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THE WEIGHT SYSTEM OF THE MULTIVARIABLE ALEXANDER POLYNOMIAL

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Abstract. We derive a formula for the weight system of the multivariable Alexander polynomial using determinants, show that it obeys known relations, and satisfies some of the same relations as the single variable polynomial. This formula will be computable in polynomial time.

1. Introduction

When Alexander first defined the multivariable Alexander polynomial [1] it was defined only up to sign and powers of the variables. In [5] Murakami used Hartleys [4] normalization of the Conway potential function to show that under an appropriate change of variables the multivariable Alexander polynomial is of finite type. A recursive definition of this weight system was given in [5] whose computation requires exponential time. We improve on this result by giving a closed form formula using determinants, which would be computable in polynomial time.

Our explicit formula makes it possible to verify that the relations from [3], which hold for the weight system of the single variable Alexander polynomial, also hold for the weight system of the multivariable Alexander polynomial.

2. Definition of the multivariable Alexander polynomial

Definition 1. A coloured link (or chord diagram) is a link along with a variable associated to each of the components.

We use the normalization of the multivariable Alexander polynomial given in [5]; which eliminates the ambiguity of signs and powers of t. This uses the matrix given by Fox free calculus on the Wirtinger presentation of the link, with the crossings labeled by the arc starting at that crossing.

To compute the multivariable Alexander polynomial, one first labels all of the arcs, and then labels the crossings by the exiting lower strand. This allows us to create a matrix with rows and columns indexed by the arcs. The non zero

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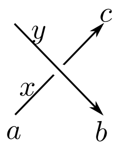
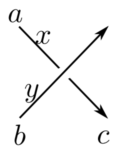
entries of the c^{th} row are given below, where x and y are the colours of the link components as marked.



$\longmapsto$

	a	b	c
c	-1	$1-x$	y



$\longmapsto$

	a	b	c
c	$-y$	$x-1$	1

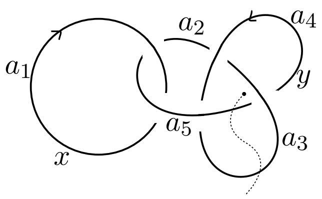
Let M_i^j denote the matrix obtained from M by deleting the i^{th} row and j^{th} column. Let $rot(k)$ denote the rotation number (Whitney degree) of the k^{th} component of the link and let $\mu(k)$ be the number of times that the k^{th} component is the over strand in a crossing. Let w_i be a word corresponding to a path from a point to the right of the a_i^{th} crossing to the unbounded region of the plane. Let t_i be the colour assigned to the i'th arc. The following definition is due to [5].

Definition 2. For a coloured oriented link L the Multivariable Alexander Polynomial is given by

$$\Delta(L) = \frac{(-1)^{i+j}\det(M_i^j)}{w_i(t_i-1)} \prod_k t_k^{\frac{rot(k)-\mu(k)}{2}}.$$

Note that for links $t_i - 1$ divides $\det(M_i^j)$ so this is indeed a polynomial, and that for knots this differs from the usual definition by a factor of $t_1 - 1$.

Example 1. For the following labeled link, we construct the given matrix.



	a_1	a_2	a_3	a_4	a_5
a_1	$y-1$	0	0	0	$1-x$
a_2	$1-y$	x	0	0	-1
a_3	$-y$	0	1	$y-1$	0
a_4	0	0	$-y$	1	$y-1$
a_5	0	0	$y-1$	$-y$	1

The marked point is to the right of the 5^{th} crossing; by following the dotted path we see that $w_5 = y^{-2}$. Checking that $\mu(1) = 1$, $\mu(2) = 4$, $rot(1) = 1$, $rot(2) = 3$, and $\det(M_5^5) = x(y-1)(1-y+y^2)$ gives the multivariable Alexander polynomial of the link as $xy(1-y+y^2)$.

3. The multivariable Alexander polynomial as a finite type invariant

We now recall how Murakami [5] shows that under an appropriate change of variables the multivariable Alexander polynomial is of finite type.

Definition 3. A singular knot (link) is a knot with double points. It represents a linear combination of knots, where each double point represents the difference between a positive and negative crossing.

