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AN INTRODUCTION TO THE VOLUME CONJECTURE AND ITS GENERALIZATIONS

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ABSTRACT. In this paper we give an introduction to the volume conjecture and its generalizations. Especially we discuss relations of the asymptotic behaviors of the colored Jones polynomials of a knot with different parameters to representations of the fundamental group of the knot complement at the special linear group over complex numbers by taking the figure-eight knot and torus knots as examples.

After V. Jones' discovery of his celebrated polynomial invariant $V(K;t)$ in 1985 [22], Quantum Topology has been attracting many researchers; not only mathematicians but also physicists. The Jones polynomial was generalized to two kinds of two-variable polynomials, the HOMFLYpt polynomial [10, 52] and the Kauffman polynomial [27] (see also [21, 3] for another one-variable specialization). It turned out that these polynomial invariants are related to quantum groups introduced by V. Drinfel′d and M. Jimbo (see for example [26, 55]) and their representations. For example the Jones polynomial comes from the quantum group $U_q(sl_2(\mathbb{C}))$ and its two-dimensional representation. We can also define the quantum invariant associated with a quantum group and its representation.

If we replace the quantum parameter q of a quantum invariant (t in $V(K;t)$) with e^h we obtain a formal power series in the formal parameter h. Fixing a degree d of the parameter h, all the degree d coefficients of quantum invariants share a finiteness property. By using this property, one can define a notion of finite type invariant [2, 1]. (See [57] for V. Vassiliev's original idea.)

Via the Kontsevich integral [35] (see also [36]) we can recover a quantum invariant from the corresponding 'classical' data. (Note that a quantum group is a deformation of a 'classical' Lie algebra.) So for example one only needs to know the Lie algebra $sl_2(\mathbb{C})$ (easy!) and its fundamental two-dimensional representation (very easy!) to define the Jones polynomial, provided that one knows the Kontsevich integral (unfortunately, this is very difficult).

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In the end of the 20th century, M. Khovanov introduced yet another insight to Quantum Topology. He categorified the Jones polynomial and defined a homology for a knot such that its graded Euler characteristic coincides with the Jones polynomial [29]. See [30] for a generalization to the HOMFLYpt polynomial.

Now we are in the 21st century.

In 2001, J. Murakami and the author proposed the Volume Conjecture [48] to relate a sequence of quantum invariants, the N-colored Jones polynomials of a knot, to the volume of the knot complement, generalizing R. Kashaev's conjecture [24]. The aim of this article is to introduce the conjecture and some of its generalizations, emphasizing a relation of the asymptotic behavior of the sequence of the colored Jones polynomials of a knot to representations of the fundamental group of the knot complement.

In Section 1 we prepare some fundamental facts about the colored Jones polynomial, the character variety, the volume and the Chern–Simons invariant. Section 2 contains our conjectures, and in Sections 3 and 4 we give supporting evidence for the conjectures taking the figure-eight knot and torus knots as examples.

1. Preliminaries

In this section we review the definition of the colored Jones polynomial, the character variety, and the volume and the Chern–Simons invariant.

1.1. **Colored Jones polynomial.** Let K be an oriented knot in the three-sphere S^3, and D its diagram. We assume that D is the image of a projection $\mathbb{R}^3 \cong S^3 \setminus \{\infty\} \to \mathbb{R}^2$ of a circle S^1 together with a finite number of double point singularities as in Figure 1.

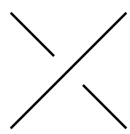


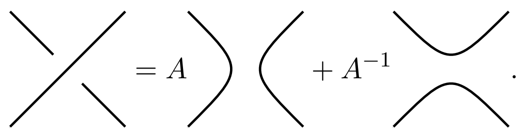
Figure 1. A crossing of a knot diagram

We call these double points with over/under data crossings.

Now we associate to the knot diagram a Laurent polynomial in A following L. Kauffman [28].

First of all we forget the orientation of D.

Then replace a crossing with a linear combination of two pairs of parallel arcs as in the right hand side of the following equality:



Now we have a linear combination of two new diagrams, which may not be knot diagrams but link diagrams.

