

QUANTUM INVARIANTS AND FREE \mathbb{Z}_{p^2} -ACTIONS ON 3-MANIFOLDS

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ABSTRACT. We give a congruence for the quantum invariant of a \mathbb{Z}_p -quotient of a 3-manifold with a \mathbb{Z}_{p^2} action. We show the congruence does not hold for quotients of 3-manifolds with a $\mathbb{Z}_5 \times \mathbb{Z}_5$ action.

1. INTRODUCTION

Let p denote an odd prime. The most simple kind of finite cyclic covers are those which are quotients of infinite cyclic covering spaces. We call such covers *simple* cyclic covers. Such covers are formed by stacking slit copies of the base. As TQFTs satisfy nice properties with respect to stacking, one can calculate quantum invariants of finite simple cyclic covers nicely from data for the base and the covering [5]. Moreover one obtains in this way congruences modulo p for the quantum invariants of simple \mathbb{Z}_p -cyclic covers of closed oriented connected 3-manifolds [7, 6]:

Theorem 1. *Suppose the infinite group \mathbb{Z} acts freely and preserving the orientation on a connected oriented 3-manifold \check{N} with a compact quotient. Then there exist m and n in \mathbb{Z} with*

$$\langle \check{N}/p\mathbb{Z} \rangle_p \equiv \kappa^m n \pmod{p \mathcal{O}_p}.$$

Here \mathcal{O}_p denotes $\mathbb{Z}[A, \kappa]$ where A is a primitive $2p$ th root of unity and $\kappa^2 = A^{-6-p(p+1)/2}$, and $\langle \rangle_p \in \mathcal{O}_p[1/p]$ denotes the invariant [2] of closed oriented 3-manifolds possessing an integral weight, as well a (possibly empty) p -admissibly colored (with colors integers in range $[0, p-2]$) fat graph. This invariant can be computed using the TQFT (Z_p, V_p) of [2]. We use a modified form of this TQFT as in [7] where p_1 -structures are replaced by integral weights on 3-manifolds and Lagrangian subspaces of the first homology of surfaces. When one raises this weight, one multiplies the invariant of a closed 3-manifold by the phase factor κ . One has that $\langle S^1 \times S^2 \text{ with weight zero} \rangle_p = 1$, but $\langle S^3 \text{ with weight zero} \rangle_p \notin \mathcal{O}_p$. Thus this is a different normalization for the invariant than used in [11].

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It is a natural to wonder whether Theorem 1 continues to hold for a more general class of p -fold covers \tilde{N} of a closed oriented 3-manifold N . However this congruence would only be possible if the values of the quantum invariant of \tilde{N} lies at least in \mathcal{O}_p .

Proposition 2. *Assume $p \neq 3$. Let \tilde{N} be a p -fold cyclic cover of a closed oriented 3-manifold N with empty colored fat graph. The following are equivalent.*

- (1) $\langle \tilde{N} \rangle_p \in \mathcal{O}_p$;
- (2) $H^1(\tilde{N}, \mathbb{Z}_p) \neq 0$;
- (3) *There is a p^2 -fold regular cover \check{N} of N with group of covering transformations G isomorphic to either \mathbb{Z}_{p^2} or $\mathbb{Z}_p \times \mathbb{Z}_p$ such that \tilde{N} is the quotient of \check{N} by a subgroup of G of order p .*

Proof. According to [11], $\langle \tilde{N} \rangle_p \notin \mathcal{O}_p$ if M is a \mathbb{Z}_p -homology sphere. Thus (1) implies (2). By [3, 8] (2) implies (1).

Clearly (3) implies (2). For the converse, let ψ be an eigenvector for the action of a generator of the group of covering transformation $\tilde{N} \rightarrow N$ on $H^1(\tilde{N}, \mathbb{Z}_p)$. As the order of this generator is p , then any eigenvalue is a p^{th} root of unity in \mathbb{Z}_p . But the only element with order a divisor of p in \mathbb{Z}_p^* is one. Therefore, one is the only eigenvalue for this generator. Thus ψ is fixed by the group of covering transformations. Let \check{N} be the \mathbb{Z}_p cover of \tilde{N} which is classified by ψ . The cover $\check{N} \rightarrow N$ is then regular with group of covering transformations a group of order p^2 . As is well-known, a group of order p^2 is either \mathbb{Z}_{p^2} , or $\mathbb{Z}_p \times \mathbb{Z}_p$. \square

All the implications in the above proof hold also in the case N contains a non-empty colored fat graph except perhaps (1) implies (2). Similarly in the case $p = 3$, all these results hold except (1) implies (2). We remark that the results of this paper which concern quantum invariants are uninteresting in the case $p = 3$, as $\langle N \rangle_3$ is always some power of κ . However some of the purely topological results may be of interest in the case $p = 3$.