We now wish to compute what happens for the multivariable Alexander polynomial on a knot with one double point. To do this we examine the following small labeled region of a knot, note that the added kink is to make the number and labeling of arcs consistent, and does not change the knot. The kink doesn't affect the normalization as it adds one to both $\mu(k)$ and $rot(k)$. The two matrices constructed as in Section 1 will differ in only the two rows indexed by a_1 and a_2. The only difference in the normalization coefficient for these two knots is the number of over crossings, which adds an extra factor of $x^{\frac{1}{2}}$ or $y^{\frac{1}{2}}$. By multiplying a matrix row by that change we can see that the matrices differ only in the following rows:

	a_1	a_2	a_3	a_4
a_1	$y^{-\frac{1}{2}}$	0	$-y^{-\frac{1}{2}}$	0
a_2	$1-x$	y	0	-1

	a_1	a_2	a_3	a_4
a_1	1	$y-1$	$-x$	0
a_2	0	$x^{-\frac{1}{2}}$	0	$-x^{-\frac{1}{2}}$

Using row operations we can change the two matrices to the following;
Now we can see that the resulting difference will contain the following rows:

	a_1	a_2	a_3	a_4
a_1'	$y^{-\frac{1}{2}}$	$x^{-\frac{1}{2}}$	$-y^{-\frac{1}{2}}$	$-x^{-\frac{1}{2}}$.
a_2'	$1-x^{\frac{1}{2}}$	$y-y^{\frac{1}{2}}$	$-x+x^{\frac{1}{2}}$	$-1+y^{\frac{1}{2}}$

Now if we use the substitution $e^{t_1}=x_1$, $e^{t_2}=x_2$ and expand e^{t_i} as a power series, omitting terms of degree greater than 1 we get:

	a_1	a_2	a_3	a_4
a_1'	$1-\frac{1}{2}t_2-\dots$	$1-\frac{1}{2}t_1-\dots$	$-1+\frac{1}{2}t_2+\dots$	$-1+\frac{1}{2}t_1+\dots$
a_2'	$-\frac{1}{2}t_1-\dots$	$\frac{1}{2}t_2+\dots$	$-\frac{1}{2}t_1-\dots$	$\frac{1}{2}t_2+\dots$

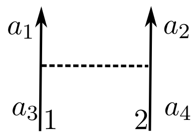
We note that all the entries in the second row are of degree one. Which means that this singular point will contribute one to the degree of the resulting polynomial. So if the knot has m singular points the power series we get from taking the determinant of the matrix starts in degree m. When we apply the same change of variables to the normalization coefficient, we notice that dividing by $x_i - 1$ will lower the index of t_i by one. Since the rest of the coefficient will have leading term 1, the resulting polynomial will have degree at least $m-1$. Hence after the expansion of $x_i = e^{t_i} = 1 + t_i + ...$ the polynomial of degree m is an $m+1$ type invariant.

So the contribution from a singular point will be the following;

	a_1	a_2	a_3	a_4
a_1'	1	1	-1	-1
a_2'	$-\frac{1}{2}t_1$	$\frac{1}{2}t_2$	$-\frac{1}{2}t_1$	$\frac{1}{2}t_2$

4. The weight system

We define a function from coloured chord diagrams to $\mathbb{Z}[t_1, ..., t_n]$. Given a coloured chord diagram, choose a marked point on one of the arcs. This divides that arc into two; we then label the arcs. The chords are indexed by the two exiting arcs. We now construct a matrix $M(D)$ whose rows and columns are indexed by the arcs, as follows, all other entries are zero. (If you change roles of the two sides, you swap the two rows below and multiply by -1 which leaves the determinant unchanged).



$\longmapsto$

	a_1	a_2	a_3	a_4
a_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$-\frac{1}{2}$
a_2	$-t_2$	t_1	$-t_2$	t_1

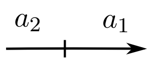
Note that the row corresponding to the marked point is zero. As before $M_i^i(D)$ is the matrix $M(D)$ with the i^{th} row and column removed.

Definition 4. The weight system for the multivariable Alexander polynomial is

$$\Delta(D) = \frac{\det(M_i^i(D))}{t_i},$$

where i is the colour of the marked point. [1]

To aid in later sections we can extend the definition to allow us to divide arcs at non chords. To divide the arc a into a_1 and a_2 the the following row would need to be added to the matrix; this will not change the value of the determinant.