We choose another crossing and replace it with a linear combination of two new diagrams. Now we have a linear combination of four diagrams.

Continue these processes until we get a linear combination of diagrams with no crossings, whose coefficients are some powers of A.

If we replace each resulting diagram with $(-A^2-A^{-2})^{\nu-1}$ where ν is the number of components of the diagram, we get a Laurent polynomial $\langle D\rangle \in \mathbb{Z}[A,A^{-1}]$, which is called the Kauffman bracket.

Now we recall the orientation of D. Let $w(D)$ be the sum of the signs of the crossings in D, where the sign is defined as in Figure 2.

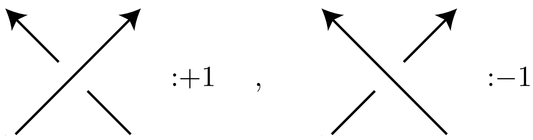


FIGURE 2. A positive crossing (left) and a negative crossing (right)

Lastly we define $V(K;t) := \left(-A^3\right)^{-w(D)} \langle D\rangle\Big|_{A:=t^{-1/4}}$. It is known that this coincides with (a version of) the Jones polynomial [22] since

$$\begin{aligned}
&t^{-1}V\left(\times_+;t\right)-tV\left(\times_-;t\right)\\
=&t^{-1}\times\left(-t^{-3/4}\right)^{-w(\times_+)}\left\langle \times_+\right\rangle - t\times\left(-t^{-3/4}\right)^{-w(\times_-)}\left\langle \times_-\right\rangle\\
=&t^{-1}\times\left(-t^{-3/4}\right)^{-w()()-1}\left\{t^{-1/4}\left\langle)(\right\rangle + t^{1/4}\left\langle \asymp\right\rangle\right\}\\
&-t\times\left(-t^{-3/4}\right)^{-w()()+1}\left\{t^{-1/4}\left\langle \asymp\right\rangle + t^{1/4}\left\langle)(\right\rangle\right\}\\
=&\left(-t^{-3/4}\right)^{-w()()}\left(t^{1/2}-t^{-1/2}\right)\left\langle)(\right\rangle\\
=&\left(t^{1/2}-t^{-1/2}\right)V\left()(;t\right).
\end{aligned}$$

In this paper we use another version $J_2(K;q)$ that satisfies

$$qJ_2\left(\times_+;q\right)-q^{-1}J_2\left(\times_-;q\right)=\left(q^{1/2}-q^{-1/2}\right)J_2\left()(;q\right)$$

with the normalization $J_2(\text{unknot};q)=1$. Note that

$$J_2(L;q)=(-1)^{\sharp(L)-1}V\left(L;q^{-1}\right)$$

for a link with $\sharp(L)$ components.

It is well known that J_2 corresponds to the two-dimensional irreducible representation of the Lie algebra $sl(2;\mathbb{C})$. The invariant corresponding to the N-dimensional irreducible representation is called the N-colored Jones polynomial and denoted by $J_N(K;q)$. It can be also defined as a linear combination of the (2-colored) Jones polynomials of $(N-1)$-parallels or less of the original knot.

For example, K. Habiro [14] and T. Lê [38] calculated the N-colored Jones polynomial for the trefoil knot T (Figure 3) and for the figure-eight knot E (Figure 4) as follows:

$$J_N(T;q) = q^{1-N}\sum_{k=0}^{N-1} q^{-kN}\prod_{j=1}^{k}\left(1-q^{j-N}\right)$$

and

$$J_N(E;q) = \sum_{k=0}^{N-1}\prod_{j=1}^{k}\left(q^{(N-j)/2}-q^{-(N-j)/2}\right)\left(q^{(N+j)/2}-q^{-(N+j)/2}\right).$$

For other formulas of $J_N(T;q)$, see [42, 40].

