ACTA MATHEMATICA VIETNAMICA 291
Volume 33, Number 3, 2008, pp. 291–320

NOTES ON YOSHIDA'S COORDINATES ON HITCHIN'S PRYM COVER

SØREN KOLD HANSEN AND DANIEL MOSKOVICH

Abstract. As the first stage of his proposed geometric quantization of the $SU(2)$ WZW model, T. Yoshida introduced coordinates on a Prym variety which covers the moduli space of semi-stable rank 2 vector bundles with trivial determinant over a Riemann surface [13]. We explain Yoshida's coordinates, and reprove their key properties using elementary combinatorial arguments.

1. Introduction

Let M_g be the moduli space of semi-stable rank 2 holomorphic vector bundles with trivial determinant over a Riemann surface Σ_g of genus $g \geq 2$. A *rank* 2 *theta function* is a holomorphic section in $H^0(M_g, \mathscr{L}^k)$ where $\mathscr{L}$ is the determinant line bundle of M_g and k is a positive integer. Rank 2 theta functions are known to correspond to conformal blocks of the $SU(2)$ WZW model [4].

Hitchin constructed a morphism π from a *Prym variety* $\mathrm{Prym}(\tilde{\Sigma}_g)$ to M_g [8]. It is a covering map of degree 2^{3g-3}, whose branching is generically simple [4, 11]. The Prym variety $\mathrm{Prym}(\tilde{\Sigma}_g)$ is associated to a simple double branched cover $\tilde{\Sigma}_g$ of Σ_g, which is in turn associated to the determinant of a generic *Higgs field* $\det\Phi \in H^0(\Sigma_g, K_\Sigma^{\otimes 2}) \simeq \mathbb{C}^{3g-3}$ where K_Σ is the canonical line bundle of Σ_g. Atiyah [2] and Hitchin [8] posed the challenge to identify rank 2 theta functions as *covariant constant* sections of $\mathrm{Prym}(\tilde{\Sigma}_g)$ with respect to varying $\det\Phi$ along closed paths. Since a Prym variety is an abelian variety, this would identify rank 2 theta functions in terms of 'standard' abelian theta functions.

In a recent paper in Annals of Mathematics, Tomoyoshi Yoshida sets out to realize the abelianization programme of Atiyah and Hitchin [13]. His strategy is to first choose a mapping from $\mathrm{Prym}(\tilde{\Sigma}_g)$ to an abelian variety $\mathrm{Prym}^0(\tilde{\Sigma}_g)$ parameterizing degree 0 line bundles over $\tilde{\Sigma}_g$ which are anti-invariant with respect to the action of the covering involution σ of $\tilde{\Sigma}_g$. Such a choice corresponds to a choice of one of 2^g possible basepoints for $\mathrm{Prym}(\tilde{\Sigma}_g)$. The variety $\mathrm{Prym}^0(\tilde{\Sigma}_g)$ can explicitly be described as an abelian variety in terms of quotients of homology groups, and may be given coordinates. However, $\mathrm{Prym}^0(\tilde{\Sigma}_g)$ is not principally polarized, so Yoshida takes a 2^{2g-3}–fold covering space $P(\tilde{\Sigma}_g)$ of $\mathrm{Prym}^0(\tilde{\Sigma}_g)$ which is principally polarized. The relationship between the spaces is given below:

Received January 31, 2008; in revised form September 15, 2008.

$$\begin{array}{ccc} & \mathrm{Prym}(\tilde{\Sigma}_g) & \longrightarrow M_g \\ & \downarrow & \\ P(\tilde{\Sigma}_g) \longrightarrow & \mathrm{Prym}^0(\tilde{\Sigma}_g) & \end{array} \tag{1}$$

The principally polarized abelian variety $P(\tilde{\Sigma}_g)$ has a natural choice of coordinates, and a frame for its bundle of holomorphic sections is given by 'standard' Riemann theta functions with characteristics. Such sections descend to sections over $\mathrm{Prym}^0(\tilde{\Sigma}_g)$ if they are invariant with respect to the covering transformations, and pull back from there to sections over $\mathrm{Prym}(\tilde{\Sigma}_g)$ if they are invariant with respect to change of basepoint from among the 2^g possibilities. Yoshida will then construct pullbacks of rank 2 theta functions by considering covariant constant sections with respect to varying $\det\Phi$ along closed paths.

Behaviour of the sections as we pull them back and push them forward through Diagram 1 is determined by the behaviour of the coordinates. The combinatorial heart of Yoshida's approach is thus the pulled-back coordinates on $\mathrm{Prym}(\tilde{\Sigma}_g)$ and their behaviour as we vary $\det\Phi$ along closed paths. These coordinates are of independent interest also— see e.g. [6], where they are used to find representation matrices for the action of a certain Heisenberg group on the space of conformal blocks following [1, 5].

In this paper we explain Yoshida's coordinates in an elementary combinatorial way. No claim of originality is made, and everything in this paper (with the exception of Lemma 1) is contained in some form or another, implicitly or explicitly, in [13], Sections 2, 3, and 4c.

The outline of the paper is as follows. In Section 2 we summarize the algebraic geometric background for Yoshida's parametrization, following [3, 8, 9, 10]. The upshot of this section is that (when a generic Φ and a compatible pants decomposition is fixed) we may identify $Prym^0(\tilde{\Sigma}_g)$ as an abelian variety with $H^0(\omega^-_{\tilde{\Sigma}_g})^*/H_1(\Sigma_g;\mathbb{Z})^-$ (Equation 18). In Section 3, $H_1(\Sigma_g;\mathbb{Z})^-$ is identified as the sum of two lattices Λ and Λ_0^*. This identification depends on choices of orientations for curves in the lattices. We prove that Λ is indeed a maximal lattice (Corollary 3.1) (for Λ_0^* the corresponding property is obvious) and calculate an explicit basis for it (Lemma 1). This tells us (in principle at least) how one may translate from an expression on $P(\tilde{\Sigma}_g) \simeq H^0(\omega^-_{\tilde{\Sigma}_g})^*/(\Lambda_0+\Lambda_0^*)$ to an expression on $Prym^0(\tilde{\Sigma}_g) \simeq H^0(\omega^-_{\tilde{\Sigma}_g})^*/(\Lambda+\Lambda_0^*)$. Next, in Section 4 we show how change of basepoint from amongst the 2^g possibilities acts by translation on coordinates by an element of $\frac{1}{2}\Lambda^*/\Lambda_0^*$ (Yoshida's *shift operator*). This tells us how to translate from an expression in coordinates over $Prym^0(\tilde{\Sigma}_g)$ to an expression over $Prym(\tilde{\Sigma}_g)$, and gives Yoshida's coordinates for $Prym(\tilde{\Sigma}_g)$. Finally, in Section 5 we 'activate' Φ and find formulae for how motions of $\det\Phi$ along closed paths act on the coordinates of $Prym(\tilde{\Sigma}_g)$. A choice of an element of the fundamental group of the space of simply branched double covers of Σ_g associated to Υ is called a *marking*. We find generators for markings in Lemma 2. The conclusion of the paper is Proposition 5.1 which show how a marking induces Yoshida's coordinates on $\mathrm{Prym}(\tilde{\Sigma}_g)$ (and therefore also a complex structure on it) by fixing the arbitrary choices we made in its construction— choice of basepoint and choice of orientations of curves in Λ and in Λ_0^*.

In future work we would like to use the combinatorial tools in this paper to reproduce Yoshida's results on expressing rank 2 theta functions in terms of sections on $Prym(\tilde{\Sigma}_g)$. We would then like to consider manifolds with boundary, when Φ is allowed to have simple poles. The author's motivation is to investigate the 3–manifold invariants (one for each quantum level) which Yoshida seems to have constructed using his abelianization results, and to expand them to full TQFTs and investigate their relationship with the $SU(2)$ Reshetikhin–Turaev invariants.

2. Background

The following section summarizes the context into which Yoshida's coordinates fit. The material it contains is standard, and is collected from [3, 8, 9, 10]. In Section 2.1 we summarize Atiyah and Hitchin's abelianization programme, which Yoshida's paper addresses, explaining how the Prym cover $\pi\colon Prym(\tilde{\Sigma}_g) \to M_g$ arises. Then in Section 2.2 we review the construction of the Prym variety, concluding with it's non-canonical identification as a quotient of homology groups in Equation 18. This construction depends on fixing a choice of pants decomposition which is compatible in a suitable sense with Φ, which allows for the explicit combinatorics of Yoshida's approach.

2.1. **Atiyah and Hitchin's Abelianization Programme.**

In this section we review the basic idea of Atiyah and Hitchin's abelianization programme for $SU(2)$ (see [2, 8]). The purpose is to give context for Yoshida's results and to fix notation.

Let M_g denote the moduli space of semi-stable rank 2 holomorphic vector bundles with trivial determinant on Σ_g, a Riemann surface of genus $g \geq 2$ with a fixed complex structure, and let $\mathscr{L}$ denote the determinant line bundle over M_g. A holomorphic section in $H^0(M_g, \mathscr{L}^k)$ is called a *rank 2 theta function* (in other places it has been called other names *e.g.* a *generalized theta function*). Rank 2 theta functions have a number of alternative descriptions, most notably as *conformal blocks* of the $SU(2)$ WZW model [4].

The cotangent space to M_g at a point E is given by

$$T_E^* M_g = H^0(\mathrm{End}(E) \otimes K_\Sigma), \tag{2}$$

where K_Σ denotes the canonical line bundle of Σ_g. Inspired by physics, Hitchin [8] considered a slightly larger space consisting of *Higgs bundles* which are pairs (E, Φ) where E is a rank 2 vector bundle with trivial determinant over Σ_g, and Φ (the *Higgs field*) is a holomorphic section in $H^0(\mathrm{End}(E) \otimes K_\Sigma)$. Stability of

(E,Φ) is defined to mean that for every Φ–invariant rank 1 sub-bundle L of E

$$\deg L < \frac{1}{2}\deg(\wedge^2 E). \tag{3}$$

For a stable E any pair (E,Φ) is stable, but the converse is not true. Nevertheless, the natural analytic structure on the cotangent space to the space of stable bundles in M_g, denoted T^*M_{gs}, induces an analytic structure on the space of stable Higgs bundles and then on the space of Higgs bundles (E,Φ). Equipped with this structure, the space of Higgs bundles is called the *Hitchin moduli space*, and is denoted $\mathcal{M}$.

The map

$$\begin{aligned} \chi\colon\ \mathcal{M} &\longrightarrow H^0(\Sigma_g, K_\Sigma^{\otimes 2}), \\ (E,\Phi) &\mapsto \det\Phi \end{aligned} \tag{4}$$

is called the *Hitchin fibration*, and has some nice properties:

(i) It is proper (the pre-image of each compact set is compact) and surjective.
(ii) The generic fibre is an abelian variety.
(iii) $\dim M_g = \dim H^0(\Sigma_g, K_\Sigma^{\otimes 2})$ and $H^0(\Sigma_g, K_\Sigma^{\otimes 2}) \simeq \mathbb{C}^{3g-3}$.
(iv) M_g is an irreducible component of $\chi^{-1}(0)$.

A point $q \in H^0(\Sigma_g, K_\Sigma^{\otimes 2})$ is a quadratic differential over Σ_g, and thus generically has $4g-4$ distinct zeros $x_1,\ldots,x_{4g-4}$ on the Riemann surface. Fix Φ such that $\det\Phi = q$. The space of solutions in $T^*\Sigma_g$ of

$$\lambda^2 = \chi(\Phi) \tag{5}$$

defines a double cover $\mathrm{pr}\colon \tilde{\Sigma}_g \to \Sigma_g$ simply branched over $x_1,\ldots,x_{4g-4}$ whose genus by the Riemann–Hurwitz formula is $4g-3$. This is the most basic example of a *spectral cover*.

The abelian variety which is the generic fibre of χ is the subvariety of the Jacobian of $\tilde{\Sigma}_g$ consisting of degree $2g-2$ line bundles E on $\tilde{\Sigma}_g$ such that pr^*E has trivial determinant. This subvariety is called the *Prym variety* and is denoted $Prym(\tilde{\Sigma}_g)$.