$\longmapsto$

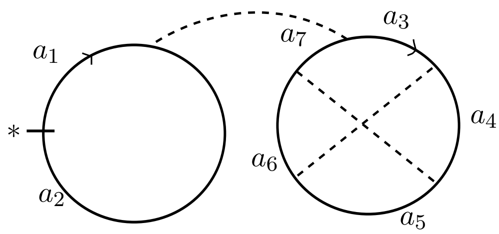
	a_1	a_2
a_1	1	-1

Theorem 1. *The above function defines a weight system for the Multivariable Alexander Polynomial under the substitution $x_i = e^{t_i} = 1 + t_i + \dots$, and is independent of the chosen marked point.*

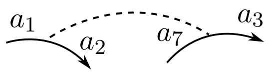
A proof of this theorem will follow a sample computation.

[1]Note for links this is divisible by t_i since $\Sigma t_i c_i = 0$ where c_i is the ith column.

5. A worked example



On the above diagram we have chosen a marked point $\star$ and labeled the arcs. We now construct a matrix indexed by those arcs. Here is one of the chords and the corresponding matrix entries.



	a_1	a_2	a_3	a_7
a_2	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$
a_3	$-t_2$	$-t_2$	t_1	t_1

We can use the rest of the chords to fill in the remainder of the table as follows:

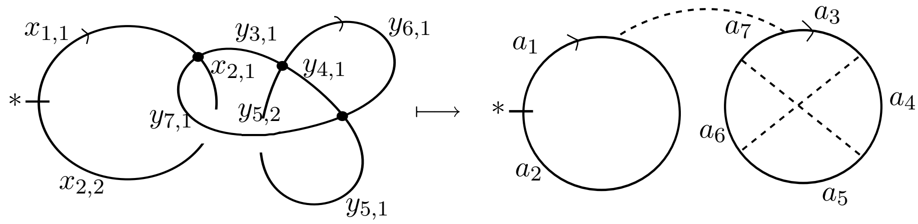
	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_1							
a_2	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$				$-\frac{1}{2}$
a_3	$-t_2$	$-t_2$	t_1				t_1
a_4			t_2	t_2	$-t_2$	$-t_2$	
a_5				t_2	t_2	$-t_2$	$-t_2$
a_6			$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	
a_7				$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

To calculate the weight of this diagram, we simply remove the marked row and column (a_1), take the determinant and divide by t_1 to get ${t_2}^2$.

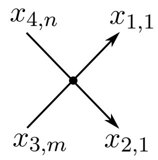
6. Proof of Theorem 1

We need to show that the weight system defined above on chord diagrams, is the same the one defined by the MVA on singular knots. That is for a given chord diagram, we must take a corresponding singular knot and show that the MVA on that knot (under the appropriate substitution, and up to the appropriate degree) is the same as the weight system evaluated at that chord diagram.

Suppose we have an m singular link and its underlying chord diagram. We first label the arcs on the chord diagram $x_1, ..., x_n$. We then lift this labeling to the arcs of the link, so that the arcs which map to x_i, will be labeled in order as $x_{i,j}$ as in the diagram below. Then we mark the same arc on the link and chord diagram to corresponding to the row and column that we will delete. Each crossing is indexed by the under arc leaving it.

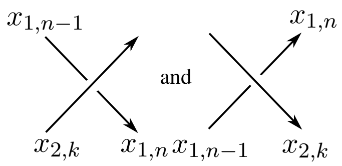


As in the proof in Section 3, we will be using the substitution $x_i = e^{t_i} = 1 + t_i + \ldots$. Since we are interested in the terms of order $m-1$, we take a row of order 1 in the rows corresponding to the double points, and terms of order zero elsewhere. We note that up to order 0 the rows corresponding to over and undercrossings are the same after this substitution. So to we will create a matrix with the following rows;



$\longmapsto$

	$x_{1,1}$	$x_{2,1}$	$x_{3,n}$	$x_{4,m}$
$x_{1,1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$
$x_{2,1}$	t_2	$-t_1$	t_2	$-t_1$



$\longmapsto$

	$x_{1,n}$	$x_{1,n-1}$	$x_{2,k}$
$x_{1,n}$	1	-1	0

We can think of the matrix as being in blocks corresponding to the arcs in the underlying chord diagram. This gives us a matrix that is mostly zeros with 1 and -1's near the diagonal. The only variables occur in the rows $x_{i,1}$ the top row of each block. Each block will be of the form shown below with zeros omitted.