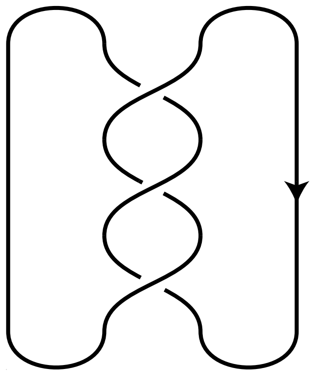


FIGURE 3. Trefoil

1.2. $SL(2;\mathbb{C})$**-Character variety of a knot group.** For a knot in S^3, let us consider representations of $\pi_1(S^3\setminus K)$ at $SL(2;\mathbb{C})$, where $SL(2;\mathbb{C})$ is the set of all 2×2 complex matrices with determinant one. The set of all characters of representations is denoted by $X(S^3\setminus K)$ or $X(K)$. We can regard $X(K)$ as the set of equivalence classes where we regard two representations equivalent if they have the same trace. It is well known that $X(K)$ is an algebraic variety (the set of the zeroes of polynomials) and called the $SL(2;\mathbb{C})$-character variety of the knot K. (See for example [6].)

Here are two families of knots whose $SL(2;\mathbb{C})$-character varieties are well known.

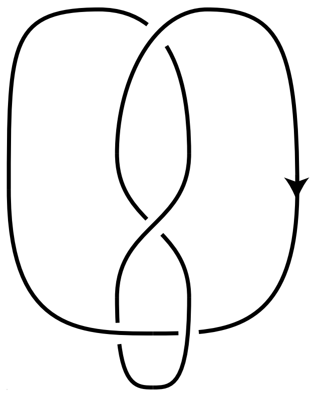


FIGURE 4. Figure-eight knot

Example 1.1 (Two-bridge knots)**.** If K is a two-bridge knot (a knot that can be put in $\mathbb{R}^3 = S^3 \setminus \{\infty\}$ in such a way that it has only two maxima), we can express its $SL(2;\mathbb{C})$-character variety as follows [37, 3.3.1 Theorem] (see also [53, Theorem 1]).

It is well known that $\pi_1(S^3 \setminus K)$ has the following presentation:

$$\langle x, y \mid \omega x = y\omega \rangle,$$

where ω is a word in x and y of the form $x^{\pm 1}y^{\pm 1}\dots x^{\pm 1}y^{\pm 1}$. Let ρ be a representation of $\pi_1(S^3 \setminus K)$ at $SL(2;\mathbb{C})$. Then for any word z in x and y we can express $\operatorname{tr}\bigl(\rho(z)\bigr)$ as a polynomial in $\xi := \operatorname{tr}\bigl(\rho(x)\bigr)$ and $\eta := \operatorname{tr}\bigl(\rho(xy)\bigr)$ by using the following formulas [58]:

$$\begin{aligned} \operatorname{tr}(AB) &= \operatorname{tr}(A)\operatorname{tr}(B) - \operatorname{tr}(A^{-1}B), \\ \operatorname{tr}(A^{-1}) &= \operatorname{tr}(A). \end{aligned} \tag{1.1}$$

We denote this polynomial by $P_z(\xi, \eta)$.

The $SL(2;\mathbb{C})$-character variety $X(K)$ is given as follows.

Theorem 1.2 (Le [37])**.** *The $SL(2;\mathbb{C})$-character variety is determined by the polynomial*

$$(2 + \eta - \xi^2)F(\xi, \eta),$$

where

$$F(\xi, \eta) := \sum_{i=0}^{k} (-1)^i P_{\omega^{(i)}}(\xi, \eta)$$

with $\omega^{(i)}$ the word obtained from ω by deleting the first i letters and the last i letters (we put $P_{\emptyset}(\xi, \eta) := 1$). Moreover the first factor $2 + \eta - \xi^2$ determines the abelian part and the second factor $F(\xi, \eta)$ determines the non-abelian part.

Example 1.3 (Torus knots)**.** A knot that can be put on the standard torus is called a torus knot. Such knots are parametrized by two coprime integers, and up to mirror image we may assume that they are both positive. We also assume that they are bigger than 1.