The above discussion concerned generic Higgs fields Φ for which the zeros of $\det\Phi$ are distinct. At the opposite extreme, when $\Phi = 0$, we recover M_g. Thus M_g appears as the 'maximal degeneration' of a family of Prym varieties. There is a natural line bundle $\mathcal{L}$ on $\mathcal{M}$ whose restriction to M_g gives $\mathscr{L}$. Sections of $\mathcal{L}$ over the fibres of χ define a vector bundle over the generic points of $H^0(\Sigma_g, K_\Sigma^{\otimes 2})$— the points whose corresponding zeros over Σ_g are all distinct. Sections of $\mathscr{L}^k$ can be pulled back to T^*M_g and then extended over all of $\mathcal{M}$, implying that rank 2 theta functions can be expressed in terms of functions over Prym varieties (Riemann theta functions). The vector bundle over $H^0(\Sigma_g, K_\Sigma^{\otimes 2})$ has a projectively flat connection, so we can compare sections (at least projectively). With respect to this connection, sections over the Prym variety $Prym(\tilde{\Sigma}_g)$ which are pullbacks of sections over M_g should be covariant constant— *i.e.* they should be (at least projectively) constant as we vary $\det\Phi$ along closed paths.

Thus, Atiyah and Hitchin's abelianization programme for $SU(2)$ is to express rank 2 theta functions explicitly in terms of covariant constant sections on $Prym(\tilde{\Sigma}_g)$. It is that problem, to which Yoshida's paper seeks to address.

2.2. **The Prym variety.**

2.2.1. *Constructing the Prym variety.* Because $\mathrm{pr}\colon \tilde{\Sigma}_g \to \Sigma_g$ is a double cover of Σ_g, the field of rational functions on $\tilde{\Sigma}_g$ is a quadratic extension of that on Σ_g. Thus Σ_g has an open covering by $U_a := \mathrm{Spec}(R_a)$ such that $\mathrm{pr}^{-1}(U_a) = \mathrm{Spec}(S_a)$ with $S_a := R_\alpha[t_a] \big/ (t_\alpha^2 - \beta_a)$ for some $\beta_a \in R_a$.

In differential geometric terms, this means that there exists a line bundle $\mathfrak{d}$ of degree $2g-2$ over Σ_g such that

$$\mathrm{pr}_* \mathcal{O}_{\tilde{\Sigma}_g} = \mathcal{O}_{\Sigma_g} \oplus \mathfrak{d}^{-1}. \tag{6}$$

The multiplication induced by the quadratic extension on fields of rational functions is

$$(p \oplus l)(q \oplus m) = (pq + \phi(l \otimes m)) \oplus (pm + ql), \tag{7}$$

where p, q are sections of $\mathcal{O}_{\Sigma_g}$ ("constants"), and l, m are sections of $\mathfrak{d}^{-1}$ for some

$$\phi\colon\ \mathfrak{d}^{-2} \xrightarrow{\sim} \mathcal{O}_{\Sigma_g - \sum_{i=1}^{4g-4} x_i} \subset \mathcal{O}_{\Sigma_g}. \tag{8}$$

Then the zeros of β_a, or equivalently the points where $\phi(\mathfrak{d}^{-2}) \neq \mathcal{O}_\Sigma$, are the branch points $\mathbf{b} = \{x_1, x_2, \ldots, x_{4g-4}\}$ of the projection map. Alternatively

$$\mathfrak{d}^{-2} = \sum_{i=1}^{4g-4} [x_i], \tag{9}$$

where here and in the future we freely confuse line bundles and the divisors to which they correspond.

Now let $J\Sigma_g$ and $J\tilde{\Sigma}_g$ denote the Jacobians of Σ_g and of $\tilde{\Sigma}_g$ correspondingly. These parameterize line bundles of degree 0 on Σ_g and on $\tilde{\Sigma}_g$ correspondingly, and may naturally be given the structure of abelian varieties (complex tori). Topologically,

$$J\Sigma_g = H^0(\omega_{\Sigma_g})^* / H_1(\Sigma_g; \mathbb{Z}), \tag{10}$$

where $H^0(\omega_{\Sigma_g})^* := \mathrm{Hom}(H^0(\omega_{\Sigma_g}), \mathbb{C})$ are dual holomorphic 1–forms over Σ_g. The space $H_1(\Sigma_g; \mathbb{Z})$ can be identified with a lattice in $H^0(\omega_{\Sigma_g})^*$ via the map:

$$\gamma\colon\ H^0(\omega_{\Sigma_g}) \longrightarrow \Sigma_g, \qquad \omega \mapsto \int_\gamma \omega. \tag{11}$$

The argument above can be repeated for $J\tilde{\Sigma}_g$.

We also consider the variety $J^{2g-2}\tilde{\Sigma}_g$ which parameterizes line bundles of degree $2g-2$ on $\tilde{\Sigma}_g$ and which is (non-canonically) isomorphic to $J\tilde{\Sigma}_g$ via the map $M \mapsto M \otimes L^{-1}$ where $M \in J^{2g-2}\tilde{\Sigma}$ and L is any fixed degree $2g-2$ line bundle over $\tilde{\Sigma}_g$.

Any pointed space maps to its Jacobian via the Abel–Jacobi map which we call μ:

$$p \mapsto \left\{\eta \mapsto \int_x^p \eta\right\} \quad \text{modulo periods,} \tag{12}$$

where x is the basepoint. This induces the *norm map* Nm such that

$$\begin{array}{ccc} \tilde{\Sigma}_g & \xrightarrow{\mu} & J\tilde{\Sigma}_g \\ \big\downarrow \text{pr} & & \big\downarrow \text{Nm} \\ \Sigma_g & \xrightarrow{\mu} & J\Sigma_g \end{array} \tag{13}$$

commutes.

From the preceding discussion and Equation 6, the determinant of a line bundle $L \in J^{2g-2}\tilde{\Sigma}_g$ is $\mathrm{Nm}(L) \otimes \mathfrak{d}^{-1}$. Hence, for a semi-stable degree $2g-2$ line bundle L over $\tilde{\Sigma}_g$ which satisfies $\mathrm{Nm}(L) = \mathfrak{d}$ we have $\mathrm{pr}_* L \in M_g$. This gives us a map

$$\pi\colon\ Prym^{ss}(\tilde{\Sigma}_g) := \left\{ L \in J^{2g-2}(\tilde{\Sigma}_g) \middle| \mathrm{Nm}(L) = \mathfrak{d} \right\} \longrightarrow M_g. \tag{14}$$

With the structure of an abelian variety induced from the Jacobian, $Prym^{ss}(\tilde{\Sigma}_g)$ (the moduli space of semi-stable degree $2g-2$ line bundles over $\tilde{\Sigma}_g$ which satisfies $\mathrm{Nm}(L) = \mathfrak{d}$) is called the (semi-simple part of the) Prym variety of Σ_g associated to $\tilde{\Sigma}_g$.

2.2.2. *Stability and semi-stability.* Let us say a few words about stability and semi-stability now so that we never need mention them again, after which we shall drop the superscript from the notation $Prym^{ss}(\tilde{\Sigma}_g)$. To recall, a vector bundle E is said to be *stable* (respectively, *semi-stable*) if its slope $\mu(E) := \frac{\deg E}{\mathrm{rank} E}$ is greater than (respectively greater or equal to) the slope of any proper subbundle E' of E. Conceptually, π should really be thought of as a map from the whole moduli space of degree $2g-2$ line bundles over $\tilde{\Sigma}_g$ which satisfies $\mathrm{Nm}(L) = \mathfrak{d}$ to M_g. The problem is that the latter space might not be a manifold. One standard way to overcome this difficulty is to use stacks rather than varieties, and the other way, which Yoshida uses, is to restrict to the subspace of M_g which is a manifold— which means that we must restrict to the semi-stable part of the

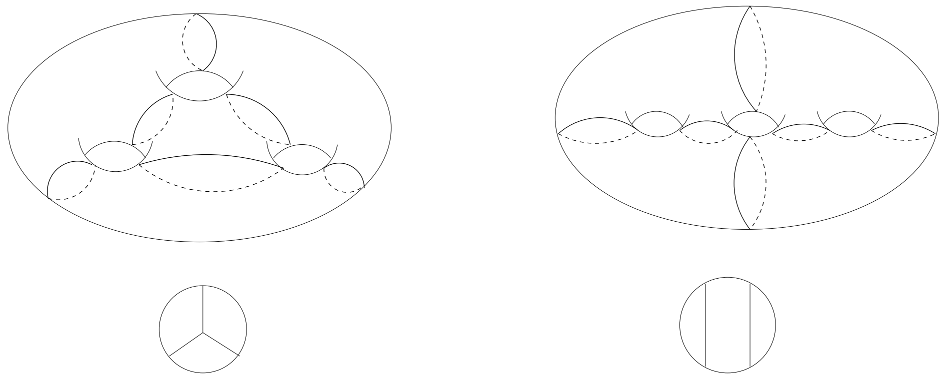


Figure 1. Two pants decompositions of a Riemann surface of type $(3,0)$ and their graphs.

source $Prym(\tilde{\Sigma}_g)$ as well. Hartog's theorem allows us to extend holomorphic sections over $Prym^{ss}(\tilde{\Sigma}_g)$ to holomorphic sections over $Prym(\tilde{\Sigma}_g)$, so for the purposes of Yoshida's paper we lose nothing be considering the semi-stable part of the Prym variety.

The map π is surjective in the sense which is required for our context. Namely, restricting to stable bundles, the map π restricts to a map $\pi_s\colon Prym^s(\tilde{\Sigma}_g) \longrightarrow M_{gs}$. This map is dominant (in other words, surjective on a Zariski open set) by [3, Corollary 1.5]. Now the fact that M_{gs} is Zariski dense in M_g combined with the result that $Prym(\tilde{\Sigma}_g) - Prym^{ss}(\tilde{\Sigma}_g)$ (respectively $Prym(\tilde{\Sigma}_g) - Prym^{s}(\tilde{\Sigma}_g)$) is a subvariety of $Prym(\tilde{\Sigma}_g)$ of codimension $\geq g+1$ (respectively $\geq g-1$) ([3, Lemme 1.2]) tells us that π is dominant, because the degree of $\mathfrak{d}$ is $2g-2$.

2.2.3. *Pants decompositions compatible with* Φ. Before we take the next step, we define a pants decomposition of Σ_g and specify the necessary compatibility condition with Φ. This constitutes the stage for Yoshida's combinatorial approach to be played out on.

A connected compact oriented surface is said to be of *type* (g,n) if it is of genus g and has n boundary components. For Σ_g, a *pants decomposition* is a choice of $3g-3$ disjointly embedded circles in Σ_g which divide it up into surfaces of type $(0,3)$ [7]. The closures of such surfaces are called *pairs of pants.* Let Σ_Υ denote the Riemann surface Σ_g equipped with the pants decomposition $\Upsilon := \{e_j, Y_i\}$ where $Y_1,\ldots,Y_{2g-2}$ are embeddings of pairs of pants into Σ_g which intersect at only at their boundary curves $e_1,\ldots,e_{3g-3}$. The pairs of pants are of course induced by the e_j's, but it is more convenient to include them explicitly as part of the data. For each pair of pants Y_i, fix an ordering of its three boundary curves, so we can write them as $e_{i(1)}$, $e_{i(2)}$ and $e_{i(3)}$ (so each boundary curve gets labeled twice).

For Yoshida's paper, Σ_Υ is fixed.

A pants decompositions of a surface of type $(g, 0)$ defines a cubic (or trivalent) graph with $2g-2$ vertices which represent pairs of pants and $3g-3$ arcs which represent their boundary components. This dual graph to a pair of pants decomposition Υ is denoted $G(\Upsilon)$, with the vertex corresponding to a pair of pants Y_i denoted v_i and the arc corresponding to a circle e_j also denoted e_j where if we are referring to a curve in the Riemann surface or to an arc in the graph should be clear from context. The choice of orientation of the curves $\{e_i\}$ induces an orientation on the edges of $G(\Upsilon)$, which we choose and fix such that each vertex has either two edged going into it and one coming out of it, or the converse.

We next explain what it means for Υ to be compatible with Φ. Recall that $\det\Phi$ has $4g-4$ distinct zeros $\mathbf{b}{:=}\{x_1, \ldots x_{4g-4}\}$, which are the branch points of $\tilde{\Sigma}_g$ a simply branched double cover of Σ_g. We require two things.

First, we require that there be a pair of points from $\mathbf{b}$ in each pair of pants. Denote by x_1^i and x_2^i the pair of points in the pair of pants Y_i for $i = 1, \ldots, 2g-2$.

