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SMOOTHING RESOLUTION FOR THE ALEXANDER–CONWAY POLYNOMIAL

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Abstract. We introduce a new kind of smoothing such that we obtain a disjoint union of framed circles and a framed path after we smooth all crossings of a $(1, 1)$-tangle diagram. By using such a smoothing, we reconstruct the Alexander–Conway polynomial in a manner similar to the way the Jones polynomial is constructed by using the Kauffman bracket polynomial.

1. Introduction

We focus on two well-known knot invariants: the Alexander–Conway polynomial [1, 4], and the Jones polynomial [8].

The Alexander polynomial is defined through the infinite cyclic cover of a knot complement. It was the only known knot polynomial until the Jones polynomial was discovered. One obtains the normalized Alexander polynomial of an oriented link by using the skein relation found by J. W. Alexander [1], and in normalized form by J. H. Conway [4]. We call it the Alexander–Conway polynomial.

The Jones polynomial came out of investigations of operator algebras. It is also determined by using a skein relation. The skein relations are often used to give simple definitions of the Alexander–Conway and Jones polynomials. The HOMFLYPT polynomial [6, 19] is a two-variable polynomial invariant motivated by these skein relations.

Since the discovery of the Jones polynomial, many link invariants have been defined, which include so-called quantum invariants. As quantum invariants, the Alexander–Conway and Jones polynomials are derived from two dimensional representations of the quantum (super)algebras $U_q[gl(1|1)]$ and $U_q[sl(2)]$, respectively. And then, each invariant is defined by using a 4×4 R-matrix, which is a solutions of the Yang–Baxter equation (see [5, 9, 12, 16, 20]).

The Jones polynomial also can be obtained via the Kauffman bracket polynomial [10]. We smooth the crossings of a link diagram until we reduce it to a disjoint union of circles:

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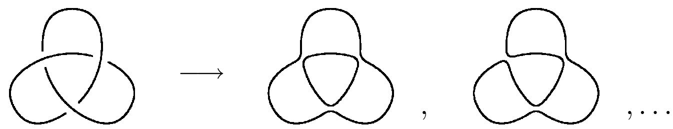
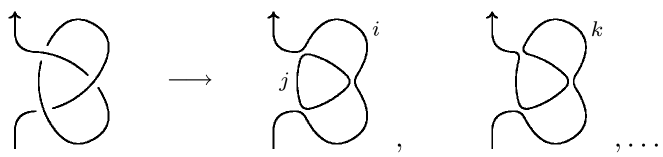


The Kauffman bracket polynomial is defined by using the weight of a crossing and the number of circles, where the weight is a value determined by the choice of smoothing. Then it is an invariant of regular isotopy. By utilizing the writhe of the link diagram, we obtain an invariant of ambient isotopy, which is the Jones polynomial.

Since smoothing is one of the simplest operations on a link diagram, the reconstruction of the Jones polynomial via the Kauffman bracket polynomial is related to important topics in knot theory: invariance of the number of crossings in a reduced alternating diagram of a knot, the Kauffman bracket skein module [17, 18], the Khovanov homology [13, 2, 3], and so on.

On the other hand, a state model for the Alexander–Conway polynomial was also given by L. H. Kauffman [11, p. 174]. A state in this state model is represented by a diagram which may have transversal intersections. In this paper, we aim to reconstruct the Alexander–Conway polynomial through a smoothing resolution formula, say, a state model in which a state is represented by a disjoint union of circles (and a path). Such a construction is expected to be related to important topics in knot theory as in the case of the Kauffman bracket polynomial. This state model can also lead to new (Khovanov type) categorification of the Alexander–Conway polynomial.

We introduce a new kind of smoothing such that we obtain a disjoint union of framed circles and a framed path after we smooth all crossings of a $(1, 1)$-tangle diagram:



Our bracket polynomial is defined by using the weight of a crossing and the framings. Then it is an invariant of regular isotopy. By utilizing the Whitney degree (rotation number) of the link diagram, we obtain an invariant of ambient isotopy, which is the Alexander–Conway polynomial.