For the torus knot $T(a,b)$ of type (a,b) with positive coprime integers a and b with $a>1$ and $b>1$, the fundamental group $\pi_1(T(a,b))$ also has a presentation with two generators and one relation:

$$\langle g,h \mid g^a = h^b\rangle.$$

Note that the meridian (a loop that goes around the knot in the positive direction) μ and the longitude (a loop that is parallel to the knot and is null-homologous in the knot complement) λ can be expressed as $\mu = g^{-c}h^d$ and $\lambda = g^a\mu^{ab}$, where we choose c and d so that $ad-bc=1$.

Then the $SL(2;\mathbb{C})$-character variety of $\pi_1(T(a,b))$ is given as follows.

Theorem 1.4 ([9, Theorem 2])**.** *The components of the non-abelian part of the $SL(2;\mathbb{C})$-character variety are indexed by k and l with $1 \leqslant k \leqslant a-1$, $1 \leqslant l \leqslant b-1$ and $k \equiv l \pmod 2$. Moreover a representation $\rho_{k,l}$ in the component indexed by (k,l) satisfies*

$$\operatorname{tr}\left(\rho_{k,l}(g)\right) = 2\cos\left(\frac{k\pi}{a}\right)$$

and

$$\operatorname{tr}\left(\rho_{k,l}(h)\right) = 2\cos\left(\frac{l\pi}{b}\right).$$

(See also [34, Theorem 1]).

1.3. **Volume and the Chern–Simons invariant of a representation.** For a closed three-manifold W one can define the Chern–Simons function cs_W on the $SL(2;\mathbb{C})$-character variety of $\pi_1(W)$ as follows. See for example [32, Section 2] for a nice review.

Let A be a $\mathfrak{sl}(2;\mathbb{C})$-valued 1-form on W satisfying $dA + A\wedge A = 0$. Then A defines a flat connection on $W \times SL(2;\mathbb{C})$ and so it also induces the holonomy representation ρ of $\pi_1(W)$ at $SL(2;\mathbb{C})$ up to conjugation. Then the $SL(2;\mathbb{C})$ Chern–Simons function cs_W is defined by

$$\mathrm{cs}_W\bigl([\rho]\bigr) := \frac{1}{8\pi^2}\int_W \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \in \mathbb{C} \pmod{\mathbb{Z}},$$

where $[\rho]$ means the conjugacy class. Note that every representation is induced as the holonomy representation of a flat connection.

P. Kirk and E. Klassen [33] gave a formula to calculate the $SL(2;\mathbb{C})$ Chern–Simons invariant if W is given as the union of two three-manifolds whose boundaries are tori.

Let M be an oriented three-manifold with boundary ∂M a torus. Given a representation ρ of $\pi_1(M)$ at $SL(2;\mathbb{C})$, one can define the Chern–Simons function

cs_M as a map from the $SL(2;\mathbb{C})$-character variety $X(M)$ of $\pi_1(M)$ to a circle bundle $E(\partial M)$ over the character variety $X(\partial M)$ of $\pi_1(\partial M) \cong \mathbb{Z} \oplus \mathbb{Z}$ [33], which is a lift of the restriction map $X(M) \to X(\partial M)$.

To describe cs_M, we consider a map

$$p\colon \mathrm{Hom}(\pi_1(\partial M), \mathbb{C}) \to X(\partial M)$$

defined by

$$p(\kappa) := \left[\gamma \mapsto \begin{pmatrix} \exp(2\pi\sqrt{-1}\kappa(\gamma)) & 0 \\ 0 & \exp(-2\pi\sqrt{-1}\kappa(\gamma)) \end{pmatrix}\right]$$

for $\gamma \in \pi_1(\partial M)$, where the square brackets mean the equivalence class in $X(\partial M)$. Note that κ and κ' define the same element in $X(\partial M)$ if and only if

$$\cos(2\pi\sqrt{-1}\kappa(\gamma)) = \cos(2\pi\sqrt{-1}\kappa'(\gamma)) \text{ for any } \gamma \in \pi_1(\partial M).$$