Secondly, to specify a double-cover of Σ_Υ simply branched over $\mathbf{b}$ is equivalent to specifying an element $\alpha \in H^1(\Sigma_g - \mathbf{b}; \mathbb{Z}_2) \simeq \mathrm{Hom}(H_1(\Sigma_g - \mathbf{b}; \mathbb{Z}_2), \mathfrak{S}_2)$ such that the evaluation of cohomology classes on homology classes $\langle \alpha, c_j\rangle$ is 1 (the non-unit of $\mathfrak{S}_2 \simeq \mathbb{Z}_2$ the symmetric group on 2 elements) for all j, where c_j denotes the class in $H_1(\Sigma_g - \mathbf{b}; \mathbb{Z}_2)$ represented by the boundary of a small disc centred at x_j. For Υ to be compatible with Φ, so that the double-cover can be built up by sticking together double-covers of pairs of pants, in addition we also require that the curves $\{e_i\}$ satisfy $\langle \alpha, [e_j]\rangle = 0$ (where $[e_j]$ is the homology class represented by e_j). The cohomology class α is called a *covering type* of Σ_Υ, and is determined by $\det\Phi$.

2.2.4. *Identification of* $Prym(\tilde{\Sigma}_g)$ *with* σ*–anti-invariant degree* 0 *line bundles over* $\tilde{\Sigma}_g$. We would like to write $Prym(\tilde{\Sigma}_g)$ explicitly as an algebraic variety. First, we transfer the problem to one concerning degree 0 line bundles. Given a pants decomposition Υ of Σ_g which is compatible with Φ, choose

$$L{:=} \sum_{i=1}^{2g-2} [x_1^i] \tag{15}$$

(one point from each pair of pants) and consider again the map $J^{2g-2}\tilde{\Sigma}_g \longrightarrow J\tilde{\Sigma}_g$ given by $M \mapsto M \otimes L^{-1}$ where $M \in J^{2g-2}\tilde{\Sigma}_g$. Denote by $Prym^0(\tilde{\Sigma}_g)$ the image of the Prym variety under this map. The line bundle $\mathfrak{d}^{-2}$, which corresponds to $\sum_{i=1}^{4g-4}[x_i]$, maps to the degree 0 line bundle corresponding to $\sum_{i=1}^{2g-2}\left([x_1^i] - [x_2^i]\right)$ which has a square root which we fix and (after tensoring with L to make it a point in $Prym(\tilde{\Sigma}_g)$) fix to be $\mathfrak{d}$.

Now, resetting the letter L, let L be any point in $Prym(\tilde{\Sigma}_g)$. We have $L = \mathfrak{d} \otimes L_0$ for some degree 0 line bundle L_0, so

$$\mathfrak{d} = \mathrm{Nm}(L) = \mathrm{Nm}(\mathfrak{d}) \otimes \mathrm{Nm}(L_0) = \mathfrak{d} \otimes \mathrm{Nm}(L_0). \tag{16}$$

Therefore $Prym(\tilde{\Sigma}_g)$ is (non-canonically) isomorphic to the connected component of the identity in $\ker \mathrm{Nm}$. Because for any divisor on $\tilde{\Sigma}_g$ it holds that $\mathrm{pr}^{-1}(\mathrm{pr}(L_0)) = \sigma L_0 \otimes L_0$, it follows that

$$\mathrm{pr}^*(\mathrm{Nm}(L_0)) = \sigma L_0 \otimes L_0. \tag{17}$$

Thus on the connected component of the identity on $\ker \mathrm{Nm}$, the covering involution σ acts by inversion. This allows us to (non-canonically) identify $Prym(\tilde{\Sigma}_g)$ with the subvariety of $J\tilde{\Sigma}_g$ consisting of the σ–anti-invariant degree 0 line bundles over $\tilde{\Sigma}_g$. Thus we have the non-canonical isomorphism

$$Prym(\tilde{\Sigma}_g) \simeq H^0(\omega^-_{\Sigma_g})^* / H_1(\Sigma_g;\mathbb{Z})^-, \tag{18}$$

where $H^0(\omega^-_{\Sigma_g})$ denotes the σ–anti-invariant part of $H^0(\omega_{\Sigma_g})$ and $H_1(\Sigma_g;\mathbb{Z})^-$ denotes the σ–anti-invariant part of $H_1(\Sigma_g;\mathbb{Z})$.

3. Λ_0 and Λ : lattices associated to cubic graphs

In Section 3.1 we identify $H_1(\Sigma_g;\mathbb{Z})^-$ as the sum of two lattices $\Lambda + \Lambda_0^*$, where Σ_g comes equipped with a pants decomposition which is compatible with Φ. This depends on a choice of orientations for the curves which make up Λ and Λ_0^*, which in this section we choose arbitrarily. The sum $\Lambda + \Lambda_0^*$ in no sense splits into the sum of a lattice and its dual, so in Section 3.2 we are led to consider $P(\tilde{\Sigma}_g) := H^0(\omega^-_{\Sigma_g})^* / (\Lambda_0 + \Lambda_0^*)$, the principally polarized abelian variety where the explicit Riemann theta functions live. This turns out to be a 2^{2g-3}–fold cover of $Prym^0(\tilde{\Sigma}_g)$. The interplay between Λ and Λ_0 is subtle and is combinatorially the most difficult part of Yoshida's paper.

As noticed by H. Fujita (see *e.g.* [6]), Λ and Λ_0 are essentially combinatorial objects related to cubic graphs (also known as trivalent graphs), and may be defined and studied in this context. Thus in Section 3.3 we take a step back from algebraic geometry and abelian varieties, and study Λ and Λ_0 'abstractly' as lattices associated to a cubic graph G. We prove that Λ is a maximal lattice (Corollary 3.1) and show that an explicit basis for it corresponds to a perfect matching for (bridgeless maximal subgraphs of) G (Lemma 1). During the proof of Corollary 3.1 we also calculate that the quotient group Λ / Λ_0 is isomorphic to $\mathbb{Z}_2^{2g-3}$.

3.1. Identifying $H_1(\Sigma_g;\mathbb{Z})^-$ as $\Lambda + \Lambda_0^*$.

In this section we identify $H_1(\Sigma_g;\mathbb{Z})^-$ as a sum of two lattices $\Lambda + \Lambda_0^*$. Recall that the pants decomposition Υ of Σ_g is compatible with Φ, so that any curve e_i in Υ satisfies $\langle \alpha, [e_j] \rangle = 0$ with respect to the covering type α. This implies that a pair of pants Y_i lifts to a surface of type $(0,6)$ in $\tilde{\Sigma}_g$. A generating set for the σ–anti-invariant classes in $H_1(\Sigma_g;\mathbb{Z})$ is represented by the curves in Figure 3 and in Figure 4. We fix an orientation for these curves arbitrarily.

Now define:

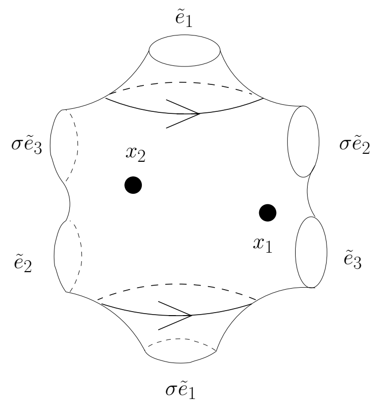


FIGURE 2. A curve representing $(\tilde{e}_i - \sigma\tilde{e}_i)$.

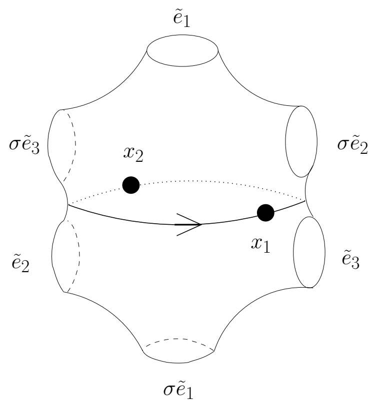


FIGURE 3. A curve representing E_i.

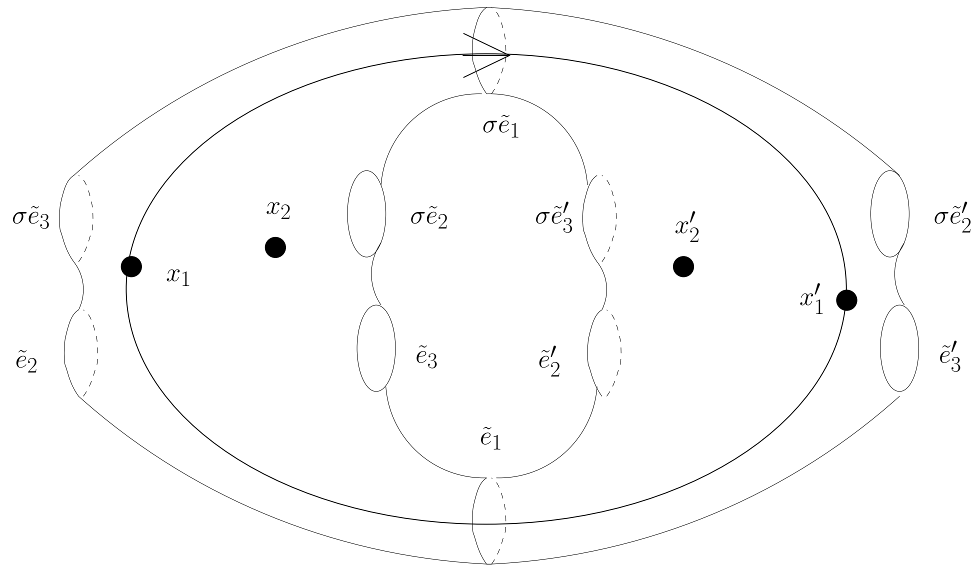


FIGURE 4. A curve representing e_i^*.

$$\Lambda := \operatorname{span}_{\mathbb{Z}} \left\{ E_i^j \right\}, \qquad \Lambda_0^* := \operatorname{span}_{\mathbb{Z}} \left\{ e_i^* \right\}. \tag{19}$$

It is not immediately clear why either of these should be a lattice, but clearly we have $H_1(\Sigma_g;\mathbb{Z})^- \simeq \Lambda + \Lambda_0^*$. To see that they are indeed lattices, and to treat them combinatorially, we map them into $\mathbb{R}^{3g-3}$.

Let $(\tilde{e}_1 - \sigma\tilde{e}_1), \ldots, (\tilde{e}_{3g-3} - \sigma\tilde{e}_{3g-3})$ be the symplectic dual curves to the e_i^*'s (the $\tilde{e}_i$ are in fact the lifts of the e_i curves of the pants decomposition to $\tilde{\Sigma}_g$).

Representatives of these are pictured in Figure 2. Note that the $(\tilde{e}_i - \sigma\tilde{e}_i)$'s are linearly independent (no sum of e_i's is zero in the homology of $\Sigma_g - \mathbf{b}$).

Define

$$\Lambda_0 := \text{span}_{\mathbb{Z}} \left\{ (\tilde{e}_i - \sigma\tilde{e}_i) \right\}. \tag{20}$$

The lattice $\mathbb{Z}^{3g-3} \subset \mathbb{R}^{3g-3}$ has a standard basis given by the vectors

$$\vec{e}_i := (0, \ldots, \overset{i}{1}, \ldots, 0). \tag{21}$$

Map $\tilde{e}_i - \sigma\tilde{e}_i$ to $\vec{e}_i$ for each i. This induces a bijection between Λ_0 and $\text{span}_{\mathbb{Z}} \left\{ \vec{e}_i \right\}$ which we also call Λ_0.

We construct another set of generators $\left\{ \vec{E}_i^j \right\}$ for $\mathbb{Z}^{3g-3} \subset \mathbb{R}^{3g-3}$ by the following formulae:

$$\begin{aligned} \vec{E}_1^j &:= \frac{1}{2}\left[-\vec{e}_{j(1)} + \vec{e}_{j(2)} + \vec{e}_{j(3)} \right], \\ \vec{E}_2^j &:= \frac{1}{2}\left[\vec{e}_{j(1)} - \vec{e}_{j(2)} + \vec{e}_{j(3)} \right], \\ \vec{E}_3^j &:= \frac{1}{2}\left[\vec{e}_{j(i)} + \vec{e}_{j(2)} - \vec{e}_{j(3)} \right]. \end{aligned} \tag{22}$$

This set of generators has cardinality $6g - 6$. Now mapping E_i^j to $\vec{E}_i^j$ for each $1 \le i \le 2g - 2$ and $j = 1, 2, 3$ is a bijection induced by the bijection above. Let $\text{span}_{\mathbb{Z}} \left\{ \vec{E}_i^j \right\}$ be called Λ as well.