	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$\cdots$	$x_{1,n-1}$	$x_{1,n}$
$x_{1,1}$	*					*
$x_{1,2}$	-1	1				
$x_{1,3}$		-1	1			
$\vdots$			$\ddots$	$\ddots$		
$x_{1,n-1}$				-1	1	
$x_{1,n}$					-1	1

Then for all of the blocks except for the marked one, we can switch to the following using column operations:

	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	...	$x_{1,n-1}$	$x_{1,n}$
$x_{1,1}$	$\Sigma *$					$*$
$x_{1,2}$	0	1				
$x_{1,3}$		0	1			
$\vdots$			$\ddots$	$\ddots$		
$x_{1,n-1}$				0	1	
$x_{1,n}$					0	1

Note that rows $x_{1,2}$ to $x_{1,n}$ are zero except with a 1 on the diagonal, so we may delete those rows and columns without changing the determinant. But we must remember that the $x_{1,1}^{th}$ column is replaced with the sum of columns $x_{1,1}$ and $x_{1,n}$. That sum will be the contribution of the arcs $x_{1,1}$ and $x_{1,n}$ from double points; note that those arcs map to a_1 in the chord diagram.

In the case that one of the arcs is marked, we must remember that we will be deleting that column before we take the determinant. We will always mark an arc that is not near a double point. Let us look at the block where one of the arcs is marked; with out loss of generality we choose $x_{1,2}$. Then when we reduce the matrix we get:

	$x_{1,1}$	$x_{1,3}$	...	$x_{1,n-1}$	$x_{1,n}$
$x_{1,1}$	$*$				$*$
$x_{1,3}$	0	1			
$\vdots$		$\ddots$	$\ddots$		
$x_{1,n-1}$			-1	1	
$x_{1,n}$				-1	1

which can then be changed using row and column operations to:

	$x_{1,1}$	$x_{1,3}$	...	$x_{1,n-1}$	$x_{1,n}$
$x_{1,1}$	$*$				$*$
$x_{1,3}$	0	1			
$\vdots$		$\ddots$	$\ddots$		
$x_{1,n-1}$			0	1	
$x_{1,n}$				0	1

Then once again we can delete the rows and columns corresponding to $x_{1,3}$, ..., $x_{1,n-1}$. But note that there is no change in the $x_{1,1}^{th}$ column. So the contribution of the $x_{1,n}^{th}$ column is lost, this is equivalent to deleting the column associated to the marked arc.

This is the matrix that was described in Theorem 1[2].

7. Testing known relations

As a test of this method, we wish to show that this function obeys certain known relations, such as the 4T relation and the recursive relations in Murakami's paper. Each of these relations involve chord diagrams that differ only in one small area. This means that the matrices formed in computing the invariant will differ in only one block. The following lemma shows that the relations need only hold among certain minors of that block.

Lemma 1 (A linear algebra lemma)**.** *Let B be an $n \times n$ block matrix of the form*

$$B = \begin{matrix} \\ m\{ \\ \\ \end{matrix}\!\!\begin{pmatrix} \overbrace{A}^{l} & 0 \\ \underbrace{0}_{k} & M \end{pmatrix}$$

where A is an $m \times l$ matrix, and $k < l$.

Then $\det(B) = \sum_{k<i_1<\ldots<i_{n-k}} (\pm)\det(A^{1,2,\ldots,k,i_1,\ldots,i_{n-k}})\det(M_{i_1,\ldots,i_{n-k}}),$

where $A^{1,2,\ldots,k,i_1,\ldots,i_{n-k}}$ refers to the matrix formed from the $1, 2, \ldots, k, i_1, \ldots, i_{n-k}$ columns of A and $M_{i_1,\ldots,i_{n-k}}$ refers to M with the $i_1, \ldots, i_{n-k}$ columns removed.

Proof. Expand B along the rows of A using cofactor expansion, and collect like terms. □

Lemma 2. *Let B_i be matrices as in the previous lemma, with the same M but different A_i. To show a relation of the form*

$$\sum a_i \det(B_i) = 0,$$

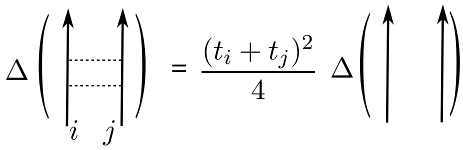
it is enough to show the relation holds on the appropriate minors the A_i's.