Now fix a generator system (μ, λ) of $\pi_1(\partial M) \cong \mathbb{Z} \oplus \mathbb{Z}$, and take its dual basis (μ^*, λ^*) of $\mathrm{Hom}(\pi_1(\partial M), \mathbb{C})$. If $\kappa = \alpha\mu^* + \beta\lambda^*$ and $\kappa' = \alpha'\mu^* + \beta'\lambda^*$, then $p(\kappa) = p(\kappa')$ if and only if

$$k\alpha + l\beta = \pm(k\alpha' + l\beta') \pmod{\mathbb{Z}},$$

for any integers k and l. This means that p is invariant under the following actions on $\mathrm{Hom}(\pi_1(\partial M); \mathbb{C})$:

$$\begin{aligned} x \cdot (\alpha, \beta) &:= (\alpha + 1, \beta), \\ y \cdot (\alpha, \beta) &:= (\alpha, \beta + 1), \\ b \cdot (\alpha, \beta) &:= (-\alpha, -\beta). \end{aligned}$$

Note that these actions form the group

$$G := \langle x, y, b \mid xyx^{-1}y^{-1} = bxbx = byby = b^2 = 1 \rangle$$

and that in fact the quotient space $\mathrm{Hom}(\pi_1(\partial M), \mathbb{C})/G$ can be identified with the $SL(2;\mathbb{C})$-character variety $X(\partial M)$.

Next let us consider the following actions on $\mathrm{Hom}(\pi_1(\partial M), \mathbb{C}) \times \mathbb{C}^*$.

$$\begin{aligned} x \cdot [\alpha, \beta; z] &:= [\alpha + 1, \beta; z\exp(2\pi\sqrt{-1}\beta)], \\ y \cdot [\alpha, \beta; z] &:= [\alpha, \beta + 1; z\exp(-2\pi\sqrt{-1}\alpha)], \\ b \cdot [\alpha, \beta; z] &:= [-\alpha, -\beta; z]. \end{aligned}$$

We denote the quotient space $(\mathrm{Hom}(\pi_1(\partial M), \mathbb{C}) \times \mathbb{C}^*)/G$ by $E(\partial M)$. Then $E(\partial M)$ becomes a $\mathbb{C}^*$-bundle over $X(\partial M)$. The Chern–Simons function cs_M is a map from $X(M)$ to $E(\partial M)$ such that the following diagram commutes, where q is the projection $q\colon E(\partial M) \to X(\partial M)$ and i^* is the restriction map

$$\begin{array}{ccc} & & E(\partial M) \\ & \overset{\mathrm{cs}_M}{\nearrow} & \downarrow q \\ X(M) & \xrightarrow{\;i^*\;} & X(\partial M) \end{array}.$$

More precisely, given a representation ρ, then

$$\mathrm{cs}_M([\rho]) := \begin{cases} \left[\alpha, \beta; \exp(2\pi\sqrt{-1}\,\mathrm{cs}_M(A))\right], & \text{if } i^*(\rho) \text{ is diagonalizable,} \\ \left[-a/2, -b/2; \exp(2\pi\sqrt{-1}\,\mathrm{cs}_M(A))\right], & \text{otherwise.} \end{cases}$$

Here in the first case

$$\rho(\mu) = \begin{pmatrix} \exp(2\pi\sqrt{-1}\alpha) & 0 \\ 0 & \exp(-2\pi\sqrt{-1}\alpha) \end{pmatrix},$$

$$\rho(\lambda) = \begin{pmatrix} \exp(2\pi\sqrt{-1}\beta) & 0 \\ 0 & \exp(-2\pi\sqrt{-1}\beta) \end{pmatrix},$$

and

$$A = \begin{pmatrix} \sqrt{-1}\alpha & 0 \\ 0 & -\sqrt{-1}\alpha \end{pmatrix} dx + \begin{pmatrix} \sqrt{-1}\beta & 0 \\ 0 & -\sqrt{-1}\beta \end{pmatrix} dy$$

near ∂M, and in the second case

$$\rho(\mu) = (-1)^a \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix},$$