Thus we are lead to:

Question 3. *Suppose G , a group of order p^2 , acts freely on a closed connected oriented 3-manifold \check{N} . Let $H \subset G$ denote a subgroup of order p . Must there exist m and n in \mathbb{Z} with*

$$\langle \check{N}/H \rangle_p \equiv \kappa^m n \pmod{p \mathcal{O}_p}?$$

We will show that the answer is “yes” if $G \approx \mathbb{Z}_{p^2}$, and “no” if $G \approx \mathbb{Z}_5 \times \mathbb{Z}_5$.

The example that we give for $\mathbb{Z}_5 \times \mathbb{Z}_5$ when modified in the most natural way will not show the answer is “no” for any $\mathbb{Z}_p \times \mathbb{Z}_p$ with $p > 5$.

The Appendix discusses a case which was omitted in the proof of a related congruence in previous paper of the first author [6].

2. SIMPLE COVERS AND WEAK TYPE- p SURGERY

Before the proof of Theorem 10 in the next section, we must prepare some tools. We let X denote the base of a \mathbb{Z}_k -cyclic cover, where k is a power of p . Then we have $\chi \in H^1(X, \mathbb{Z}_k)$ which classifies the cover together with a choice of generating covering transformation. Consider the Bockstein homomorphism β_k associated to the short exact sequence of coefficients:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_k \rightarrow 0.$$

It fits into a long exact sequence.

$$H^1(X) \xrightarrow{\times k} H^1(X) \longrightarrow H^1(X, \mathbb{Z}_k) \xrightarrow{\beta_k} H^2(X).$$

Clearly $\chi \in H^1(X, \mathbb{Z}_k)$ classifies a simple cover if and only if $\beta_k(\chi) = 0$.

For the rest of this paper N will denote a closed oriented connected 3-manifold. Recall that the Poincaré duality isomorphism from $H^2(N)$ to $H_1(N)$ is given by capping with the fundamental class: $\cap[N]$. Let ρ denote the inverse isomorphism from $H_1(N)$ to $H^2(N)$.

Lemma 4. *Suppose $\mathcal{G} = \cup_i^n \gamma_i$ is a link. The cover given by χ restricted to the complement of \mathcal{G} is simple if and only if $\beta_k(\chi)$ is in the span of $\{\rho([\gamma_i])\}$. The cover restricted to γ_i is characterized by $\chi(\gamma_i) \in \mathbb{Z}_k$.*

Proof. Let $\nu(\mathcal{G})$ denote a closed tubular neighborhood of \mathcal{G} . Let $N_{\mathcal{G}}$ denote N with the interior of $\nu(\mathcal{G})$ deleted. Consider the long exact sequence of the pair $(N, N_{\mathcal{G}})$:

$$\begin{array}{ccccc} H^2(N, N_{\mathcal{G}}) & \xrightarrow{j} & H^2(N) & \longrightarrow & H^2(N_{\mathcal{G}}) \\ \uparrow \text{excision isomorphism inverse} & & & & \\ H^2(\nu(\mathcal{G}), \partial\nu(\mathcal{G})) & \nearrow & & & \\ \uparrow \text{Thom isomorphism} & & & & \\ H^0(\mathcal{G}) & & & & \end{array}$$

We have that $H^0(\mathcal{G})$ is free on the components. The diagonal map is defined so that the diagram commutes. Using the dual chain complex construction of the Poincaré duality isomorphism e.g. [10, Thm 65.1], we see that the image of the generator of $H^0(\gamma_i)$ under the diagonal map is $\rho[\gamma_i]$. Thus the image of j is spanned by the $\rho[\gamma_i]$ in $H^2(N)$. As the horizontal sequence is exact, $\beta_k(\chi)$ restricted to $H^2(N_{\mathcal{G}})$ is zero. The cover of the simple closed curve is classified by the map from the infinite cyclic group $H_1(\gamma)$ to \mathbb{Z}_k , which maps the generator to $\chi(\gamma_i)$. □

Example 5. A connected \mathbb{Z}_p -cover of the lens space $L(p^2, 1)$ is simple on the complement of a simple closed curve representing p times a generator of $H_1(L(p^2, 1))$, but the covering of the curve is trivial. If instead we consider a simple closed curve representing a generator of $H_1(L(p^2, 1))$, this same covering is simple on

the complement of the curve and non-trivial on the curve. However, if we consider connected sum of p copies of $L(p^2, 1)$ and the cover given by a character χ which sends the standard generator on each summand to $1 \in \mathbb{Z}_p$, then there is no simple closed curve which is covered non-trivially but whose complement is covered simply.