2. A magnetic link/tangle

We introduce a magnetic link/tangle diagram in which an "orientation" $\triangleleft$ is given at a point called node. In our reconstruction for the Alexander–Conway polynomial, we reduce a $(1, 1)$-tangle diagram into magnetic tangle diagrams without a crossing, and the node orientation contributes to a framing the value $\pm 1/2$. A magnetic link/tangle without node orientation appears in constructions of the oriented state model for the Jones polynomial [11, p. 74], Miyazawa's

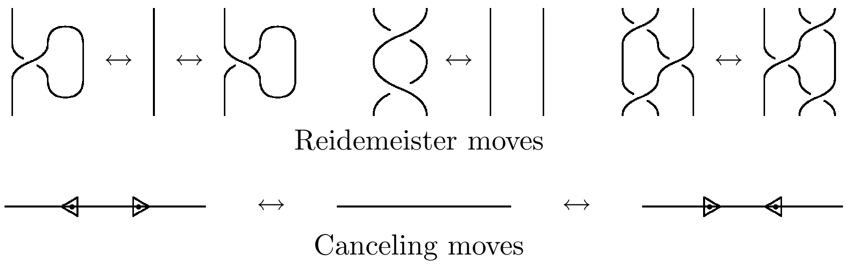


FIGURE 1

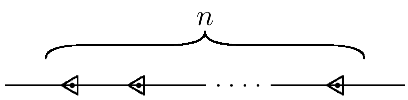
polynomial for virtual links [14, 15], the virtual magnetic skein module [7], and so on.

A *magnetic link/tangle diagram* is an oriented link/tangle diagram on $\mathbb{R}^2$ which may have oriented 2-valent vertices

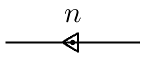
 and 

which we call nodes. A magnetic link/tangle diagram may also have a 4-valent vertex which is represented as a crossing in the diagram. Two diagrams are called *equivalent* if one can be transformed into the other by a finite sequence of Reidemeister moves and the canceling moves (Figure 1). We omit orientations of strands in Figure 1. A *magnetic link/tangle* is an equivalence class of magnetic link/tangle diagrams under the moves.

We denote the join of n nodes with the same orientation

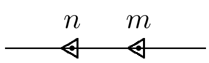


by the triangle labeled n

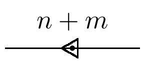


.

For a positive integer n, a triangle labeled $-n$ indicates one labeled n with the reversed orientation. Then, by the canceling moves,



and



are equivalent for any integers n and m, which implies that a circle/path is parameterized by an integer.

3. Smoothing resolution formula

In this section, we introduce a bracket polynomial of a magnetic $(1,1)$-tangle diagram, which is a single-input, single-output magnetic tangle diagram as shown in Figure 2. We show a relationship between the Alexander–Conway polynomial and the bracket polynomial, and evaluate the bracket polynomial of the trefoil knot as an example of this relationship.

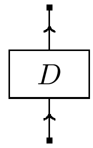
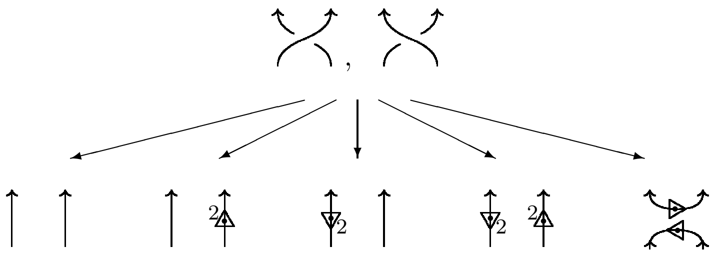


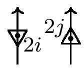
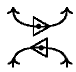
FIGURE 2

We define a bracket polynomial $\langle D \rangle$ of a magnetic (1,1)-tangle diagram D as a state sum. A *state* σ of D is an assignment of an element in the set $\{E_{00}, E_{01}, E_{10}, E_{11}, E_\infty\}$ to each crossing:



E_{00} E_{01} E_{10} E_{11} E_∞ .

We denote by D_σ the magnetic (1,1)-tangle diagram which is obtained by replacing each crossing with


or


in accordance with a state σ. Let $C(D_\sigma)$ be the set of connected components of D_σ. It consists of some circles and a path. For a state σ, we denote the path by p_σ.