$$\rho(\lambda) = (-1)^b \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix},$$

and

$$A = \begin{pmatrix} -\sqrt{-1}a/2 & c\exp\left(\sqrt{-1}(ax+by)\right)/(2\pi) \\ 0 & \sqrt{-1}a/2 \end{pmatrix} dx$$

$$+ \begin{pmatrix} -\sqrt{-1}b/2 & d\exp\left(\sqrt{-1}(ax+by)\right)/(2\pi) \\ 0 & \sqrt{-1}b/2 \end{pmatrix} dy$$

near ∂M for $a, b \in \mathbb{Z}$ and $c, d \in \mathbb{C}$, where (x, y) is the coordinate such that $(x, y) \mapsto (\exp(\sqrt{-1}x), \exp(\sqrt{-1}y))$ is the universal cover $\mathbb{R}^2 \to \partial M$. See [33, Section 3] for more details.

If a closed three-manifold W is given as $M_1 \cup M_2$ by using two three-manifolds M_1 and M_2 whose boundaries are tori, where we identify ∂M_1 with $-\partial M_2$. We give the same basis for $\pi_1(\partial M_1)$ and $\pi_1(-\partial M_2)$ and let ρ_i be the restriction to $\pi_1(M_i)$ of a representation ρ of $\pi_1(W)$ $(i = 1, 2)$. Then we have

$$\mathrm{cs}_W\left([\rho]\right) = z_1 z_2,$$

where $\mathrm{cs}_{M_i}([\rho_i]) = [\alpha, \beta, z_i]$ with respect to the common basis [33, Theorem 2.2].

For a three-manifold M with hyperbolic metric one can define the ($SO(3)$) Chern–Simons invariant $\mathrm{CS}(M)$ as follows [5]. Let A be the Levi-Civita connection (an $so(3)$-valued 1-form) defined by the hyperbolic metric. Then put

$$\mathrm{cs}(M) := \frac{1}{8\pi^2} \int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \in \mathbb{R} \pmod{\mathbb{Z}}.$$

In this paper we use another normalization $\mathrm{CS}(M) := -2\pi^2 \,\mathrm{cs}(M)$ so that $\mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M)$ can be regarded as a complexification of the volume $\mathrm{Vol}(M)$. Note that if M has a cusp (that is, if M is homeomorphic to the interior of a manifold with torus boundary), cs is defined modulo 1/2 and so CS is defined modulo π^2. It is known that the imaginary part of the $SL(2;\mathbb{C})$ Chern–Simons function of the holonomy representation is the volume with respect to the hyperbolic metric and the real part is the $SO(3)$ Chern–Simons invariant up to multiplications of some constants [59, Lemma 3.1] (see also [33, p. 554]).

When M has a cusp and possesses a complete hyperbolic structure of finite volume, we can deform the structure to incomplete ones by using a complex parameter u around 0, where $u = 0$ corresponds to the complete structure. Then T. Yoshida [59, Theorem 2] (see also [51, Conjecture]) proved that there exists a complex analytic function $f(u)$ around 0 such that if the corresponding (incomplete) hyperbolic manifold can be completed to a closed manifold M_u by adding a geodesic loop γ, then

$$\begin{aligned}(1.2)\quad &\mathrm{Vol}(M_u) + \sqrt{-1}\,\mathrm{CS}(M_u) - \bigl\{\mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M)\bigr\}\\ &\equiv \frac{f(u)}{\sqrt{-1}} - \frac{\pi}{2}\bigl\{\mathrm{length}(\gamma) + \sqrt{-1}\,\mathrm{torsion}(\gamma)\bigr\} \pmod{\pi^2\sqrt{-1}\mathbb{Z}},\end{aligned}$$

where $\mathrm{length}(\gamma)$ and $\mathrm{torsion}(\gamma)$ are the length and the torsion of γ respectively. (The torsion measures how much a normal vector is twisted when it travels around γ.)

We can interpret (1.2) in terms of Kirk and Klassen's Chern–Simons function cs_M.