Proof. Follows from previous lemma. □

When we wish to show relations between chord diagrams differing in a region, we note that the matrices used to compute their weight will be of the above form. Using lemma 2 let us show the following relation holds:

To use the previous lemma we make both sides of the relation have the same number of arcs, so we divide the arcs in the left term in three. The matrices involved are:

[2] $\Delta(D)$ is independent of the choice of marked point. The sum of the columns in the matrix $M(D)$ is 0, changing the column deleted changes the determinant by a factor of -1. If you divide every arc in the diagram so that each arc touches only one chord, and sum the columns which end at chords multiplied by the label of that arc gives the row for the marked point multiplied by the label of the marked point. So changing the marked point will change the determinant by $\pm t_i$, which is canceled by the coefficient.



$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
$-t_2$	t_1	0	0	$-t_2$	$-t_1$	0
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$-t_2$	t_1	$-t_2$	t_1	0	0	0
0	0	a_1	a_2	a_3	a_4	M

1	0	0	0	-1	0	0
0	1	0	0	0	-1	0
-1	0	$+1$	0	0	0	0
0	-1	0	1	0	0	0
0	0	a_1	a_2	a_3	a_4	M

For the first matrix in the example above the minors in question are as follows:

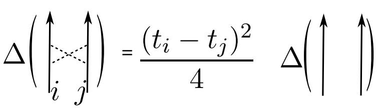
$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -t_2 & t_1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -t_2 & t_1 & -t_2 & t_1 \end{vmatrix} = \left(\frac{t_1+t_2}{2}\right)^2 \qquad \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ -t_2 & t_1 & 0 & -t_2 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ -t_2 & t_1 & t_1 & 0 \end{vmatrix} = \left(\frac{t_1+t_2}{2}\right)^2$$

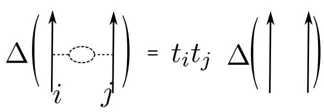
$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ -t_2 & t_1 & 0 & -t_2 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ -t_2 & t_1 & -t_2 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ -t_2 & t_1 & 0 & -t_1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ -t_2 & t_1 & t_1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ -t_2 & t_1 & 0 & -t_1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ -t_2 & t_1 & -t_2 & 0 \end{vmatrix} = -\left(\frac{t_1+t_2}{2}\right)^2 \qquad \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -t_2 & t_1 & -t_2 & -t_1 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -t_2 & t_1 & 0 & 0 \end{vmatrix} = \left(\frac{t_1+t_2}{2}\right)^2.$$

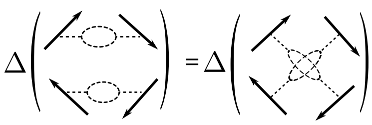
The corresponding minors for the second matrix are $\{1, 0, -1, 1, 0, 1\}$. You can see that the first are precisely $\left(\frac{t_1+t_2}{2}\right)^2$ times those of the second, so the relation holds.

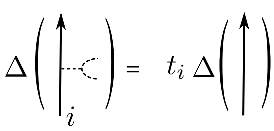
If we cross the chords we get the following two relations:





This allows us to derive the following useful relations:

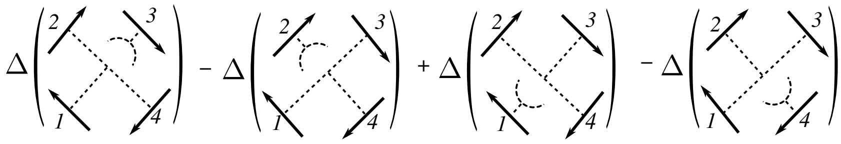




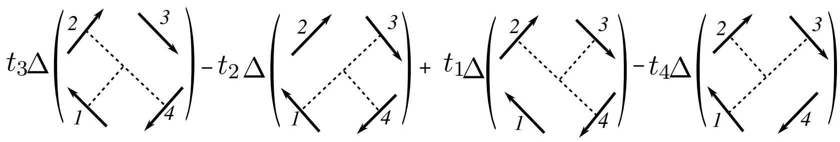
The first relation is a direct consequence of the previous. Since it doesn't matter how these components connect we can omit the connections from our diagrams. We can draw elements as in the second relation with the understanding that there will always be an even number of such elements in a diagram.