The dual lattices of Λ_0 and of Λ with respect to the standard inner product on $\mathbb{R}^{3g-3} \oplus \mathbb{R}^{3g-3}$ are denoted Λ_0^* and Λ^*. The dual of $\vec{e}_i$ we call $\vec{e}_i^{\,*}$ (Yoshida calls it $\tilde{f}_k^*$) and the dual of $\vec{E}_i$ we call $\vec{E}_i^*$. Note that we have:

$$\Lambda^* = \left\{ \sum_{l=1}^{3g-3} n_l \vec{e}_l^{\,*} \;\middle|\; n_{i(1)} + n_{i(2)} + n_{i(3)} \in 2\mathbb{Z} \text{ for all } i \right\}, \tag{23}$$

where the coefficients are integers. Again, the vector representations of the lattices and the curve representations are isomorphic.

3.2. **The Principally Polarized Abelian Variety $P(\tilde{\Sigma}_g)$.** It is not even obvious yet that Λ is a maximal lattice (this will be proven in Section 3.3), but it is obvious that Λ and Λ_0^* are not dual and that their sum doesn't split into dual lattices. This means that $Prym^0(\tilde{\Sigma}_g)$ is not principally polarized, which is bad news if one wants to write Riemann theta functions explicitly on it.

However, $\Lambda_0 \subseteq \Lambda$ because:

(24)
$$\begin{cases} \vec{E}^i_1 + \vec{E}^i_2 = \vec{e}_{i(3)}, \\ \vec{E}^i_2 + \vec{E}^i_3 = \vec{e}_{i(1)}, \\ \vec{E}^i_3 + \vec{E}^i_1 = \vec{e}_{i(2)}, \end{cases}$$

where $i = 1, \ldots, 2g-2$. We may thus define the quotient Λ / Λ_0 which induces a finite sheeted covering

(25)
$$P(\tilde{\Sigma}_g) := H^0(\omega^-_{\tilde{\Sigma}_g})^* / (\Lambda_0 + \Lambda^*_0)$$

of $Prym^0(\tilde{\Sigma}_g) := H^0(\omega^-_{\tilde{\Sigma}_g})^* / (\Lambda + \Lambda^*_0)$ (in the proof of Lemma 3.1 we will see that this is in fact a 2^{2g-3}–fold covering).

3.3. **A Basis for Λ.** Clearly Λ_0 is a maximal lattice in $\mathbb{R}^{3g-3}$. In this section we prove that Λ is also a maximal lattice in $\mathbb{R}^{3g-3}$, in other words that $\Lambda \simeq \text{span}_{\mathbb{Z}} \left\{ \vec{D}_i \right\}$ where $\left\{ \vec{D}_i \right\}$ is a subset of $\left\{ \vec{E}^j_i \right\}$ of cardinality $3g-3$. We prove this twice, first in Lemma 1 where we construct the $\vec{D}_i$'s explicitly and show that a choice of a basis $\left\{ \vec{D}_i \right\}$ for Λ is equivalent to a choice of perfect matchings for bridgeless components of the cubic graph G (a set of non-adjacent arcs incident to every vertex). For planar graphs this corresponds to Yoshida's concept of a *grouping*. We then give a simpler and shorter proof (due to Fujita) that Λ is a lattice in Corollary 3.1 in which we calculate the quotient $\Lambda / \Lambda_0 \simeq \mathbb{Z}_2^{2g-3}$ which shows that $P(\tilde{\Sigma}_g)$ is a 2^{2g-3}–fold covering of $Prym^0(\tilde{\Sigma}_g)$.

Lemma 1. *A basis for Λ has $3g-3$ elements, and a choice of basis for Λ corresponds to a perfect matching for $G_1, \ldots, G_r$, the bridgeless components of G with separating arcs erased.*

Proof. Assume first that G is bridgeless. Choose a perfect matching

$$R := \{r_1, \ldots, r_{g-1}\} \subset E.$$

Such a matching exists by Petersen's Theorem [12, Theorem VII.29]. Hiding the arcs in R leaves us with a set of cycles $G - R := \{C_1, \ldots, C_k\}$. Orient all arcs on the cycles counter-clockwise, and fix a orientation of the arcs in R arbitrarily (for the moment). Re-index the edges of G as follows:
where the thick (vertical) line represents an arc in R, with either orientation, and $e_{i(j)}$ is shorthand for $e_{v_i(j)}$.

We consider a number of cases in ascending order of generality.

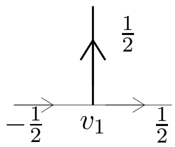
G **is Hamiltonian.** For pedagogical reasons we begin by considering the Hamiltonian case. Assume the vertices around the Hamiltonian cycle C are indexed cyclically, so that traveling clockwise around the cycle we go from vertex v_1 to vertex v_2 to vertex v_3, *etc.* Assume also that r_1 begins at vertex v_1, and ends at some vertex v_m.

We assemble a set of elements in Λ of cardinality $3g-3$, and show that together they generate Λ over $\mathbb{Z}$. To simplify notation set $\vec{E}_i^j{:=}\vec{E}_i^{v_j}$ for all $1 \leq i \leq 2g-2$ and $j = 1, 2, 3$. Call r_1 the *special edge* of G. We begin selecting elements for our generating set. Choose $\vec{E}_2^i$ and $\vec{E}_3^i$ for each v_i that is an outgoing vertex for one of $\{r_2, \ldots, r_{g-1}\}$, and choose $\vec{E}_1^1$ and $\vec{E}_3^1$. Choose $\vec{E}_3^j$ for all remaining vertices v_j of G. We have three generators corresponding to each element of R, thus our set of generators has the correct cardinality $3g-3$.

If r_l goes from v_i to v_j, we obtain the relation:

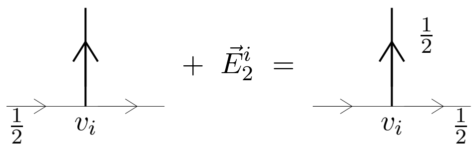
$$\vec{E}_2^i + \vec{E}_3^i = \vec{e}_{i(1)} = \vec{e}_{j(1)} = \vec{E}_2^i + \vec{E}_3^i. \tag{26}$$

Thus we can generate $\vec{E}_2$ and $\vec{E}_3$ for all endpoints of edges $r_2, \ldots, r_{g-1}$, *i.e.* for all vertices which are not endpoints of r_1. For v_1, draw $\vec{E}_2^1$ as follows:

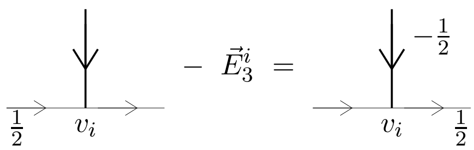


For all $i = 1, 2, \ldots, 2g-2$ have the following relations:

(i)



(ii)



Recall that $\vec{E}_2^1 = \frac{1}{2}(\vec{e}_{1(1)} - \vec{e}_{1(2)} + \vec{e}_{1(3)})$, and use the above relation for each vertex v_i in C, for $i = 2, 3, \ldots, 2g-2$, starting from $\frac{1}{2}\vec{e}_{1(3)} = \frac{1}{2}\vec{e}_{2(2)}$. At the last step, we obtain $\frac{1}{2}\vec{e}_{(2g-2)(3)} = \frac{1}{2}\vec{e}_{1(2)}$ which cancels with $-\frac{1}{2}\vec{e}_{1(2)}$ which was a summand in $\vec{E}_2^1$, plus a sum of $\frac{1}{2}\vec{e}_{i(1)}$ for each vertex v_i from which a red edge is outgoing and $-\frac{1}{2}\vec{e}_{i(1)}$ for each vertex v_i from which a red edge is incoming. These contributions cancel out since each red edge has both an outgoing vertex and an incoming vertex, the final cancelation being with $\frac{1}{2}\vec{e}_{1(1)}$ the final summand of $\vec{E}_2^1$. Define:

$$V_1' := \left\{ v_i \in V \,\middle|\, \text{the red edge is outgoing from } v_i \right\}, \tag{27}$$
$$V_2' := \left\{ v_i \in V \,\middle|\, \text{the red edge is incoming to } v_i \right\}.$$

Then we have obtained the following key archetypal relation between elements of Λ:

$$\sum_{\substack{v_i \in V_1' \\ v_j \in V_2'}} \vec{E}_2^i - \vec{E}_3^j = 0. \tag{28}$$

All elements in the above equation are in our set of generators or have been generated by them, save $\vec{E}_2^1$. Thus we may generate $\vec{E}_2^1$. Thus by Equation 26 we may in turn generate $\vec{E}_2^m$ for any m.

Thus we have generated $\vec{E}_2$ and $\vec{E}_3$ for all vertices. Finally, we have

$$\vec{E}_1^i + \vec{E}_2^i = \vec{e}_{i(3)} = \vec{e}_{(i+1)(2)} = \vec{E}_1^{i+1} + \vec{E}_3^{i+1} \tag{29}$$

which allows us to generate $\vec{E}_1$ for all vertices by taking $\vec{E}_1^1$ and using it to generate $\vec{E}_1^2$, then using that to generate $\vec{E}_1^3$, *etc.* The generation of Λ for the Hamiltonian bridgeless case is complete.

G **is bridgeless.** We drop the requirement that G be Hamiltonian. Choose a set of edges $D := \{d_1, \ldots, d_{k-1}\} \in R$ such that d_i goes from C_i to C_{i+1} (assume without the limitation of generality that such a choice exists, otherwise we re-index the cycles). The red edge adjacent to v_1 (assume without the limitation of generality that it is outgoing from v_1) is called the *special edge* and we denote it r_1 as before. Choose as generators $\vec{E}_2^i$ and $\vec{E}_3^i$ for all v_i ($i > 1$) from which an arc in $R - D$ is outgoing, and choose $\vec{E}_1^i$ and $\vec{E}_2^i$ for all v_i ($i > 1$) a starting point of an arc in D. Choose $\vec{E}_1^1$ for v_1 if $r_1 = d_1$, otherwise choose $\vec{E}_3^1$ for v_1 and choose $\vec{E}_1^i$ and $\vec{E}_2^i$ for the starting point of d_1. Choose $\vec{E}_1^m$ and $\vec{E}_3^m$ for the endpoint v_m of d_{k-1}. Choose $\vec{E}_3^j$ for all remaining vertices j of G. We have three generators corresponding to each element of R, thus our set of generators has the correct cardinality $3g - 3$.

By Equation 28 (which is true for the same reason here as it was true in the Hamiltonian case) we may generate $\vec{E}_2^1$. By Equation 26 we may generate $\vec{E}_2$ and $\vec{E}_3$ for all endpoints of edges in $R - D$, *i.e.* for all vertices which are not endpoints of $d_1, \ldots d_{k-1}$.

Thus for $k > 1$ (otherwise we are back in the Hamiltonian case), we may generate $\vec{E}_2$ and $\vec{E}_3$ for all vertices on C_1 save one, v_i the starting point of d_1. However $\vec{E}_1^i$ is an element in our set of generators, hence by Equation 29 we may generate $\vec{E}_1$ for all vertices by taking $\vec{E}_1^i$ and using it to generate $\vec{E}_1^{i+1}$ (numbering the vertices counter-clockwise along C_1 as in the Hamiltonian case), then using that to generate $\vec{E}_1^{i+2}$, *etc.* For n_1 the number of vertices in C_1, the final relation is:

$$\vec{E}_1^{n_1} + \vec{E}_2^{n_1} = \vec{e}_{n_1(3)} = \vec{e}_{i(2)} = \vec{E}_1^i + \vec{E}_3^i, \tag{30}$$

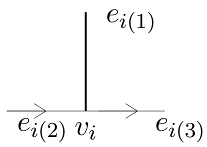
allowing us to generate $\vec{E}_3^i$. Thus we have generated all elements of Λ for all vertices on C_1. For v_l the endpoint of d_1, we have additionally

$$\vec{E}_2^i + \vec{E}_3^i = \vec{e}_{i(1)} = \vec{e}_{l(1)} = \vec{E}_2^l + \vec{E}_3^l, \tag{31}$$

allowing us to generate $\vec{E}_3^l$, and repeat everything for C_2. Repeat for C_3, C_4,... until C_k. Thus we have generated Λ for the bridgeless cubic graph case.