We now wish to learn how to pick $\mathcal{G} = \cup_i^n \gamma_i$ as above with control over $\chi(\gamma_i)$. To understand the general case, we must discuss the relation between the Bockstein and the linking form. We must also relate our characters in $H^1(N, \mathbb{Z}_k)$ to characters in $H^1(N, \mathbb{Q}/\mathbb{Z})$. For this last issue, we have a commutative diagram of coefficients with short exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times k} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_k \longrightarrow 0 \\ & & \parallel & & \times \frac{1}{k} \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \end{array}$$

The residue class of one in \mathbb{Z}_k is sent to $1/k \in \mathbb{Q}/\mathbb{Z}$. We let the Bockstein associated to the lower sequence be denoted β . We obtain in this way a commutative diagram with exact rows:

$$\begin{array}{ccccc} H^1(N, \mathbb{Z}_k) & \xrightarrow{\beta_k} & H^2(N) & \xrightarrow{\times k} & H^2(N) \\ \downarrow & & \parallel & & \downarrow \\ H^1(N, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\beta} & H^2(N) & \longrightarrow & H^2(N, \mathbb{Q}) \end{array}$$

Thus given a character $\chi : H_1(N) \rightarrow \mathbb{Z}_k$, if we compose with the standard inclusion \mathbb{Z}_k to \mathbb{Q}/\mathbb{Z} and then apply β , we get $\beta_k(\chi)$. Also β maps onto the torsion subgroup of $H^2(N)$, denoted $\text{Tor}(H^2(N))$. Note that $\beta(\chi) \cap [N]$ is the Poincare dual of the Bockstein of χ . We are interested in the bilinear form:

$$b : H^1(N, \mathbb{Q}/\mathbb{Z}) \times H^1(N, \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

given by

$$b(\chi_1, \chi_2) = \chi_1(\beta(\chi_2) \cap [N]).$$

So using [12, p.254]

$$b(\chi_1, \chi_2) = \chi_1 \cap (\beta(\chi_2) \cap [N]) = (\chi_1 \cup \beta(\chi_2)) \cap [N].$$

Lemma 6. *The form b is symmetric.*

Proof.

$$\begin{aligned} 0 &= \beta(\chi_1 \cup \chi_2) \text{ by exactness as } H^3(N, \mathbb{Z}) \rightarrow H^3(N, \mathbb{Q}) \text{ is one to one} \\ &= \beta(\chi_1) \cup \chi_2 - \chi_1 \cup \beta(\chi_2) \text{ as } \beta \text{ is a derivation} \\ &= \chi_2 \cup \beta(\chi_1) - \chi_1 \cup \beta(\chi_2) \end{aligned}$$

So $b(\chi_2, \chi_1) - b(\chi_1, \chi_2) = (\chi_2 \cup \beta(\chi_1)) \cap [N] - (\chi_1 \cup \beta(\chi_2)) \cap [N] = 0. \quad \square$

As b vanishes on $\text{kernel}(\beta) \times H^1(N, \mathbb{Q}/\mathbb{Z})$ and

$$H^1(N, \mathbb{Q}/\mathbb{Z})/\text{kernel}(\beta) \approx \text{Tor}(H^2(N)),$$

there is an induced nonsingular symmetric form [4, p.75–76]

$$\mathbf{b} : \text{Tor}(H^2(N)) \times \text{Tor}(H^2(N)) \rightarrow \mathbb{Q}/\mathbb{Z}$$

such that

$$b(\chi_1, \chi_2) = \mathbf{b}(\beta(\chi_1), \beta(\chi_2)).$$

In fact, transferred to $\text{Tor}(H_1(N))$ by the Poincare duality isomorphism, one can check that \mathbf{b} becomes the well known linking form on $\text{Tor}(H_1(N))$.

Wall [13] has classified non-singular symmetric forms on finite abelian groups. Such forms have an orthogonal decomposition into primary summands associated to each prime. The p -primary summand is isomorphic to a direct sum of elementary forms of two types: A_{p^t} and B_{p^t} . The underlying group of both is \mathbb{Z}_{p^t} . The form on A_{p^t} sends (x, y) to xy/p^t . The form on B_{p^t} sends (x, y) to $n_t xy/p^t$, where n_t is a non-square unit in \mathbb{Z}_{p^t} . It is pleasant that we do not have to deal with the even prime, which is also studied by Wall but which is more complicated.

Lemma 7. *If $\chi \in H^1(N, \mathbb{Z}_p)$ is nonzero, then we can pick a link $\mathcal{G} = \cup_i^n \gamma_i$ in N , so that the cover given by χ restricted to the complement of \mathcal{G} is simple and χ on each γ_i is nonzero.*

Proof. We reinterpret χ to lie in $H^1(N, \mathbb{Q}/\mathbb{Z})$. We can view $\beta(\chi)$ as an element of the p -primary subgroup of $\text{Tor}(H^2(N))$ which we assume is already decomposed and identified in the way described by Wall. Let n be the number of summands where $\beta(\chi)$ projected into that summand is nonzero. The image of $\beta(\chi)$ under each of these projections has order p . It is always possible to find an element x_i of this summand which pairs under the torsion form with the projection of $\beta(\chi)$ to yield $1/p \in \mathbb{Q}/\mathbb{Z}$. Let γ_i be Poincare dual to x_i . Then $\beta(\chi)$ is in the span of the Poincare duals of the $\{\gamma_i\}$. Moreover for each $1 \leq i \leq n$,

$$\chi(\gamma_i) = b(\chi, \beta^{-1}(x_i)) = \mathbf{b}(\beta(\chi), x_i) = 1/p.$$