For any integer k, set

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}},$$

and

$$d := \sqrt{-1}(q - q^{-1}).$$

For a state σ, we define the *weight* $\mathrm{wt}(v;\sigma)$ of a crossing v by

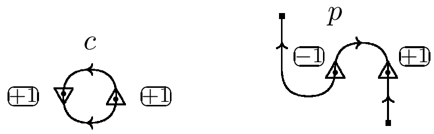
$$\mathrm{wt}(v;\sigma) = \begin{cases} a_{ij}/d & \text{if } v \text{ is a positive crossing to which } E_{ij} \text{ is assigned,} \\ a'_{ij}/d & \text{if } v \text{ is a negative crossing to which } E_{ij} \text{ is assigned,} \\ 1 & \text{otherwise,} \end{cases}$$

where

$$(a_{00}, a_{01}, a_{10}, a_{11}) = (\, q^{-2}, -q^{-1}, -q, 1),$$
$$(a'_{00}, a'_{01}, a'_{10}, a'_{11}) = (\;\, q^{2}\;, -q^{-1}, -q, 1).$$

We define the bracket polynomial $\langle D \rangle$ of a magnetic (1,1)-tangle diagram D as the following state sum:

$$\langle D \rangle = \sum_{\sigma\colon \text{state}} \Big(\prod_{v\colon \text{crossing}} \mathrm{wt}(v;\sigma) \Big) \Big([\vec{p}_\sigma + 1] \prod_{c \in C(D_\sigma) \setminus \{p_\sigma\}} [\vec{c}] d \Big),$$



$\vec{c} = (1+1)/2 = 1,$
$\vec{p} = (-1+1)/2 = 0.$

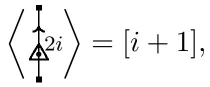
FIGURE 3

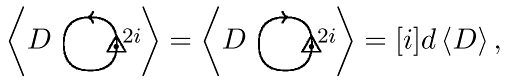
where the integer $\vec{x}$ is defined as follows. We assign a sign $+1$ or -1 to each node on a circle/path x:

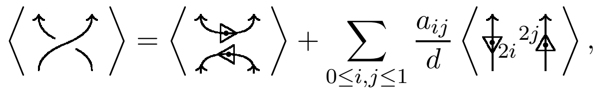
where the dotted area indicates inside/left-hand side of the circle/path x. Sum up signs of nodes on the circle/path x. Then the integer $\vec{x}$ is half of this sum (see Figure 3).

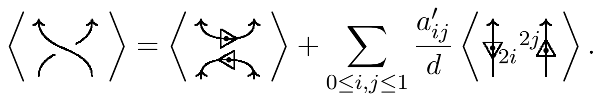
By the definition of the bracket polynomial $\langle D \rangle$ of a magnetic (1,1)-tangle diagram D, we have the following proposition.

Proposition 1. *Let D be a magnetic $(1,1)$-tangle diagram. The bracket polynomial $\langle D \rangle$ is characterized by the following relations up to the canceling moves:*

(1) 

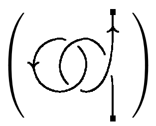
(2) 

(3) 

(4) 

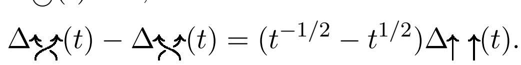
The equalities (3) and (4) are relations among the bracket polynomials of magnetic (1,1)-tangle diagrams which are identical except in the neighborhood of a point where they are the magnetic tangle diagrams depicted in the brackets.

The Whitney degree $\mathrm{rot}(D)$ of an oriented link/tangle diagram D is the total turn of the tangent vector to the curve as one traverses it in the given direction. For example,

rot  $= 2.$

For an oriented link L, the Alexander–Conway polynomial $\Delta_L(t)$ is defined by the following relations:

$\Delta_{\bigcirc}(t) = 1,$

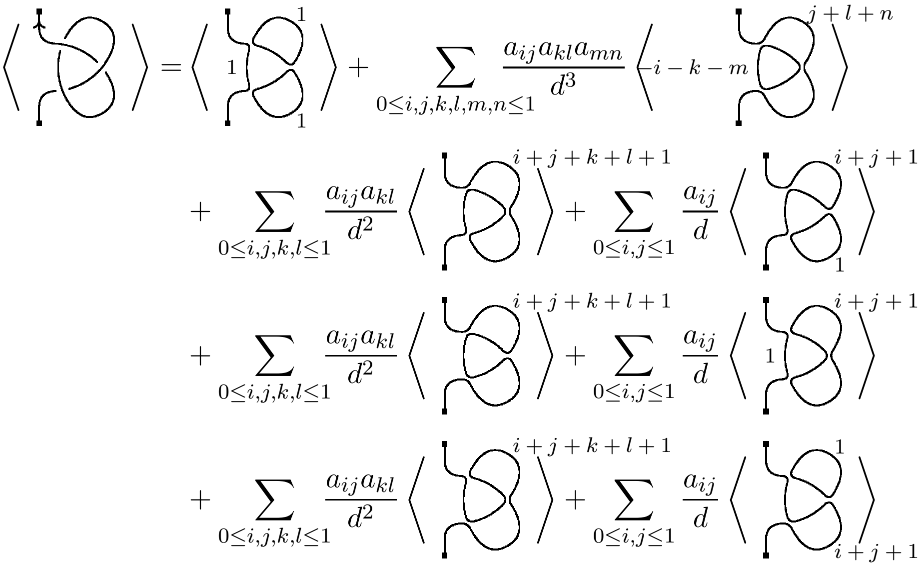


We obtain the Alexander–Conway polynomial from the bracket polynomial by utilizing the Whitney degree.

Theorem 2. *Let T be an oriented $(1,1)$-tangle represented by a diagram D. We denote by $\widehat{T}$ a link which is obtained by closing the $(1,1)$-tangle T. Then*

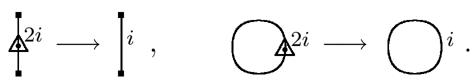
$$\Delta_{\widehat{T}}(t) = \left(q^{-\mathrm{rot}(D)} \langle D \rangle \right)\Big|_{q=\sqrt{-1}t^{1/2}}.$$

We give a proof of Theorem 2 in Section 5. At the end of this section, we evaluate the bracket polynomial of the trefoil knot as an example of Theorem 2:

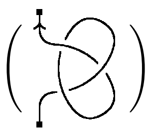


$$\begin{aligned}
=&[2][1]d[1]d + \sum_{0\le i,j,k,l,m,n\le 1} \frac{a_{ij}a_{kl}a_{mn}}{d^3}[j+l+n+1][-i-k-m]d \\
&+ 3\sum_{0\le i,j,k,l\le 1} \frac{a_{ij}a_{kl}}{d^2}[i+j+k+l+2] \\
&+ 2\sum_{0\le i,j\le 1} \frac{a_{ij}}{d}[i+j+2][1]d + \sum_{0\le i,j\le 1} \frac{a_{ij}}{d}[2][i+j+1]d \\
=&-q^3 - q - q^{-1},
\end{aligned}$$

where we use the simplified notation, omitting triangles, as follows:



And the Whitney degree of the $(1,1)$-tangle diagram is equal to 1:

rot  = 1.

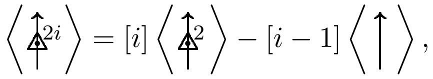
By Theorem 2, the Alexander–Conway polynomial of the trefoil knot is

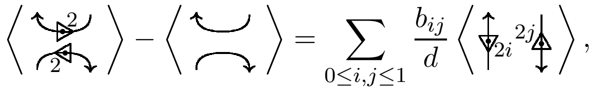
$$t - 1 + t^{-1}.$$

4. Some relations

Two relations, given in the following proposition, are used to show the invariance of the bracket polynomial.

Proposition 3. *We have the following equalities:*

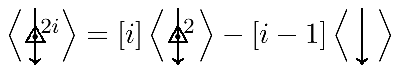
(5)


(6)


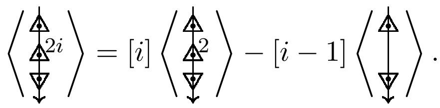
where

$$(b_{00}, b_{01}, b_{10}, b_{11}) = (-[2], 2, 2, -[2]).$$

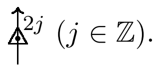
We remark that the equality



follows from the equality (5) immediately:



Proof. We show the equality (5). Let $D^{(2j)}$ ($j \in \mathbb{Z}$) be magnetic (1,1)-tangle diagrams which are identical except in the neighborhood of a point x where they are