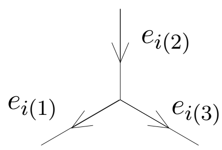
G **has separating arcs.** Finally, we consider the case where we have separating arcs. Such a graph minus the separating arcs breaks apart into bridgeless cubic graphs $G_1, \ldots, G_l$ where by convention we define the closed loop with no vertices as a trivial bridgeless cubic graph. Choose a perfect matching for each of these, and again refer to arcs in these matchings as 'red'.

The first stage is to re-index arcs of G. For the vertices in $G_1, \ldots, G_l$ such an indexing is induces by the indexing of each of these subgraphs as in the bridgeless case. For a separating edge s connecting to an edge G_i at vertex v_i:



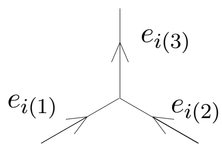
for both endpoints of s.

where the thick (vertical) line represents s. When three separating edges s_1, s_2, and s_3 connect at vertex v_i, and s_1 is incoming and s_2 and s_3 are outgoing from v_i:



where s_1, s_2 and s_3 meet.

and when s_1 and s_2 are incoming and s_3 is outgoing:



where s_1, s_2 and s_3 meet.

To obtain the analogue of Equation 28 for the general case, define:

(32)
$V_1^{''}{:=}\ \left\{v_i \in V \middle| v_i \text{ is adjacent to exactly one separating edge, which is outgoing}\right\}$
$V_2^{''}{:=}\ \left\{v_i \in V \middle| v_i \text{ is adjacent to exactly one separating edge, which is outgoing}\right\}$
and define:

(33)
$V_1^{'''}{:=}\ \left\{v_i \in V \middle| v_i \text{ is adjacent to one incoming and two outgoing separating edges}\right\}$
$V_2^{'''}{:=}\ \left\{v_i \in V \middle| v_i \text{ is adjacent to one outgoing and two incoming separating edges}\right\}$
Then define:

$$V_1{:=}\ V_1' \cup V_1^{''} \cup V_1^{'''},\quad V_2{:=}\ V_2' \cup V_2^{''} \cup V_2^{'''}. \tag{34}$$

The analogue of Equation 28 in the general case is:

$$\sum_{\substack{v_i \in V_1 \\ v_j \in V_2}} \vec{E}_2^i - \vec{E}_3^j = 0. \tag{35}$$

Now we choose generators for the general case. First, denote the genuses of $G_1, \ldots, G_l$ by $g_1, \ldots, g_l$ respectively, with the genus of a trivial bridgeless cubic graph set to 1 by convention. If G_1 is non-trivial, for vertices in G_1, the generators are those we would obtain by choosing generators of G_1 as a stand-alone bridgeless graph. If G_1 is trivial, choose no generators for it. We obtain $3g_1 - 3$ generators. For vertices in $G_2, \ldots, G_n$, make the same choices we would have made for them as 'stand-alone' bridgeless graphs, but with the special edge 'forgotten'. In other words, if G_r is Hamiltonian ($r > 1$), choose $\vec{E}_2^i$ and $\vec{E}_3^i$ for all v_i that is an outgoing vertex for one of $\{r_1, \ldots, r_{g-1}\}$ (note that this time we include r_1), and choose $\vec{E}_1^1$ and $\vec{E}_2^1$. Choose $\vec{E}_3^j$ for all remaining vertices j of G. If G_r is not Hamiltonian, we choose generators for v_1 as if it were any other starting point for a red arc, in D or otherwise, and choose all other generators precisely as before. Finally, if G_r is trivial, choose $\vec{E}_1^i$ for the vertex v_1 connecting it to a separating arc. In all cases we obtain $3g_r - 2$ generators.

It remains to choose generators for vertices adjacent to separating arcs. For $v_i \in V_1^{''} \cup V_1^{'''}$ choose $\vec{E}_2^i$ and for $v_i \in V^{''} \cup V_2^{'''}$ choose $\vec{E}_3^i$.

Using Equation 35 we may obtain one additional generator for G_1, putting it on 'equal footing' with $G_2, \ldots, G_r$.

For G_r with $r \geq 1$ with only a single separating arc adjacent to them, we can generate all of Λ restricted to G_r by arguments very similar to the preceding arguments, although the proof now breaks down into many cases:

If G_r is Hamiltonian and s is adjacent to a red arc (an arc is red if it was in the chosen perfect matching of G_r) and connects to the G_r at vertex v_i, call this red arc d, and for all red arcs except for d we may generate $\vec{E}_2$ and $\vec{E}_3$ for all incoming and outgoing vertices by Equation 26. We may generate all $\vec{E}_1$'s by Equation 29, a-priori for all vertices except v_1 and v_i. Now we know $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$ for two vertices adjacent to v_1 so therefore by Equations 26 and 29 we may deduce them for v_1, and repeating the argument we may also deduce them for v_i.

The second case is when G_r is Hamiltonian and s connects to an arc on C at vertex v_i, which we may assume to be the next vertex along C in a clockwise direction from v_1. Now we generate $\vec{E}_2$ and $\vec{E}_3$ for all vertices on G_r by Equation 26, and $\vec{E}_1$ for all vertices on C by Equation 29, where again we deduce $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$ from having generated them on both vertices adjacent to v_i.

Next, we have the non-Hamiltonian case, where s connects up to a red arc between cycles (between a cycle and itself is completely analogous to the Hamiltonian situation), which we may assume to be d_{k-1}. Generate elements of Λ on the cycles as we did before, using Equation 31 to move between cycle C_j and C_{j+1} for $j < k-1$. Here, because there is no special edge, we may generate all of Λ restricted to the first cycle C_1 just as we may generate it restricted to all others. Again, we have generated $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$ for two vertices adjacent to v_i and thus we may generate them for v_i.

If G_r is non-Hamiltonian and s connects to a cycle (assume C_k), the argument is virtually identical to the Hamiltonian case and is omitted.

Consider a separating arc s which connects to G_r at v_i, for whose vertex which is not on G_r we know $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$. Then by the analogue of Equation 26 we may generate $\vec{E}_2^i$ and $\vec{E}_3^i$. Thus if G_r connects via trees of separating edges to G_j's which are adjacent to only a single separating edge each, we may generate Λ restricted to G_r by the preceding arguments.

We generate all $\vec{E}_j$'s for vertices at which three separating edges meet by inductively generating all $\vec{E}_j$'s for two adjacent vertices by considering the appropriate cases of the ones listed above.

Having considered all cases, we have generated Λ for all cubic graphs using a generating set with $3g-3$ elements, and shown that the generating set is determined uniquely by the required perfect matchings. □

Because Λ generates $\mathbb{R}^{3g-3}$ over $\mathbb{R}$, it follows from Lemma 1 that Λ is a maximal lattice in $\mathbb{R}^{3g-3}$, in other words that $\Lambda \simeq \operatorname{span}_{\mathbb{Z}} \left\{\vec{D}_i\right\}$ where $\left\{\vec{D}_i\right\}$ is a subset of $\left\{\vec{E}_i^j\right\}$ of cardinality $3g-3$. This claim however has a simpler proof as pointed out by H. Fujita.

Corollary 3.1. *Λ is a maximal lattice in $\mathbb{R}^{3g-3}$.*

Proof.

(i) Λ_0 is a maximal lattice.

(ii) $\Lambda_0 \subseteq \Lambda$.

(iii) The quotient Λ/Λ_0 is finite. In this quotient we have the following relations $\vec{E}_i^k + \vec{E}_j^k = 0$ for $k = 1, 2, \ldots, 2g-2$ and $i, j = 1, 2, 3$ (equal or not). Thus for each vertex we have a single generator for Λ/Λ_0, and twice that generator is zero. There are no other relations, since such a relation would have to involve all pairs of pants, and would therefore be equivalent modulo the above relations to Equation 35 which holds already in Λ. Thus we have $[\Lambda : \Lambda_0] = 2^{2g-3}$ and

$$\Lambda/\Lambda_0 \simeq \mathbb{Z}_2^{2g-3}.$$

(iv) Λ generates $\mathbb{R}^{3g-3}$ over $\mathbb{R}$.

□

4. Change of basepoints

In Section 3 we discussed pushing forward coordinates from $P(\tilde{\Sigma}_g)$ to $Prym^0(\tilde{\Sigma}_g)$. This section concerns the next step, pulling back coordinates from $Prym^0(\tilde{\Sigma}_g)$ to $Prym(\tilde{\Sigma}_g)$.

Recall that the non-canonical isomorphism from $Prym(\tilde{\Sigma}_g)$ to $Prym^0(\tilde{\Sigma}_g)$ was given by multiplying points of $Prym(\tilde{\Sigma}_g)$ by a fixed square root of the degree 0 line bundle $\sum_{i=1}^{2g-2}\left([x_1^i]-[x_2^i]\right)$. This square root $\mathfrak{d}$ is determined by a divisor $\sum_{i=1}^{2g-2}[p_i]$ such that $\sum_{i=1}^{2g-2}\left([x_1^i]-[p_i]\right)$ is equal to $\frac{1}{2}\sum_{i=1}^{2g-2}\left([x_1^i]-[x_2^i]\right)$, which are represented via the Abel–Jacobi map by the dual 1–forms $\left\{\eta \mapsto \sum_{i=1}^{2g-2}\int_{x_1^i}^{p_i}\eta\right\}$ and $\left\{\eta \mapsto \frac{1}{2}\sum_{i=1}^{2g-2}\int_{x_1^i}^{x_2^i}\eta\right\}$ correspondingly, modulo periods, where η is a σ–anti-invariant holomorphic 1–form. Since this square root must live in $Prym^0(\tilde{\Sigma}_g)$, it is σ–anti-invariant, therefore modulo periods:

$$\sum_{i=1}^{2g-2}\int_{x_1^i}^{p_i}\eta = -\sigma\sum_{i=1}^{2g-2}\int_{x_1^i}^{p_i}\eta = \sum_{i=1}^{2g-2}\int_{x_1^i}^{\sigma p_i}\eta; \tag{36}$$

so

$$\sum_{i=1}^{2g-2}p_i = \sum_{i=1}^{2g-2}\sigma p_i. \tag{37}$$

Because $H_1(\Sigma_g;\mathbb{Z})^- \simeq \Lambda + \Lambda_0^*$ induces coordinates on $Prym^0(\tilde{\Sigma}_g)$, we can search for p_i's on Λ and on Λ_0^*. We prove by contradiction that they cannot be on Λ. If a point p_i were on Λ, it would be on a path in Y_i from x_1^i to x_2^i. Thus each p_i would be in a distinct pair of pants. Since there are no further branch points in Y_i, the points p_i and σp_i would be distinct, which would contradict 37 because there are only $2g-2$ points p_i which coincide with the $2g-2$ points σp_i.

Thus the p_i's are on Λ_0^*. For closed paths $\gamma_i = \gamma_i^1 + \gamma_i^2$ representing elements of Λ_0^*, where the γ_i^1's go from x_1^i to p_i and the γ_i^2's go from p_i to x_1^i correspondingly, we have

$$\sum_{i=1}^{2g-2} \left(\int_{x_1^i}^{p_i} \eta + \int_{p_i}^{x_1^i} \eta \right) = 0, \tag{38}$$

along the γ_i's. Combined with (36) this gives us that

$$\sum_{i=1}^{2g-2} \left(\int_{x_1^i}^{p_i} \eta + \int_{\sigma p_i}^{x_1^i} \eta \right) = 0, \tag{39}$$

where the first integral is along γ_i and the second is along $\sigma\gamma_i$. This allows us to combine γ_i and $\sigma\gamma_i$, which is the lift of a closed path on the base Σ_Υ (and shows that p_i must be a branch point).

Loops on the base corresponding to elements of Λ_0^* are cycles on $G(\Upsilon)$. It remains only to notice that $H_1(G(\Upsilon); \mathbb{Z}_2) \simeq \frac{1}{2}\Lambda^* / \Lambda_0^*$ [6, Proposition 3.5(i)].