□

If γ is a simple closed curve in N , and \mathcal{T}_γ is the boundary of a tubular neighborhood ν_γ of \mathcal{T}_γ , let $\mu_\gamma \subset \mathcal{T}$ denote a meridian of γ . We may also pick a longitude λ_γ . This is a curve on \mathcal{T}_γ which is homologous to γ (with some orientation) in ν_γ . For d an integer bigger than 1, by weak type- d surgery along γ in N [8], we mean the process of removing ν_γ and regluing it so a curve on \mathcal{T} homologous to $n\mu + dl\lambda$ for some integers n , and l bounds in the reglued solid tori. Suppose λ' is another choice of longitude. Then $\lambda' = \lambda + x\mu$, so $n\mu + dl\lambda = (n - dlx)\mu + dl\lambda'$. Thus the notion of weak type- d surgery does not depend on which longitude was chosen. A weak type- d surgery can be undone with another. The equivalence relation generated by weak type- d surgery is called weak d -congruence.

The following Proposition is a weakened form of [8, Prop 2.14]. We include it here as the proof helps to prove Proposition 9 and to motivate the proof of Lemma 12.

Proposition 8. *If $H_1(N, \mathbb{Z}_p)$ is non-zero, then N is weakly p -congruent to a 3-manifold M with $H^1(M)$ non-zero.*

Proof. We pick a nonzero $\chi \in H^1(N, \mathbb{Z}_p)$ and apply Lemma 7. Let $\hat{\chi} \in H^1(N \setminus \mathcal{G}, \mathbb{Z})$ be a lift to \mathbb{Z} of the restriction of χ to $N \setminus \mathcal{G}$. Let T_i be the boundary of a tubular neighborhood of U_i of γ_i equipped with a meridian μ_i , and a chosen longitude $\lambda_i \in H_1(T_i)$. The character $\hat{\chi}$ induces a character $\hat{\chi}_i : H_1(T_i) \rightarrow \mathbb{Z}$. Suppose the image of $\hat{\chi}_i$ is $k_i\mathbb{Z}$. Note that k_i and p must be relatively prime. Then $\hat{\chi}_i([\mu_i]) = ps_ik_i$ and $\hat{\chi}_i([\lambda_i]) = -n_ik_i$ for integers s_i, k_i, n_i with s_i, n_i relatively prime and with n_i prime to p . Then $\hat{\chi}_i$ vanishes on the homology class of a curve μ'_i representing $n_i[\mu_i] + ps_i[\lambda_i]$. Thus $\hat{\chi}$ extends uniquely to M obtained by doing a weak type- p surgery along each curve γ_i . As this extension will be nonzero, $H^1(M, \mathbb{Z})$ is nonzero. By definition, M is weakly p -congruent to N . \square

By a p -surface F , we mean [8] the result of attaching, by a map q , the whole boundary of an oriented surface \hat{F} to a collection of circles $\{S_i\}$ by a map which when restricted to the inverse image under q of each S_i is a pt_i -fold (possibly disconnected) covering space of S_i . If each component of each $q^{-1}S_i$ is itself a covering space of S_i with degree divisible by p , we say F is a good p -surface. The p -cut number of a 3-manifold N , denoted $c_p(N)$, is the maximum number of disjoint piecewise linearly embedded good p -surfaces that we can place in N with a connected complement. The following Proposition allows us to interpret [8, Theorem 4.1] as a corollary of [8, Theorem 4.2].

Proposition 9. *If $H_1(N, \mathbb{Z}_p)$ is non-zero, then $c_p(N) > 0$.*

Proof. Continuing with the proof of Proposition 8, let $\chi' = (1/k)\chi$, $U = \cup U_i$ appearing in this proof. In $N \setminus \text{Int } U$, we can find a connected surface F with a connected complement which is dual to $\chi'|_U$ and meets each T_i in μ'_i . F maybe completed to a good p -surface with a connected complement by adding the mapping cones of the projections from μ'_i to γ \square

3. $G = \mathbb{Z}_{p^2}$

Theorem 10. *Suppose \mathbb{Z}_{p^2} acts freely on closed oriented connected 3-manifold \check{N} . If H is the subgroup of \mathbb{Z}_{p^2} of order p , then there exists m and n in \mathbb{Z} with*

$$\langle \check{N}/H \rangle_p \equiv \kappa^m n \pmod{p \mathcal{O}_p}.$$

Proof. In this situation, we say \check{N}/H is a somewhat simple \mathbb{Z}_p -cyclic cover of $\check{N}/\mathbb{Z}_{p^2}$, which we will denote by N . A basic example of somewhat simple \mathbb{Z}_p -cyclic cover which is not a simple \mathbb{Z}_p -cyclic cover is the lens space $L(p, q)$ which is a somewhat simple \mathbb{Z}_p -cyclic cover of $L(p^2, q)$.