Then we may describe the equality (5) as

$$\langle D^{(2i)} \rangle = [i]\langle D^{(2)} \rangle - [i-1]\langle D^{(0)} \rangle.$$

By the equalities (1) and (2), we have

$$\langle D^{(2j)} \rangle = \sum_{\sigma\colon \text{state}} \Big\{ \prod_{v\colon \text{crossing}} \text{wt}(v;\sigma) \Big\} \langle D^{(2j)}_\sigma \rangle.$$

Hence it is sufficient to show the equality

$$\langle D^{(2i)}_\sigma \rangle = [i]\langle D^{(2)}_\sigma \rangle - [i-1]\langle D^{(0)}_\sigma \rangle \tag{7}$$

for each state σ, where we identify states of $D^{(2i)}$, $D^{(2)}$ and $D^{(0)}$ by the same symbol σ.

We first suppose that the point x is on the path p_σ. Then we have two cases shown in Figure 4. In the case (i), the equality (7) follows from the equality (1) and the following equality:

$$[k+i+1] = [i][k+1+1] - [i-1][k+1],$$

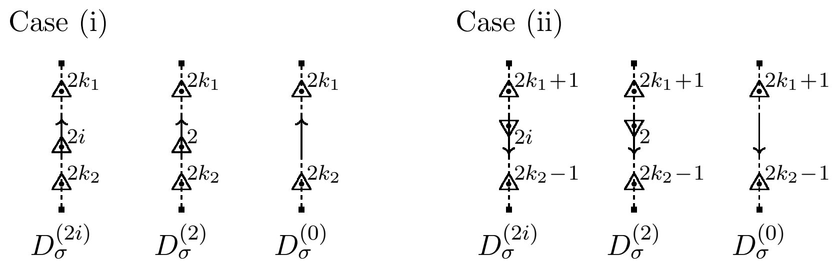


FIGURE 4

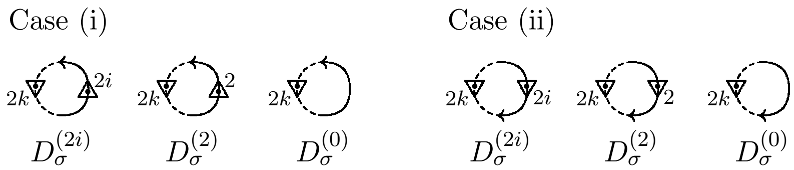


FIGURE 5

where $k = k_1 + k_2$. In the case (ii), the equality (7) follows from the equality (1) and the following equality:

$$[k - i + 1] = [i][k - 1 + 1] - [i - 1][k + 1],$$

where $k = k_1 + k_2$.

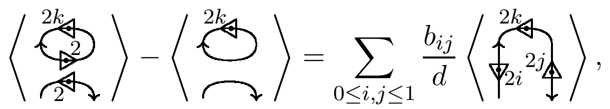
We next suppose that the point x is on a circle. Then we have two cases shown in Figure 5. In the case (i), the equality (7) follows from the equality (2) and the following equality:

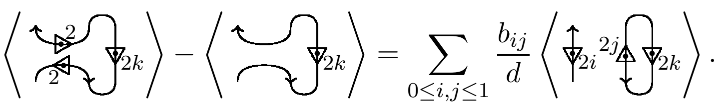
$$[k + i]d = [i][k + 1]d - [i - 1][k]d.$$

In the case (ii), the equality (7) follows from the equality (2) and the following equality:

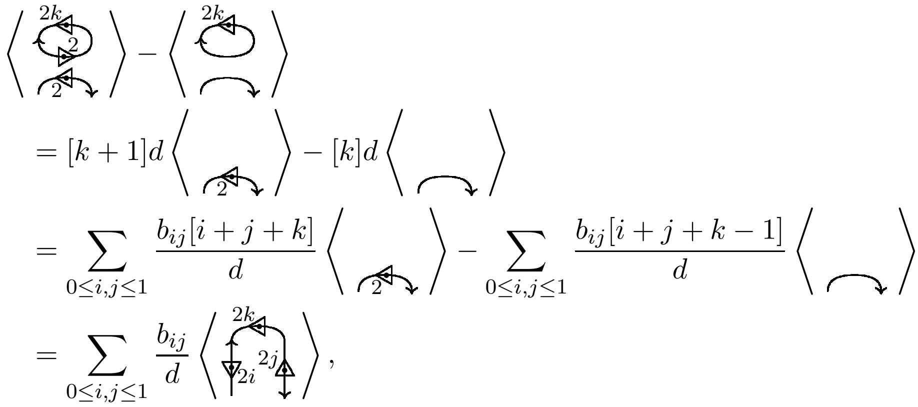
$$[k - i]d = [i][k - 1]d - [i - 1][k]d.$$