8. Testing relations using mathematica

Obviously for more complicated relations we are unable to compute by hand all of the minors involved. The solution to this is to use Mathematica to do all of the computations for us. For example, here is a computation that shows the 4Y relation holds:



Using the previous relation we can rewrite the relation as



In the following calculation, we have used the substitution $x = t_1$, $y = t_2$, $z = t_3$ and $w = t_4$, and a common factor of $\frac{1}{8}$ has been factored from all of the matrices.

```
Mat[1]={
      {1,1,0,0,0,-1,-1,0,0},
      {-x,y,0,0,0,y,-x,0,0},
      {-1,0,1,0,1,0,0,0,-1},
      {0,0,0,-1,0,0,0,1,0},
      {-w,0,-w,0,y,0,0,0,y}};

Mat[2]={
      {w,0,0,0,-y,0,w,0,-y},
      {-1,1,1,0,0,-1,0,0,0},
      {x,-y,x,0,0,-y,0,0,0},
      {0,0,0,-1,0,0,0,1,0},
      {1,0,0,0,1,0,-1,0,-1}};

Mat[3]={
      {z,0,0,-w,0,0,0,-w,z},
      {-1,1,0,0,1,-1,0,0,0},
      {0,0,-1,0,0,0,1,0,0},
      {1,0,0,1,0,0,0,-1,-1},
      {x,-w,0,0,x,-w,0,0,0}};

Mat[4]={
      {x,-w,0,0,0,-w,0,0,x},
      {1,1,0,0,0,-1,0,0,-1},
      {0,0,-1,0,0,0,1,0,0},
      {-1,0,0,1,1,0,0,-1,0},
      {z,0,0,-w,z,0,0,-w,0}};
```

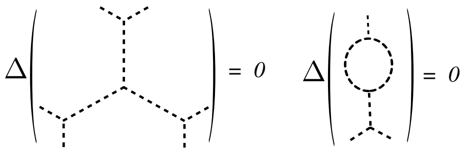
```
Mat[5]={
      {z,0,0,-w,0,0,0,-w,z},
      {0,-1,0,0,0,1,0,0,0},
      {-y,0,w,0,-y,0,w,0,0},
      {1,0,0,1,0,0,0,-1,-1},
      {-1,0,1,0,1,0,-1,0,0}};

Mat[6]={
      {1,0,1,0,0,0,-1,0,-1},
      {0,-1,0,0,0,1,0,0,0},
      {-y,0,w,0,0,0,w,0,-y},
      {-1,0,0,1,1,0,0,-1,0},
      {z,0,0,-w,z,0,0,-w,0}};

Mat[7]={
      {1,1,0,0,0,-1,-1,0,0},
      {-x,y,0,0,0,y,-x,0,0},
      {z,0,z,-y,0,0,0,-y,0},
      {-1,0,1,1,0,0,0,-1,0},
      {0,0,0,0,-1,0,0,0,1}};

Mat[8]={
      {z,0,0,-y,0,0,z,-y,0},
      {-1,1,1,0,0,-1,0,0,0},
      {x,-y,x,0,0,-y,0,0,0},
      {1,0,0,1,0,0,-1,-1,0},
      {0,0,0,0,-1,0,0,0,1}};


z*(Minors[Mat[1],5]-Minors[Mat[2],5])-y*(Minors[Mat[3],5]-
Minors[Mat[4],5])+x*(Minors[Mat[5],5]-Minors[Mat[6],5])-
w*(Minors[Mat[7],5]-Minors[Mat[8],5]);

L=Simplify[%];

Take[Transpose[L],70]

{{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},
{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},
{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},
{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},{0},
{0},{0},{0},{0},{0},{0}}
```

From our lemma, since the relation holds for the minors, it holds for the determinants of the original matrices. So the relation holds for the weight system.

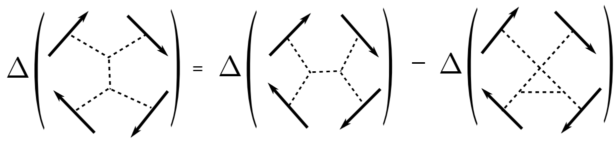
9. Additional relations

In [3], additional relations where shown to hold for the weight system of the Alexander polynomial, these can be shown to hold for the multivariable Alexander polynomial weight system as well.

No internal vertices



IHX



We use the same method to verify that the five relations in Murakami's paper [5] hold. This shows that we have indeed given an alternate directly computable definition for the same weight system.

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