We need a version of Lemma 7 for characters in $H^1(N, \mathbb{Z}_{p^2})$.

Lemma 11. *If $\chi : H_1(N) \rightarrow \mathbb{Z}_{p^2}$ is an epimorphism, then we can pick a link $G = \cup_i^n \gamma_i$ in N , so that the cover given by χ restricted to the complement of G is simple and for each i , $\chi(\gamma_i)$ is either one, or is p .*

Proof. Again we reinterpret χ to lie in $H^1(N, \mathbb{Q}/\mathbb{Z})$, view $\beta(\chi)$ as an element of the p -primary subgroup of $\text{Tor}(H^2(N))$ which we assume is already decomposed and identified in the way described by Wall.

Let $0 \leq i \leq n$ index the summands where $\beta(\chi)$ projected into that summand is nonzero. Let β_i the projection of $\beta(\chi)$ in the i th summand. If β_i has order p^2 , it is possible to find a element x_i of this summand which pairs under the torsion form with β_i to yield $1/p^2 \in \mathbb{Q}/\mathbb{Z}$. If β_i has order p , we can find a element x_i of this summand which pairs under the torsion form with β_i to yield $1/p \in \mathbb{Q}/\mathbb{Z}$. Let γ_i be Poincare dual to x_i . Then $\beta(\chi)$ is in the span of $\{\rho[\gamma_i]\}$. Moreover for each $1 \leq i \leq n$, $\chi(\gamma_i)$ is either $1/p^2$ or $1/p$. Now reinterpret χ to take values in \mathbb{Z}_{p^2} . □

Lemma 12. *Suppose $\chi : H_1(N) \rightarrow \mathbb{Z}_{p^2}$ is an epimorphism. Let \tilde{N} denote the associated p -fold cover classified by $\pi_p \circ \chi$, where $\pi_p : \mathbb{Z}_{p^2} \rightarrow \mathbb{Z}_p$ is reduction modulo p . Then N is weakly p -congruent to a manifold M with a simple \mathbb{Z}_p covering \tilde{M} . \tilde{M} may be obtained by weak type- p surgery on a link in \tilde{N} .*

Proof. Apply Lemma 11. Let $\epsilon_i \in \mathbb{Z}$ be 1 if $\chi(\gamma_i) = 1$ and ϵ_i be $p \in \mathbb{Z}$ if $\chi(\gamma_i) = p$. Let $\hat{\chi} \in H^1(N \setminus \mathcal{G}, \mathbb{Z})$ be a lift of χ to \mathbb{Z} . Then $\hat{\chi}_i([\mu_i]) = p^2 s_i k_i$ and $\hat{\chi}_i([\lambda_i]) = -\epsilon_i n_i k_i$ for integers s_i, k_i, n_i with s_i, n_i relatively prime and with both n_i and k_i prime to p . Then $\hat{\chi}$ vanishes on the homology class of a curve μ'_i representing $n_i[\mu_i] + \frac{p^2}{\epsilon_i} s_i[\lambda_i]$.

Thus we can do weak $\frac{p^2}{\epsilon_i}$ -type surgery to N along each γ_i to obtain a weakly p -congruent manifold M such that $\hat{\chi}$ extends to $\hat{\chi}' : H_1(M) \rightarrow \mathbb{Z}$. This induces a simple \mathbb{Z}_p -cover \tilde{M} of M . We have that \tilde{M} is obtained from \tilde{N} by a sequence of surgeries along the curves in \tilde{N} which lie over the γ_i . If $\epsilon_i = 1$ then γ_i is covered by a single curve in \tilde{N} and we perform weak type- p surgery along this curve. If $\epsilon_i = p$ then γ_i is covered by p disjoint curves in \tilde{N} and we perform weak type- p^2 surgeries (which are also weak type- p surgeries) along each of these curves. □

By [8, Theorem 3.8], $\langle \tilde{N} \rangle_p$ is, up to phase,

$$\left\langle \tilde{M} \text{ with some } \mathbb{Z}_p\text{-equivariant integrally colored framed link} \right\rangle_p.$$

Since \tilde{M} is a simple \mathbb{Z}_p -cover, by [7, 11.1], this last expression must satisfy the stated congruence. □