Along each cycle there are two choices of p_i. If p_i is a branch point x_1^j along a cycle, we may choose instead p_i' to be x_2^j. Notice that these choices are separated by a translation by $\gamma_i^1 + \sigma\gamma_i^2$ which represents and element of $\frac{1}{2}\Lambda^* / \Lambda_0^*$. Combined, we get an action of $\frac{1}{2}\Lambda^* / \Lambda_0^*$ on $Prym^0(\tilde{\Sigma}_g)$ (or rather on $H^0(\omega^-_{\tilde{\Sigma}_g})^*$) by translation. The coordinates on $Prym(\tilde{\Sigma}_g)$ are the coordinates on $Prym^0(\tilde{\Sigma}_g)$ modulo this action, and thus we have induced coordinates on $Prym(\tilde{\Sigma}_g)$.

Because Λ_0 is of dimension $3g-3$ (and so $\frac{1}{2}\Lambda_0 / \Lambda_0 \simeq \mathbb{Z}_2^{3g-3}$) and because $\Lambda / \Lambda_0 \simeq \mathbb{Z}_2^{2g-3}$, taking the dual gives $\frac{1}{2}\Lambda^* / \Lambda_0^* \simeq \mathbb{Z}_2^g$ so there are 2^g choices of basepoint for $Prym^0(\tilde{\Sigma}_g)$.

5. Varying $\det \Phi$ along a closed path

In the earlier sections we explained Yoshida's coordinates for a fixed Φ and a compatible pants decomposition Υ. In this section we 'activate' Φ by allowing $\det \Phi$ to move along closed paths. These closed paths are elements of W_Υ the fundamental group of the space of double covers associated to a pants decomposition $\mathcal{B}_\Upsilon$. We find generators for W_Υ in Lemma 2. Note that our set of generators is smaller than Yoshida's. These generators can be represented by an embedded graph in Σ_Υ which Yoshida calls a *marking.*

We explicitly calculate the actions of these generators on lifts of pairs of pants. The conclusion in Section 5.3 is a collection of formulae for the action of W_Υ on Λ, on Λ_0, and on the basepoint of $\mathrm{Prym}(\tilde{\Sigma}_g)$. We conclude in Proposition 5.1 by showing how a marking induces Yoshida's coordinates on $\mathrm{Prym}(\tilde{\Sigma}_g)$ (and therefore also a complex structure on it) by fixing the arbitrary choices we made in its construction— choice of basepoint and choices of orientations of curves in Λ and in Λ_0^*.

5.1. $\mathcal{B}_\Upsilon$ **: The Space of Double Coverings Associated to a Pants Decomposition** Υ**.** Recall the definition of the pants decomposition Υ from Section 2.2.3. In that section Φ was fixed and Υ was required to be compatible with it. Here Υ is fixed while Φ (or rather $\det\Phi$) is free to move along closed paths whose points are all compatible with Υ.

By the discussion in Section 2.2.3, Φ gives us two pieces of information with which we may construct $\tilde{\Sigma}_g$:

(i) An element $\mathbf{b}{:=}\{x_1,\dots x_{4g-4}\}$ in the configuration space of $4g-4$ unordered points on Σ_Υ, such that there is a pair of points in each pair of pants:

$$
(40)\quad B_{4g-4}(\Sigma_g)_\Upsilon{:=} \left\{ (x_1,\cdots,x_{4g-4}) \in \Sigma_g\times\cdots\times\Sigma_g \;\middle|\; \begin{array}{l}\{x_{2i-1},x_{2i}\} \in Y_i\,;\\ x_i\neq x_j \text{ if } i\neq j.\end{array} \Big/ \mathfrak{S}_2\times\cdots\times\mathfrak{S}_2 \right\},
$$

where $\mathfrak{S}_2$ denotes the symmetric group on 2 letters.

(ii) A covering type α of Σ_Υ which is an element of

$$
(41)\quad \breve{H}^1(\Sigma_g-\mathbf{b};\mathbb{Z}_2){:=} \left\{ \beta\in H^1(\Sigma_g-\mathbf{b};\mathbb{Z}_2) \;\middle|\; \begin{array}{l}\langle\beta,c_j\rangle=1\,;\\ \langle\beta,[e_l]\rangle=0.\end{array} \quad\text{for all}\quad \begin{array}{l}1\leq j\leq 4g-4\,;\\ 1\leq l\leq 3g-3.\end{array} \right\}.
$$

Thus the set of all double–covers of Σ_Υ (as a complex manifold) is parameterized by

$$
\mathcal{B}_\Upsilon{:=}\left\{\tilde{\mathbf{b}}{:=}(\mathbf{b},\alpha)\in B_{4g-4}(\Sigma_g)_\Upsilon\times\breve{H}^1(\Sigma_g-\mathbf{b};\mathbb{Z}_2)\right\} \tag{42}
$$

5.2. W_Υ **: The Fundamental Group of** $\mathcal{B}_\Upsilon$**.** Fix a base-point $(\mathbf{b}_0,\alpha_0)\in\mathcal{B}_\Upsilon$. We vary $\mathbf{b}_0$ in Σ_g along an element of $H_1(\Sigma_g;\mathbb{Z})$ represented by a parameterized loop $\ell(t){:=}\ \{x_1(t),\dots,x_{4g-4}(t)\}$ with $t\in[0,1]$ and $\ell(0)=\ell(1)=\mathbf{b}_0$. A loop in $\mathcal{B}_\Upsilon$ is a pair $(\ell(t),\alpha(t))$ where ℓ is a loop in $B_{4g-4}(\Sigma_g)_\Upsilon$ based at $\mathbf{b}_0$ and $\alpha(t)\in\breve{H}^1(\Sigma_g-\ell(t);\mathbb{Z}_2)$ with $t\in[0,1]$ and $\alpha(0)=\alpha(1)=\alpha_0$. Thus we obtain the map:

$$
\mathrm{ev}\colon\ \pi_1(B_{4g-4}(\Sigma_g)_\Upsilon,\mathbf{b}_0)\longrightarrow\breve{H}^1(\Sigma_g-\mathbf{b}_0;\mathbb{Z}_2), \tag{43}
$$

which sends the homotopy class represented by the curve ℓ to $\alpha_i-\alpha_0$. The fundamental group $W_\Upsilon{:=}\pi_1(\mathcal{B}_\Upsilon)$ is isomorphic to $\ker(\mathrm{ev})$.

Let Y be a pair of pants in Σ_g with boundary components e_1, e_2, and e_3, on which we fix some orientation. Since we are working in $B_{4g-4}(\Sigma_g)_\Upsilon$, we have that $\mathbf{b}\cap Y$ is a two point set, whose elements we denote $\{x_1,x_2\}$. Define the path

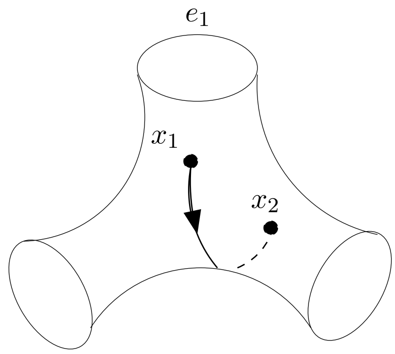


FIGURE 5. The path p_{e_1}.

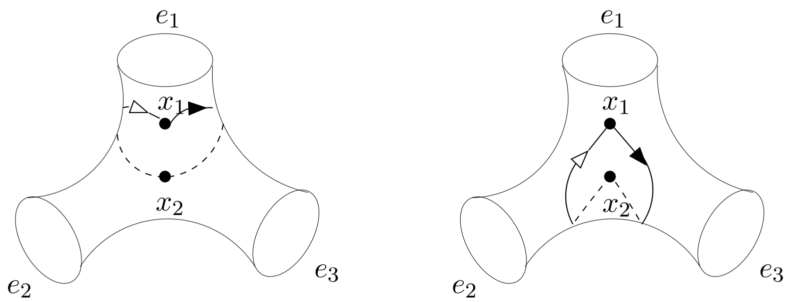


FIGURE 6. The loops t_{e_1} and k_{e_1}. The path followed by x_1 is indicated by the dark arrow while the path followed by x_2 is indicated by the white arrow.

$p_{e_i} = p_{e_i}(s)$ with $s \in [0,1]$ to be an embedded arc in the interior of Y which we denote $\mathrm{int}(Y)$ such that $p_{e_i}(0) = x_1$ and $p_{e_i}(1) = x_2$ which is "antipodal to e_i"— see Figure 5.

For a pair of pants Y, define the following closed paths in $B_2(Y)$ (restrictions to $B_2(Y)$ of closed loops in $B_{4g-4}(\Sigma_g)_\Upsilon$). Subscripts are to be understood modulo 3 (counter-clockwise in Figure 6):

(i) $t_{e_l} := \left\{p_{e_{l+1}}(s), p_{e_{l-1}}(1-s)\right\}_{s\in[0,1]}$,

(ii) $k_{e_l} := t_{e_{l-1}} t_{e_l} t_{e_{l+1}}$.

A superscript in the notation for the above paths indicates which pair of pants the path is in. The main statement about W_Υ is [13, Lemma 2.3] (note that we use fewer generator than he does):

Lemma 2. *W_Υ is generated by*

$$\left\{ \left\{ t_{e_l}^{Y_i} t_{e_l}^{Y_j} \right\} \cup \left\{ (t_{e_l}^{Y_i})^{\pm 2} \right\} \cup \left\{ k_{e_l}^{Y_i} \right\} \right\}.$$

To prove this statement we need a preliminary lemma (essentially [13, Lemma 2.2]).

Lemma 3. *The homology classes $\{[e_i]\} \subseteq H_1(\Sigma;\mathbb{Z}_2)$ are generated by a Lagrangian family of curves isomorphic to $\mathbb{Z}_2^g$ (g curves with orientations).*

Proof. If first we assume $G(\Upsilon)$ to be bridgeless (2–connected) then by Petersen's Theorem [12, Theorem VII.29] it has a perfect matching (a choice of $g-1$ pairwise disjoint edges $m_1,\ldots,m_{g-1}$). Hiding these edges leaves us with a collection of cycles $C_1,\ldots,C_k$ (if $k=1$ then $G(\Upsilon)$ is Hamiltonian). We arrange for there to be two outgoing and one incoming edge, or two incoming and one outgoing edge for each vertex. Choose a direction (clockwise or counterclockwise) for each cycle and for each of the edges in the matching, such that m_i is oriented from C_j to C_k for $j<k$, and connect together preserving orientations. If there are separating edges, add them in now with any orientation, and we still have the desired property.

The relations between elements in $\{[e_i]\}$ are as follows:

because

$$\left\{\, [e_i] \;\middle|\; 1\le i\le 3g-3; -[e_{j(1)}]+[e_{j(2)}]+[e_{j(3)}]=0 \text{ for all } 1\le j\le 2g-2 \right\},$$

where we take $j(1)$ to be the one ingoing (outgoing) edge.

To obtain a basis for the curves in the pants decomposition, since separating arcs are homologically trivial, we may assume that $G(\Upsilon)$ is bridgeless (2–connected). We choose edges $\bar{d}_1,\ldots,\bar{d}_{k-1}$ (and an ordering of the cycles) such that $\bar{d}_i$ connects C_i and C_{i+1}. Assume for simplicity that $\bar{d}_i = m_{g-i}$. Now choose a basis $d_1,\ldots,d_g$ for $\{[e_i]\}$ such that if $k>1$ then $d_i=m_i$ for $1\le i\le g-k$, and otherwise d_i is an edge in C_{i-g+k} adjacent to $\bar{d}_{i-g+k}$ for $i<g$ and an edge on C_k adjacent to $\bar{d}_{k-1}$ for $i=g$. If $k=1$ the rule is simpler— take $d_i=m_i$ for all $1\le i\le g-1$ and d_g to be an arc in C_1.

The family of curves $\{d_i\}$ which is isomorphic to $\mathbb{Z}_2^g$ clearly generates $\{[e_i]\}$, and is minimal because no member of the set is generated by all of the others. □

Proof of Lemma 2. We first note that the listed elements are in fact in $W_\Upsilon = \ker(\mathrm{ev})$. The elements $\left\{t_{e_l}^{Y_i} t_{e_l}^{Y_j}\right\}$ and $\left\{(t_{e_l}^{Y_i})^{\pm 2}\right\}$ are in ker(ev) because two times any element in $H^1(\Sigma_g-\mathbf{b}_0;\mathbb{Z}_2)$ is trivial. The elements $\left\{k_{e_l}^{Y_i}\right\}$ are in the kernel of the evaluation map because a covering type in $\breve{H}^1(\Sigma_g-\mathbf{b}_0;\mathbb{Z}_2)$ is determined by its image on the dual Lagrangian $\{d_i^*\}$, and we can choose a representative of a curve in the dual Lagrangian to be disjoint from the paths defining the k_{e_i}'s.