We show the equality (6). As in the above case, it is sufficient to show the equality (6) for each state. Then we show the following equalities:

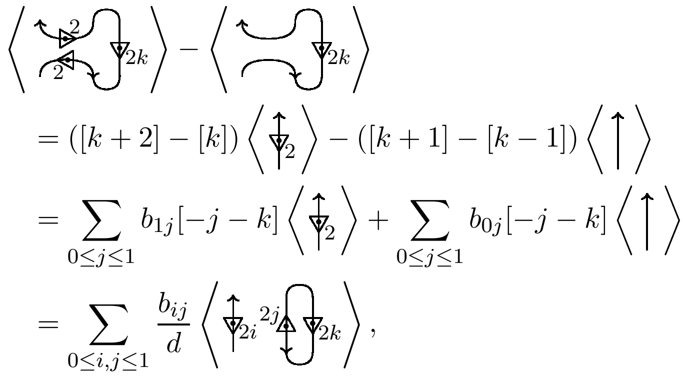
(8) 

(9) 

We have the equality (8) as follows:



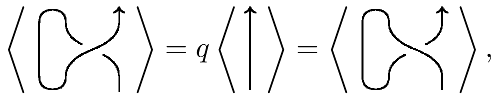
where the first equality follows from the equality (2), and the last equality follows from the equality (5). We have the equality (9) as follows:

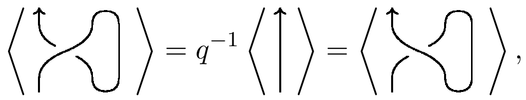


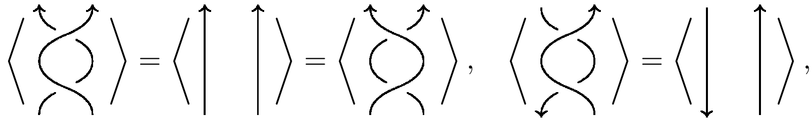
where the first equality follows from the equality (5) and the last equality follows from the equality (2). ☐

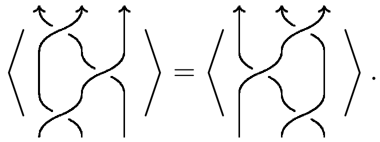
5. Proof of Theorem 2

Lemma 4. *We have the following equalities:*

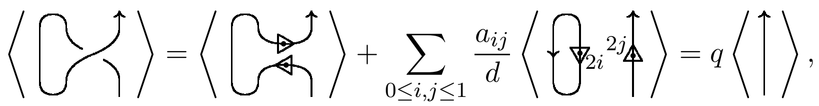
(10) 

(11) 

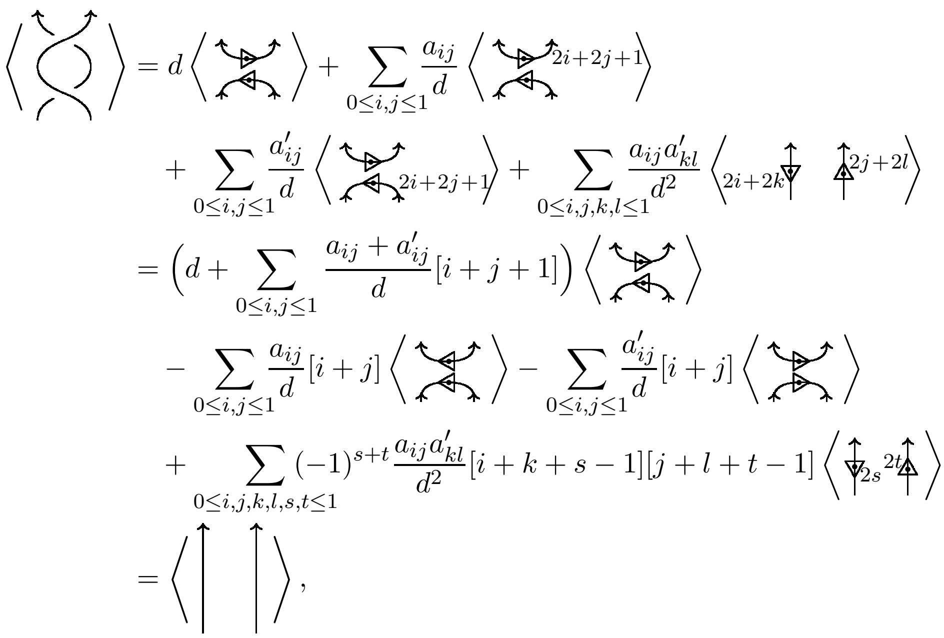
(12) 