4. $G = \mathbb{Z}_p \times \mathbb{Z}_p$

Let L_p be the two component link obtained by replacing one component of a Hopf link with a $(p, 1)$ cable with framing p on the cabled component and framing zero on the uncabled component. Let M_p be obtained by performing framed surgery on L_p . The linking matrix of the framed link is $\begin{bmatrix} 0 & p \\ p & p \end{bmatrix}$. Thus $H_1(M_p) = \mathbb{Z}_5 \oplus \mathbb{Z}_5$. Let \check{M}_p denote the maximal abelian cover of M , i.e. the covering space whose fundamental group is the commutator subgroup. It is a

regular $\mathbb{Z}_p \oplus \mathbb{Z}_p$ cover of M . Let L'_p be a (p, p) -torus link with an unknot going around it (a satellite of the Hopf link with the $(5, 5)$ torus link the companion of one component of the Hopf link.). We give L' the framing $+1$ on each component of the cabled part and the zero framing on the uncabled component. Let \tilde{M}_p be the 3-manifold obtained from S^3 by doing surgery along L'_p with weight zero. \tilde{M}_p is a \mathbb{Z}_p -cyclic cover of M , and thus a quotient of \tilde{M} by an order p cyclic subgroup of $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

Below will use the z^i basis [1] for the Kauffman skein of a genus one handlebody.

Let H_n denote the Kauffman bracket of an n component positive Hopf link where every component has framing one. By [9, Prop 11 (ii)], we have for $n \geq 1$,

$$H_n = (A^2 - A^{-2})^{-1} \sum_{r=0}^{n-1} \binom{n-1}{r} A^{(n-2r+1)^2-1} (A^{2(n-2r+1)} - A^{-2(n-2r+1)})$$

Note $H_0 = 1$, so this is not valid for $n = 0$.

Theorem 13. $\mathbb{Z}_5 \times \mathbb{Z}_5$ acts freely on \tilde{M}_5 . There is an order 5 cyclic subgroup H of $\mathbb{Z}_5 \times \mathbb{Z}_5$ such that there are no m and n in \mathbb{Z} with

$$\langle \tilde{M}_5/H \rangle_5 \equiv \kappa^m n \pmod{5 \mathcal{O}_5}$$

Proof. We let $\mathcal{O}_5 = \mathbb{Z}[\zeta_{20}]$ with $A = \zeta_{20}^2$ and $\kappa = \zeta_{20}^{-1}$. Then by formula for η near the beginning of [2, §2], $\eta = \frac{1}{5}(2\zeta_{20} + \zeta_{20}^3 + \zeta_{20}^5 - 3\zeta_{20}^7)$.

One has $\Omega_5 = 1 + \delta z$, where $\delta = -A^{-2} - A^2$. Replacing a component with framing one with $t^{-1}\Omega_5 = 1 - A^{-3}\delta z$, has the same effect as first changing the framing to zero and then replacing with Ω_5 . Here t denotes the twist map on the Kauffman skein of the solid torus [1]. We let $\langle L'(\Omega_5) \rangle$ be the Kauffman bracket evaluation of the linear combination over \mathcal{O}_5 that we obtain if we replace each component of L' by Ω_5 . One has that $\langle \tilde{M}_5 \rangle_5 = \eta_5^7 \langle L'(\Omega_5) \rangle$.

To compute $\langle L'(\Omega_5) \rangle$, we compute a linear combination of 2^6 brackets of Hopf links with zero to six components and varying framings. If we expand out the zero framed component first, we obtain:

$$\langle L'_5(\Omega_5) \rangle = \sum_{k=0}^5 \binom{5}{k} \delta^k H(k) - A^{-3}\delta \sum_{k=0}^5 \binom{5}{k} \delta^k H(k+1).$$

We obtain

$$\langle \tilde{M}_5 \rangle_5 = -2\zeta_{20} + 4\zeta_{20}^3 - \zeta_{20}^5 - 2\zeta_{20}^7$$

Comparing this with the finite list of elements of $\mathcal{O}_5/5\mathcal{O}_5$ which are images of numbers of the form $n\kappa^m$ for any $n, m \in \mathbb{Z}$, we conclude that $\langle \tilde{M} \rangle_5$ does not have this form. □

Now we consider $p = 7$. We let $\mathcal{O}_7 = \mathbb{Z}[\zeta_{14}]$ with $A = \zeta_{14}$ and $\kappa = A^4$. Then $\eta = \frac{1}{7}(-2\zeta_{14} - \zeta_{14}^2 - 2\zeta_{14}^3 + 2\zeta_{14}^4 + \zeta_{14}^5)$. We have $\Omega_7 = (2 - \delta^2) + \delta z + (\delta^2 - 1)z^2$, and $t^{-1}\Omega_7 = (1 + A^6 - A^6\delta^2) - A^{11}\delta z + A^6(\delta^2 - 1)z^2$. We have that $\langle \tilde{M}_7 \rangle_7 = \eta^9 \langle L'_7(\Omega_7) \rangle$. Using multinomial coefficients, we have

$$\begin{aligned} \langle L'_7(\Omega_7) \rangle &= (1 + A^6 - A^6\delta^2) \sum_{i+j+k=7} \binom{7}{i, j, k} (2 - \delta^2)^i \delta^j (\delta^2 - 1)^k H(j + 2k) \\ &\quad - A^{11}\delta \sum_{i+j+k=7} \binom{7}{i, j, k} (2 - \delta^2)^i \delta^j (\delta^2 - 1)^k H(j + 2k + 1) \\ &\quad + A^6(\delta^2 - 1) \sum_{i+j+k=7} \binom{7}{i, j, k} (2 - \delta^2)^i \delta^j (\delta^2 - 1)^k H(j + 2k + 2). \end{aligned}$$

