-w_0)}$$

(5)

$$Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^3_{\mathbb{P}^1}\times Hilb^0_{\mathbb{P}^1}$$

$$\frac{(h+w_0+w_1)^{16}}{(h+w_0-w_2)^4(h+w_1-w_2)^4(w_2-w_3)(h-w_2+w_3)^2(w_0-w_3)(w_1-w_3)}$$

(6)

$$Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^0_{\mathbb{P}^1}\times Hilb^3_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}$$

$$\frac{(h+w_0+w_3)^{16}}{(h+w_0-w_2)^4(h+w_3-w_2)^4(w_2-w_1)(h-w_2+w_1)^2(w_0-w_1)(w_3-w_1)}$$

(7)
$$\frac{Hilb^3_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^0_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\\(h+w_1+w_2)^{16}}{(h+w_1-w_0)^4(h+w_2-w_0)^4(w_0-w_3)(h-w_0+w_3)^2(w_1-w_3)(w_2-w_3)}$$

(8)
$$\frac{Hilb^3_{\mathbb{P}^1}\times Hilb^{0}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\\(h+w_2+w_3)^{16}}{(h+w_2-w_0)^4(h+w_3-w_0)^4(w_0-w_1)(h-w_0+w_1)^2(w_2-w_1)(w_3-w_1)}$$

(9)
$$\frac{Hilb^3_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^0_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\\(h+w_1+w_3)^{16}}{(h+w_1-w_0)^4(h+w_3-w_0)^4(w_0-w_2)(h-w_0+w_2)^2(w_1-w_2)(w_3-w_2)}$$

(10)
$$\frac{Hilb^0_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{3}_{\mathbb{P}^1}\\(h+w_1+w_2)^{16}}{(h+w_1-w_3)^4(h+w_2-w_3)^4(w_3-w_0)(h-w_3+w_0)^2(w_1-w_0)(w_2-w_0)}$$

(11)
$$\frac{Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{0}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{3}_{\mathbb{P}^1}\\(h+w_0+w_2)^{16}}{(h+w_0-w_3)^4(h+w_2-w_3)^4(w_3-w_1)(h-w_3+w_1)^2(w_0-w_1)(w_2-w_1)}$$

(12)
$$\frac{Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{0}_{\mathbb{P}^1}\times Hilb^{3}_{\mathbb{P}^1}\\(h+w_0+w_1)^{16}}{(h+w_0-w_3)^4(h+w_1-w_3)^4(w_3-w_2)(h-w_3+w_2)^2(w_0-w_2)(w_1-w_2)}$$

(13)
$$\frac{Hilb^{1}_{\mathbb{P}^1}\times Hilb^{2}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\\(H+h+w_2+w_3)^{16}}{(h+w_1-w_0)^2(H+2h+w_0-w_1)(h+w_2-w_0)^2(h+w_3-w_0)^2(H+w_3-w_1)^3(H+w_2-w_1)^3}$$

(14)
$$\frac{Hilb^{1}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{2}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\\(H+h+w_1+w_3)^{16}}{(h+w_2-w_0)^2(H+2h+w_0-w_2)(h+w_1-w_0)^2(h+w_3-w_0)^2(H+w_1-w_2)^3(H+w_3-w_2)^3}$$

(15)
$$\frac{Hilb^{1}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{2}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\\(H+h+w_1+w_2)^{16}}{(h+w_3-w_0)^2(H+2h+w_0-w_3)(h+w_2-w_0)^2(h+w_1-w_0)^2(H+w_2-w_3)^3(H+w_1-w_0)^3}$$

(16)
$$\frac{Hilb^{t+1}_{\mathbb{P}^1}\times Hilb^{1}_{\mathbb{P}^1}\times Hilb^{2}_{\mathbb{P}^1}\times Hilb^{t+1}_{\mathbb{P}^1}\\(H+h+w_0+w_3)^{16}}{(h+w_2-w_1)^2(H+2h+w_1-w_2)(h+w_0-w_1)^2(h+w_3-w_1)^2(H+w_0-w_2)^3(H+w_3-w_2)^3}$$

(17)

$$Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{1} \times Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{2}$$

$$\frac{(H+h+w_0+w_2)^{16}}{(h+w_3-w_1)^2(H+2h+w_1-w_3)(h+w_0-w_1)^2(h+w_2-w_1)^2(H+w_0-w_3)^3(H+w_2-w_3)^3}$$

(18)

$$Hilb_{\mathbb{P}^1}^{2} \times Hilb_{\mathbb{P}^1}^{1} \times Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{t+1}$$

$$\frac{(H+h+w_2+w_3)^{16}}{(h+w_3-w_1)^2(H+2h+w_1-w_3)(h+w_2-w_1)^2(h+w_3-w_1)^2(H+w_2-w_0)^3(H+w_3-w_0)^3}$$

(19)

$$Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{2} \times Hilb_{\mathbb{P}^1}^{1} \times Hilb_{\mathbb{P}^1}^{t+1}$$

$$\frac{(H+h+w_0+w_3)^{16}}{(h+w_0-w_2)^2(H+2h+w_2-w_0)(h+w_0-w_2)^2(h+w_3-w_2)^2(H+w_0-w_1)^3(H+w_3-w_1)^3}$$

(20)

$$Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{1} \times Hilb_{\mathbb{P}^1}^{2}$$

$$\frac{(H+h+w_0+w_3)^{16}}{(h+w_0-w_2)^2(H+2h+w_2-w_0)(h+w_0-w_2)^2(h+w_3-w_2)^2(H+w_0-w_1)^3(H+w_3-w_1)^3}$$

(21)

$$Hilb_{\mathbb{P}^1}^{2} \times Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{1} \times Hilb_{\mathbb{P}^1}^{t+1}$$

$$\frac{(H+h+w_0+w_1)^{16}}{(h+w_0-w_2)^2(H+2h+w_2-w_0)(h+w_1-w_2)^2(h+w_3-w_2)^2(H+w_1-w_0)^3(H+w_3-w_0)^3}$$

(22)

$$Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{2} \times Hilb_{\mathbb{P}^1}^{1}$$

$$\frac{(H+h+w_0+w_1)^{16}}{(h+w_2-w_1)^2(H+2h+w_1-w_2)(h+w_0-w_3)^2(h+w_1-w_3)^2(H+w_0-w_2)^3(H+w_1-w_2)^3}$$

(23)

$$Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{2} \times Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{1}$$

$$\frac{(H+h+w_0+w_2)^{16}}{(h+w_1-w_3)^2(H+2h+w_3-w_1)(h+w_0-w_3)^2(h+w_2-w_3)^2(H+w_0-w_1)^3(H+w_2-w_3)^3}$$

(24)

$$Hilb_{\mathbb{P}^1}^{2} \times Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{t+1} \times Hilb_{\mathbb{P}^1}^{1}$$

$$\frac{(H+h+w_1+w_2)^{16}}{(h+w_0-w_3)^2(H+2h+w_3-w_0)(h+w_2-w_3)^2(h+w_1-w_3)^2(H+w_1-w_0)^3(H+w_2-w_0)^3}$$

Once we have all the summands, we take the direct image by the morphism $\pi_1 : \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^3$ and $\pi_{12} : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 \times \mathbb{P}^1$ for the first kind of components and for the second kind of components respectively. The only terms surviving in the first case are those in h^3, and in the second case the terms in $H^2 h$. We have used Maple program to make the computations. This degree is 128. This result coincides with the one obtained by means of the Vafa-Intriligator formula, [2] and indeed it follows easily from Vafa-Intriligator formula that the degree P_d coincides with 2^{2d+1}.

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