(13) 

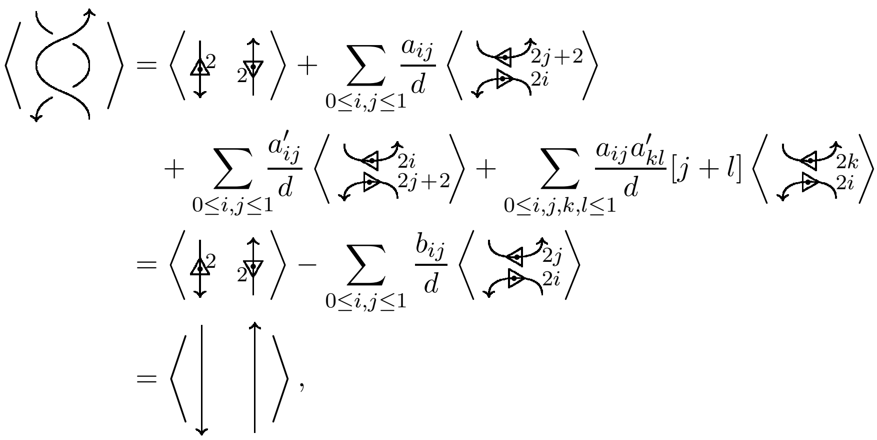
Proof. We have the first equality of (10):



where the first equality follows from the equality (3), and the second equality follows from the equality (2). The second equality of (10) and the equalities of (11) are similarly obtained. We have the first equality of (12):

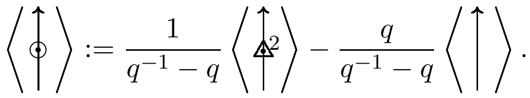


where the first equality follows from the equalities (2)–(4), and the second equality follows from the equality (5). The second equality of (12) is similarly obtained. We have the third equality of (12):

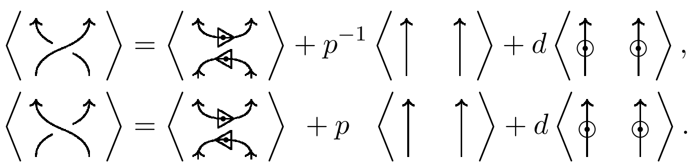


where the first equality follows from the equalities (2)–(4), and the second equality follows from the equality (5), and the last equality follows from the equality (6).

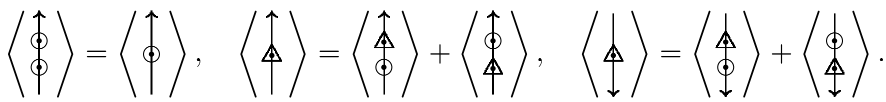
Before we verify the equality (13), we introduce a notation $\odot$ in the bracket for simplicity in calculation:



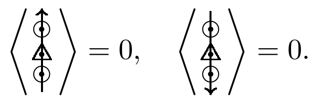
Set $p := q/\sqrt{-1}$. The benefit from this notation is that we may reduce the number of terms in the smoothing resolution formula:



The following equalities are also used to verify the equality (13):