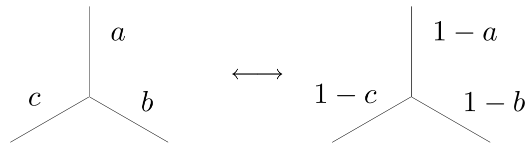
We now prove that the families of elements listed in the lemma generate ker(ev). First note that since a covering type in $\breve{H}^1(\Sigma-\mathbf{b}_0;\mathbb{Z}_2)$ is determined by its image on the dual Lagrangian $\{d_i^*\}$, and because for any covering type α we know that $\langle \alpha, c_j \rangle = 1$ for all j, adding an element c_j to a d_i^* (which is the action of t_{e_i} since it changes the cyclic order of (x_1^i, d_i^*, x_2^i)) will reverse its image in $\mathbb{Z}_2$ under α.

We prove that

$$A := \ker(\mathrm{ev}) \Big/ \left\{ \left\{ t_{e_l}^{Y_i} t_{e_l}^{Y_j} \right\} \cup \left\{ (t_{e_l}^{Y_i})^{\pm 2} \right\} \cup \left\{ k_{e_l}^{Y_i} \right\} \right\} \tag{44}$$

is trivial. First note that any element $s \in A$ is generated by the t_{e_i}'s, because it is isomorphic to a subgroup of $\pi_1(B_{4g-4}(\Sigma_g)_\Upsilon, \mathbf{b}_0)$ which is generated by the e_i's (there is a correspondence between paths in the configuration space and the curves on Σ along which the points vary). In addition, A is commutative— moves which take place within different pairs of pants clearly commute, and inside a single pair of pants, $t_{e_i} t_{e_j} = t_{e_k}^{-1}$ in the quotient group for all $i \neq j$ (because we are taking the modulo by all k_{e_l}).

We begin by assuming that $G(\Upsilon)$ is bridgeless (2–connected). On $G(\Upsilon)$ we may label each arc e_i by the number of t_{e_i}'s (with sign determined by the power of the t_{e_i}'s) parallel to e_i (since we are taking the modulo by $\left\{ t_{e_l}^{Y_i} t_{e_l}^{Y_j} \right\}$ these labels may be placed on arcs and not on half-arcs). We know that all labels may be taken to be 1 or 0 (because $\left\{ (t_{e_l}^{Y_i})^{\pm 2} \right\}$ are elements we are taking the modulo by), and that the sum of all labels around any cycle in $G(\Upsilon)$ (such cycles represent words in the d_i^*'s) must be even (note that if we were to have separating arcs there are no such cycles passing through them and thus no restriction on the label). Taking the modulo by $\left\{ k_{e_l}^{Y_i} \right\}$ gives us the following *explosion* relation in A (note that it is independent of the auxiliary numbering of the arcs):



The beautiful conclusion of this proof is due to Carsten Thomassen. We first reduce the number of 1's as far as possible by explosions on vertices. Now, because $s \in \ker(\mathrm{ev})$ there exists no cycle in the graph such that the sum of labels along that cycle is odd. Thus, connected subgraphs whose arcs are all labeled 0 form induced subgraphs (subgraphs G' such that any arc in $G(\Upsilon)$ connecting two vertices in G' is also in G'). Such subgraphs are connected by arcs labeled 1. Now choose one such induced subgraph G', and set off explosions on all its vertices. All its arcs have their labels reversed twice (so not at all), while all arcs connecting G' to another such induced subgraph have their labels reversed from 1 to 0. This process may be repeated until all 1's are canceled, and thus s must be trivial.

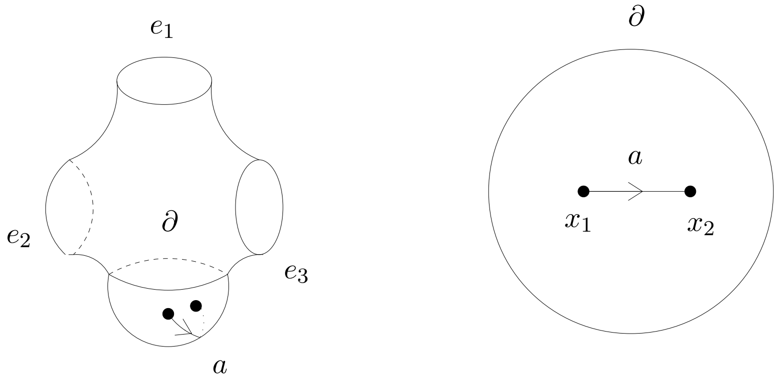


FIGURE 7. Cutting a disc out of a pair of pants.

In the remaining case, if $G(\Upsilon)$ has a separating arc a, performing an explosion on each vertex of a connected component of $G(\Upsilon) - a$ reverses the label on a, and thus the label of a also vanishes and here also s is trivial. □

In the above proof, we saw that an element of the abelianization of W_Υ is represented by an integer labeling of the arcs of $G(\Upsilon)$. An element of W_Υ itself can be represented by an embedding of $G(\Upsilon)$ into Σ_Υ constructed as follows:

First, for each pair of pairs of pants Y_i an Y_j, whose common boundary we denote e_k, we have a unique geodesic from x_1^i and x_1^j which we call e_k^* (this is slightly misleading because it is not the symplectic dual to e_k, but it does serve the same purpose). The collection of all such arcs, which comprises an embedding of G_Υ into Σ_Υ, we call the *base marking* and denote $\mathfrak{m}_0$.

Second, any element $\mathfrak{m} \in W_\Upsilon$ corresponds to a product of a collection of homology classes $C_1, \ldots, C_{2g-2}$ which represent curves in pair of pants $Y_1, \ldots, Y_{2g-2}$ via a homomorphism given by composing the evaluation map on W_Υ with the restriction to $\check{H}^1(\Sigma_g - \mathbf{b}_0; \mathbb{Z}_2)$ of the Poincaré duality isomorphism:

$$D\colon\ H^1(\Sigma_g - \mathbf{b}_0; \mathbb{Z}_2) \longrightarrow H_1(\Sigma_g - \mathbf{b}_0; \mathbb{Z}_2) \tag{45}$$

$$t_{e_i} \mapsto e_i. \tag{46}$$

Define now:

$$f_k := e_k^* \cdot C_1 \cdot C_2, \tag{47}$$

for e_k^* in $Y_1 \cup Y_2$. This is well defined as an element of $H_1(\Sigma_g - \mathbf{b}_0; \mathbb{Z}_2)$ and can be represented as a simple curve in Y_i whose endpoints coincide with those of e_k^*. Repeating for all k, we obtain a new embedding of $G(\Upsilon)$ into Σ_Υ. We can recover the product of $C_1, \ldots, C_{2g-2}$ and therefore any element of W_Υ from such an embedding, and hence such an embedding provides a compact way of writing an element of W_Υ. This embedding, and also the element of W_Υ which it represents, are both denoted $\mathfrak{m}$ and are called a *marking*.

Since a marking represents an element of W_Υ, the set of triples

$$\tilde{\mathcal{B}}_\Upsilon := \left\{ \tilde{\mathbf{b}} := (\mathbf{b}, \alpha, \mathfrak{m}) \in B_{4g-4}(\Sigma_g)_\Upsilon \times \breve{H}^1(\Sigma_g - \mathbf{b}; \mathbb{Z}_2) \times W_\Upsilon \right\} \tag{48}$$

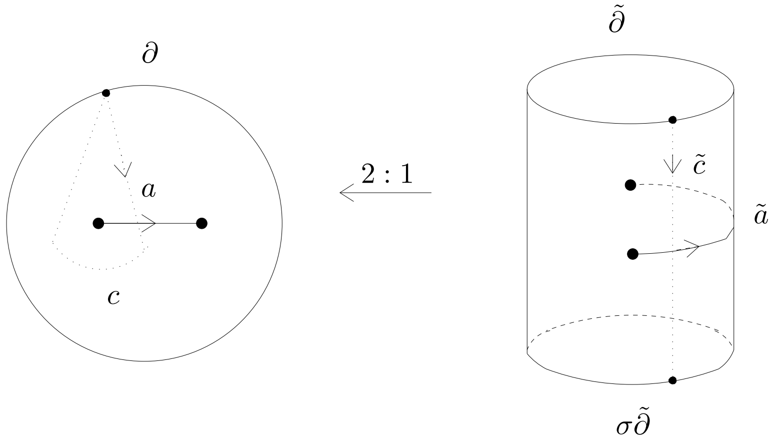
is a universal cover of $\mathcal{B}_\Upsilon$ the set of all double–covers of Σ_Υ simply branched over $4g-4$ points.

The generators of W_Υ given above are closed paths in $\mathcal{B}_\Upsilon$. We next measure the holonomy diffeomorphisms on a point in $\mathcal{B}_\Upsilon$ (a double cover) we obtain by moving along each of these paths. By construction, it is enough to understand such a diffeomorphism on a single pair of pants.

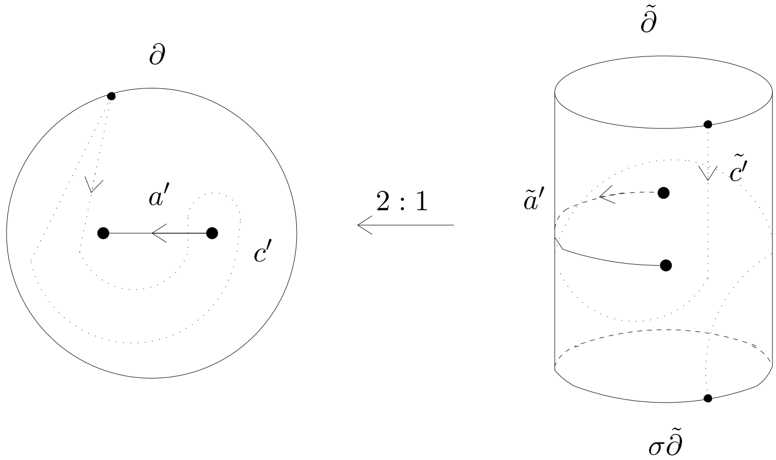
Let Y be a pair of pants with notation as before, and let $\tilde{Y}$ be a 2–fold covering of Y branched over $x_1 \cup x_2$. The lifts of e_i to $\tilde{Y}$ are denoted $\{\tilde{e_i}, \sigma\tilde{e_i}\}$ with $i \in \{1,2,3\}$. For each $e_i \in \Upsilon$, the path k_{e_i} induces a diffeomorphism κ_{e_i} on $\tilde{Y}$ as follows:

For k_{e_i} we cut a disc out of Y, and examine what happens to paths in the disc when we act by k_{e_i}. Let ∂ be the boundary of the disc, let a be a fixed cut between x_1 ad x_2, as in Figure 7. Let c be a small loop around x_1 with base-point on ∂, and let E be a small loop around a. We write the base space on the left, and the double-cover on the right(where we take only one of the pre-images of a). These are transformed by the action of k_{e_i} (which rotates a counterclockwise by π) as follows:

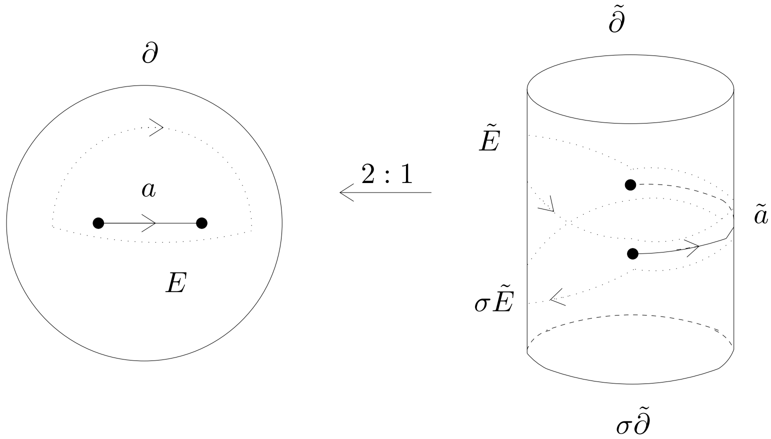
- We look at only one preimage of c



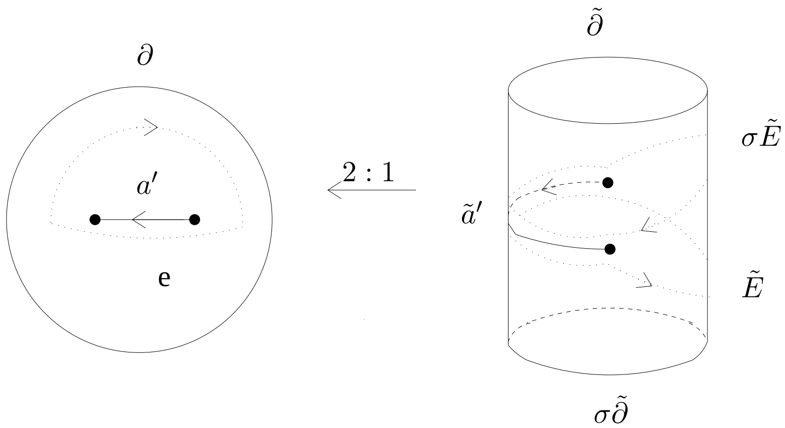
transforms to



- We look at both preimages of E

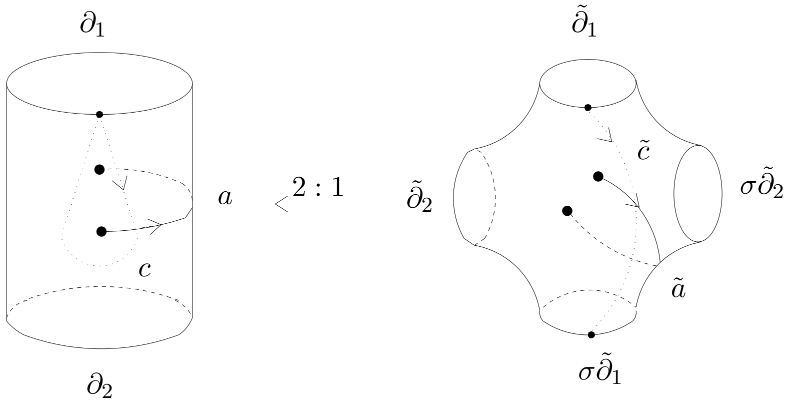


transforms to

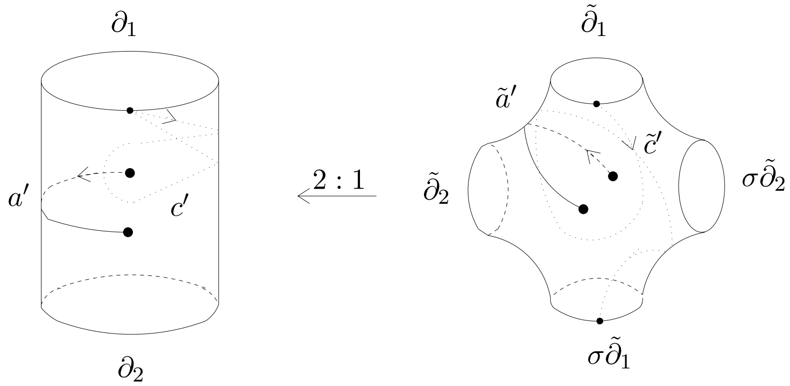


For t_{e_i} (which rotates a counterclockwise by π) the analogous moves (where ∂_2 can be taken to be e_i) are as follows:

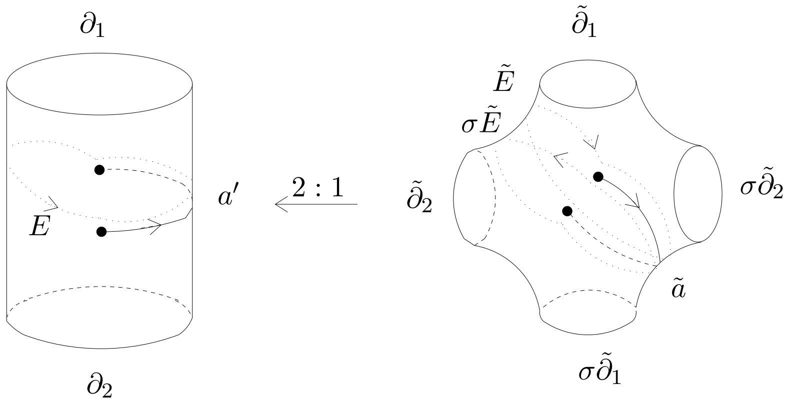
- We look at only one preimage of c



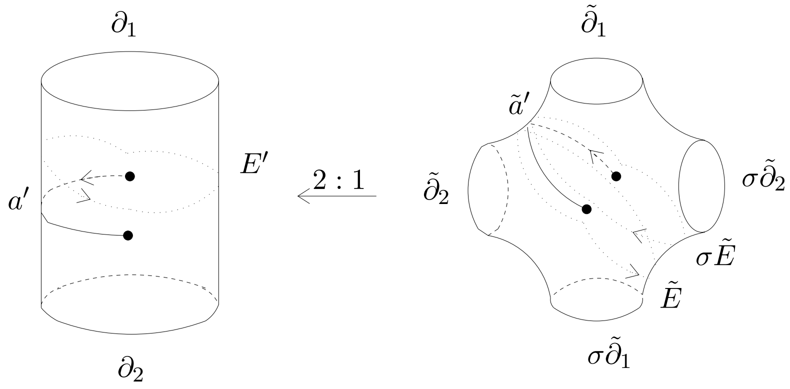
transforms to



- We look at both preimages of E



transforms to



The actions of t_{e_i} and of k_{e_i} may be extended to all of Σ_g. Yoshida calls these extensions τ and κ, but we stick with t and k to simplify notation.

5.3. **Actions on Coordinates of** $\mathrm{Prym}(\tilde{\Sigma}_g)$**.** Using the pictures in the previous section, we can read-off the action of t_{e_1} on Λ_0 (defined on a pair of pants and extended to the whole double cover). For the generators of W_Υ we obtain the map:

(49)
$$t^{Y_1}_{e_1}t^{Y_2}_{e_1}\colon\ (\vec{e}_{1(1)},\vec{e}_{1(2)},\vec{e}_{1(3)},\vec{e}_{2(1)},\vec{e}_{2(2)},\vec{e}_{2(3)})\mapsto(-\vec{e}_{1(1)},\vec{e}_{1(2)},\vec{e}_{1(3)},-\vec{e}_{2(1)},\vec{e}_{2(2)},\vec{e}_{2(3)}).$$

Since $\vec{e}_{1(1)}=\vec{e}_{2(1)}$, the element $t^{Y_1}_{e_1}t^{Y_2}_{e_1}$ acts on Λ_0 by multiplying a single element by -1. The generators $(t_{e_1})^{\pm 2}$ and k_{e_1} act trivially on Λ_0. All $t^{Y_1}_{e_1}t^{Y_2}_{e_1}$ actions combine to form a $\mathbb{Z}_2^{3g-3}$–action on Λ_0, and this is the statement of [13, Lemma 2.4].

Similarly we can read-off the actions of each of the generators of W_Υ on Λ:

(50)
$$\begin{array}{rccc} t^{Y_1}_{e_1}t^{Y_2}_{e_1}\colon & \left(\vec{E}^1_1,\vec{E}^1_2,\vec{E}^1_3,\vec{E}^2_1,\vec{E}^2_2,\vec{E}^2_3\right) & \mapsto & \left(\vec{E}^1_1,-\vec{E}^1_3,-\vec{E}^1_2,\vec{E}^2_1,-\vec{E}^2_3,-\vec{E}^2_2\right)\\ (t_{e_1})^{\pm 2}\colon & \left(\vec{E}_1,\vec{E}_2,\vec{E}_3\right) & \mapsto & \left(\vec{E}_1,\vec{E}_2,\vec{E}_3\right)\\ k_{e_1}\colon & \left(\vec{E}_1,\vec{E}_2,\vec{E}_3\right) & \mapsto & \left(-\vec{E}_1,-\vec{E}_2,-\vec{E}_3\right) \end{array}$$

The $\vec{E}_1$ classes are unchanged under the $t^{Y_1}_{e_1}t^{Y_2}_{e_1}$ action because $\vec{E}^1_2+\vec{E}^1_3=\vec{E}^2_2+\vec{E}^2_3$.

What about the basepoint? If p_i is a branch point x^j_1 half way between x^i_1 and x^i_2 along a path representing an element in $\frac{1}{2}\Lambda^*/\Lambda^*_0$, then k_{e_j} changes p_i to x^j_2, and vice versa. None of the other actions change the basepoint.

Proposition 5.1. *A marking* $\mathfrak{m}$ *uniquely induces coordinates for* $\mathrm{Prym}(\tilde{\Sigma}_g)$*.*

Proof. We just need to show that we can make a canonical choice of orientations of curves in Λ and a canonical choice of basepoint of $\mathrm{Prym}(\tilde{\Sigma}_g)$ for the base marking $\mathfrak{m}_0$. Then Equations 49, 50, and the discussion of actions on basepoints will tell us how any marking uniquely induces coordinates for $\mathrm{Prym}(\tilde{\Sigma}_g)$.

The curves $\{e_k^*\}$ which make up $\mathfrak{m}_0$ lift to closed curves in $\tilde{\Sigma}_g$. The orientations of these curves induce orientations on the $(\tilde{e}_i - \sigma\tilde{e}_i)$'s by the right hand rule, which in turn induce orientations on the elements of Λ.

For the basepoint, choose each p_i to be half way from x_1^i to x_2^i along a curve representing an element in Λ^* in the direction of the orientation of that curve. □

Acknowledgements

This paper grew out of a seminar (co-organized by K. Ueda and the second author) in 2006 at the Research Institute of Mathematical Sciences in Kyoto, whose purpose was to try to understand Yoshida's paper. The second author would like to thank the participants of seminar, K. Ueda, M. Tange, K. Kuroda, H. Iritani, and especially H. Fujita for valuable input. Thanks also to C. Thomassen for help with the proof of Lemma 2. Last but not least, the authors heartfelt thanks go to T. Yoshida for writing such an interesting paper and for substantial help in understanding various points in the construction.

The second author also thanks the Danish Technical University (DTU) where he was based at the time most of this paper was written, and the organizers of the International Conference on Quantum Topology in Hanoi which was uncommonly educational, enjoyable, and productive.

The first author was partially funded by NSF grant DMS-0604994.

References

[1] J. E. Andersen and G. Masbaum, Involutions on moduli spaces and refinements of the Verlinde formula, *Math. Ann.* **314** (1999), 291–326.
[2] M. Atiyah, *Geometry and Physics of Knots*, Lezione Lincee, Cambridge University Press, 1990.
[3] A. Beauville, Fibrés de rang 2 sur une courbe, fibré déterminant et fonctions Thêta, *Bull. Soc. Math. France* **116** (1988), 431–448 (French), Translated into English by D. Moskovich, at http://www.sumamathematica.com/BeauvilleENG.pdf.
[4] A. Beauville, M. S. Narasimhan and S. Ramanan, Spectral curves and the generalized Theta divisor, *J. Reine Angew. Math.* **398** (1989), 169–179.
[5] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, Topological quantum field theories derived from the Kauffman bracket, *Topology* **34** (4) (1995), 883–927.
[6] H. Fujita, A combinatorial realization of the Heisenberg action on the space of conformal blocks, *arXiv:math.*GT/0708.3309, 2007.
[7] A. Hatcher and W. Thurston, A presentation for the mapping class group of a closed orientable surface, *Topology* **19** (3) (1980), 221–237.
[8] N. J. Hitchin, The self–duality equations on a Riemann surface, *Proc. London Math. Soc.* **55** (3) (1987), 59–126.
[9] N. J. Hitchin, Stable bundles and integrable systems, *Duke Math. J.* **54** (1987), 91–114.
[10] D. Mumford, Prym varieties I, *Contributions to Analysis*, Academic Press, 1974, pp. 325–350.

[11] C. Teleman, Branching of Hitchin's Prym cover for $SL(2)$, *arXiv:math.*AG/0712.0163, 2007.
[12] W. T. Tutte, *Graph Theory*, Encyclopedia of Math. and its Appl. vol. 21, Addison-Wesley, 1984.
[13] T. Yoshida, An abelianization of $SU(2)$ WZW model, *Ann. of Math.* **163** (2006), 1–47.

Technical University of Denmark, Department of Mathematics, Matematiktorvet, Building 303S, 2800 Kongens Lyngby, Denmark
E-mail address: S.K.Hansen@mat.dtu.dk

Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan
E-mail address: dmoskovich@gmail.com