It follows that:

$$\langle \tilde{M}_7 \rangle_7 = 7(176993 + 397520A - 318640A^2 - 220548A^3 - 98084A^4 + 495621A^5).$$

Thus $\langle \tilde{M}_7 \rangle_7 \in 7\mathcal{O}_7$. Notice that $H_1(\tilde{M}_p) = \mathbb{Z}^{p-1}$. By [3, 4.3]

$$\langle \tilde{M}_p \rangle_p \in (1 - \zeta_p)^{\lceil (p-1)(p-3)/6 - (p-3)/2 \rceil} \mathcal{O} = (1 - \zeta_p)^{\lceil (p^2 - 7p + 12)/6 \rceil} \mathcal{O}.$$

As $\lceil (p^2 - 7p + 12)/6 \rceil \geq p - 1$ for $p \geq 11$, we have $\langle \tilde{M}_p \rangle_p \in p\mathcal{O}_p$, for $p \geq 11$. Thus these examples for $p \geq 7$ are consistent with a “Yes” answer to Question 3. To find examples allowing one to say “No”, one should begin by looking for 3-manifolds with first homology $\mathbb{Z}_p \times \mathbb{Z}_p$ with a p -fold cover with the dimension of first homology with \mathbb{Z}_p -coefficients less than $6\frac{p-1}{p-3} + 3$. Of course \tilde{M}_7 did satisfy this equation but did not provide a “No” answer.

We used Mathematica [14] for many of the computations in this section.

APPENDIX. COVERS OF DEGREE PRIME TO THE ORDER OF THE PHASE FACTOR BY PATRICK M. GILMER

In the proof of [6, Theorem 1’], on page 171 Proposition 2 only applies when the cover restricted to γ is non-trivial. The case where the cover restricted to γ is trivial was not addressed. In view of Example 5, this case may arise.

Theorem 1’ concerns the quantum invariants $\langle N \rangle_{2r}$ where r is prime to p . We continue to assume p is an odd prime. The congruence given is not modulo phase but exact. As the extra structures used to resolve the “framing anomaly” in [6] are the p_1 -structures of [2] rather than integral weights and Lagrangians, 3-manifolds, in this appendix, will be equipped with p_1 -structures. We note that p_1 -structures are more natural in this context, as covering spaces are equipped with p_1 -structures induced from the base. Also 3-manifolds are allowed to have possibly empty admissibly colored fat graphs. The quantum invariant $\langle \rangle_{2r}$ takes values in $\mathbb{Z}[\frac{1}{2r}, \xi]$ where ξ is a primitive $4r$ th root of unity if r is even and a primitive $8r$ th root of unity if r is odd. The colors of this theory are from the set C of the integers from zero to $r - 2$. The following result addresses the missing case in the proof of [6, Theorem 1’]. In fact [6, Theorem 1’] has the same conclusion as Theorem 14 under the weaker hypothesis: N_p is a connected \mathbb{Z}_p -covering of N .

Theorem 14. *Let N_p be a connected non-simple \mathbb{Z}_p -covering of N given by $\chi \in H^1(N, \mathbb{Z}_p)$. Suppose γ is a simple closed curve in N such that $\chi[\gamma] = 0$, and the cover restricted to $N \setminus \gamma$ is simple, then*

$$\langle N_p \rangle_{2r} \equiv \langle N \rangle_{2r}^p \pmod{p \mathbb{Z}[\frac{1}{2r}, \xi]}.$$

Proof. Let ν denote a closed tubular neighborhood of γ , $\mathcal{T} = \partial\nu$, $E = -N \setminus \text{Int}(\nu)$. Let E_p be the given cover restricted to E . This cover restricted to \mathcal{T} is a disjoint union of p tori $\mathcal{T}_p = \coprod_{i=1}^p \mathcal{T}_i$. Similarly this cover restricted to ν is a disjoint union of p solid tori $\nu_p = \coprod_{i=1}^p \nu_i$ with $\partial\nu_i = \mathcal{T}_i$. We index these so the covering transformation specified by χ , sends ν_i to ν_{i+1} for all i .