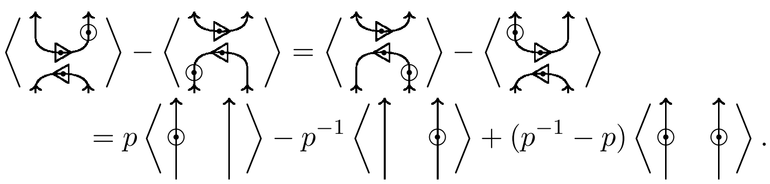


By using these equalities, we have

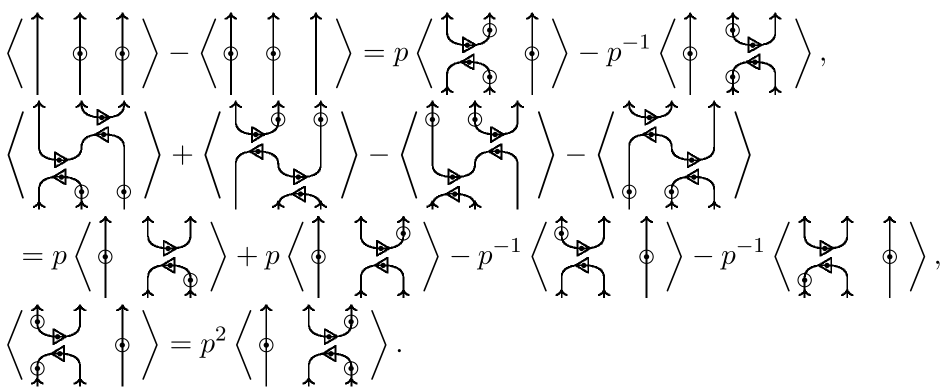


Furthermore the equality (6) implies

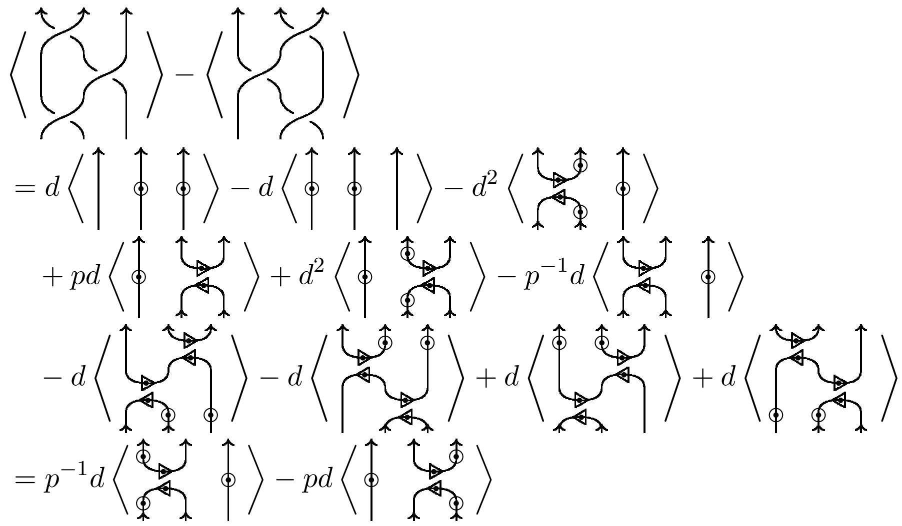
(14)



Now, we show the equality (13). By the equality (14), we have the following equalities



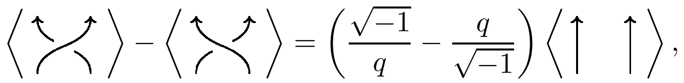
Then we have



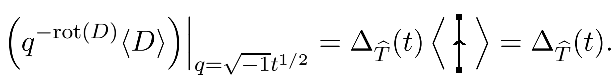
$= 0.$

∎

Proof of Theorem 2. Our bracket polynomial is a regular isotopy invariant of an oriented $(1,1)$-tangle since the equalities (10)–(13) imply the invariance of the bracket polynomial under the Reidemeister moves with the other orientations. Then, by the equalities (10) and (11), $q^{-\mathrm{rot}(D)}\langle D\rangle$ is an isotopy invariant of T. Since the equalities (3) and (4) imply



the isotopy invariant $q^{-\mathrm{rot}(D)}\langle D\rangle$ satisfies the defining relation of the Alexander–Conway polynomial. Hence we have



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