Let $\hat{\chi} : H_1(E) \rightarrow \mathbb{Z}$ be a surjective lift of χ . Let μ' be a simple closed curve in \mathcal{T} which generates the kernel of $\hat{\chi}$ composed with the map induced by inclusion $H_1(\mathcal{T}) \rightarrow H_1(E)$. Let H be a handlebody with boundary \mathcal{T} such that μ' bounds a disk in H . We let $N' = E \cup_{\mathcal{T}} H$. The cover on E extends to a simple cover N'_p of N' . Let $H(j)$ denote H with the core colored j , $N'(j) = E \cup_{\mathcal{T}} H(j)$, and $N'(j)_p$ denote N'_p with the p circles covering the core all colored j . We let H_i denote the component of the cover of H with $\partial H_i = \mathcal{T}_i$, and let $H(j)_i$ denote H_i with the core colored j . We will use $\{[H(j)] \mid 0 \leq j \leq r-1\}$ as a basis for $V_p(\mathcal{T})$. It is orthogonal with respect to the Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ on $V_p(\mathcal{T})$.

Let \mathcal{S} denote the set of sequences of colors of length p . We denote the i th term of the sequence σ by σ_i . Let $\tau : \mathcal{S} \rightarrow \mathcal{S}$ be the cyclical shift map with $\tau(\sigma)_i = \sigma_{i-1 \pmod{p}}$. The orbits of \mathcal{S} under powers of the shift map are of two types. There are singletons given by constant sequences, and orbits with cardinality p made up of non-constant sequences. We index the constant sequences by the set of colors C . We denote the sequence which is constantly j by \tilde{j} .

If $\sigma \in \mathcal{S}$, let $H(\sigma)$ denote

$$H(\sigma_1)_1 \otimes H(\sigma_2)_2 \otimes \cdots \otimes H(\sigma_p)_p \in V_p\left(\prod_{i=1}^p \mathcal{T}_i\right) = V_p(\mathcal{T}_p).$$

Using orthonormality,

$$(1) \quad [\nu] = \sum_{j \in C} \langle [\nu], [H(j)] \rangle_{\mathcal{T}} [H(j)] \in V_p(\mathcal{T})$$

and thus:

$$(2) \quad [\nu_p] = [\nu] \otimes [\nu] \otimes \cdots \otimes [\nu] = \sum_{\sigma \in \mathcal{S}} \left(\prod_{i=1}^p \langle [\nu], [H(\sigma_i)] \rangle_{\mathcal{T}} \right) [H(\sigma)] \in V_p(\mathcal{T}_p).$$

Again by orthonormality, we have:

$$(3) \quad [E] = \sum_{j \in C} \langle [E], [H(j)] \rangle [H(j)] = \sum_{j \in C} \langle N'(j) \rangle_p [H(j)] \in V_p(\mathcal{T}),$$

$$(4) \quad [E_p] = \sum_{\sigma \in \mathcal{S}} \langle [E], [H(\sigma)] \rangle [H(\sigma)] = \sum_{\sigma \in \mathcal{S}} \langle N'_p(\sigma) \rangle_p [H(\sigma)] \in V_p(\mathcal{T}_p),$$

where $N'_p(\sigma)$ is N'_p with the i th lift of the core colored σ_i for all i . Thus by 1 and 3

$$(5) \quad \langle N \rangle_p = \langle [\nu], E \rangle_{\mathcal{T}} = \sum_{j \in C} \langle [\nu], [H(j)] \rangle_{\mathcal{T}} \langle N'(j) \rangle_p,$$

and by 2 and 4

$$(6) \quad \langle N_p \rangle_p = \langle [\nu_p], E_p \rangle_{\mathcal{T}_p} = \sum_{\sigma \in S} \left(\prod_{i=1}^p \langle [\nu], [H(\sigma_i)] \rangle_{\mathcal{T}} \right) \langle N'_p(\sigma) \rangle_p.$$

Note that $N'_p(\sigma)$ is diffeomorphic to $N'_p(\tau(\sigma))$, and thus $\langle N'_p(\sigma) \rangle_p$ is constant on orbits of τ . Also $\prod_{i=1}^p \langle [\nu], [H(\sigma_i)] \rangle_{\mathcal{T}}$ is constant on orbits of τ . Since the non-constant orbits of τ have order p , we have:

$$(7) \quad \begin{aligned} \langle N_p \rangle_p &\equiv \sum_{j \in C} \left(\prod_{i=1}^p \langle [\nu], [H(j)] \rangle_{\mathcal{T}} \right) \langle N'_p(\tilde{j}) \rangle_p \\ &\equiv \sum_{j \in C} \left(\langle [\nu], [H(\tilde{j})] \rangle_{\mathcal{T}} \right)^p \langle N'_p(j) \rangle_p \pmod{p \mathbb{Z}[\frac{1}{2r}, \xi]}. \end{aligned}$$

As [6, Theorem 1'] is already proved for simple covers and $N'_p(j)$ is a simple cover of $N'(j)$, we have that:

$$(8) \quad \langle N'_p(j) \rangle_p \equiv (\langle N'(j) \rangle_p)^p \pmod{p \mathbb{Z}[\frac{1}{2r}, \xi]}.$$

Substituting this in 7 and comparing with 5, the result follows □

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