For simplicity we assume that $M := S^3 \setminus K$ is the complement of a hyperbolic knot K, that is, we assume that M has a complete hyperbolic structure with finite volume. Then u defines an (incomplete) hyperbolic structure, and so gives the holonomy representation ρ of $\pi_1(M)$ at $PSL(2;\mathbb{C})$. (Recall that the orientation preserving isometry group of the hyperbolic space $\mathbf{H}^3$ is $PSL(2;\mathbb{C})$. Since the universal cover of M is $\mathbf{H}^3$, the lift of an element in $\pi_1(M)$ defines an isometric translation in $\mathbf{H}^3$, giving an element in $PSL(2;\mathbb{C})$.) We choose a lift of ρ to $SL(2;\mathbb{C})$ [7] and denote it also by ρ.

We can regard $\exp(u)$ as the ratio of the eigenvalues of the image of the meridian $\mu \in \pi_1(M)$ by the representation ρ, that is, the eigenvalues of $\rho(\mu) \in SL(2;\mathbb{C})$ are $\exp(u/2)$ and $\exp(-u/2)$. We can also define $v(u)$ so that $\exp\bigl(v(u)\bigr)$ is the ratio of the eigenvalues of the image of the longitude λ. Then from [33, pp. 553–556] $\mathrm{cs}_K := \mathrm{cs}_M$ can be expressed in terms of $f(u)$ as follows.

$$(1.3)\qquad \mathrm{cs}_K\bigl([\rho]\bigr) = \left[\frac{u}{4\pi\sqrt{-1}}, \frac{v(u)}{4\pi\sqrt{-1}}; \exp\left(\frac{\sqrt{-1}}{2\pi} f(u)\right)\right].$$

Note that here we use (μ, λ) for the basis of $\pi_1(\partial M)$.

Remark 1.5. Note that here we are using the $SL(2;\mathbb{C})$ theory; not $PSL(2;\mathbb{C})$. So we have to divide the term appearing in exp of (1.3) by -4 [33, pp. 553–556]. See [33, p. 543].

Note also that we are using the normalization of W. Neumann and D. Zagier [51] for $f(u)$. So Yoshida's (and Kirk and Klassen's) $f(u)$ is our $f(u)\times\dfrac{2}{\pi\sqrt{-1}}$.

Remark 1.6. From (1.3), one can define $f(u)$ modulo $4\pi^2\mathbb{Z}$, but in this paper we define it only modulo $\pi^2\mathbb{Z}$. So we may have to say that we are using $PSL(2;\mathbb{C})$ theory, and $SL(2;\mathbb{C})$ theory for normalization.

If the incomplete metric is completed by adding a geodesic loop γ, then the resulting manifold K_u is obtained from S^3 by (p,q)-Dehn surgery along K for some coprime integers p and q satisfying $p\,u+q\,v(u)=2\pi\sqrt{-1}$. Then from [51, Lemma 4.2] we have

$$\text{length}(\gamma)+\sqrt{-1}\,\text{torsion}(\gamma)=-r\,u-s\,v(u)\pmod{2\pi\sqrt{-1}\mathbb{Z}}.$$

where r and s are integers satisfying $ps-qr=1$.

Therefore we can express the right hand side of (1.2) in terms of u and $v(u)$.

$$\begin{aligned}(1.4)\qquad \text{Vol}(K_u)+\sqrt{-1}\,\text{CS}(K_u)-\bigl\{\text{Vol}(K)+\sqrt{-1}\,\text{CS}(K)\bigr\}&\\ \equiv\frac{f(u)}{\sqrt{-1}}+\frac{\pi}{2}\bigl(r\,u+s\,v(u)\bigr)\pmod{\pi^2\sqrt{-1}\mathbb{Z}},&\end{aligned}$$

where $\text{Vol}(K):=\text{Vol}(S^3\setminus K)$ and $\text{CS}(K):=\text{CS}(S^3\setminus K)$. Note that this formula can also be obtained from (1.3) [33].

We also have from [51, (34), p. 323]

$$\text{length}(u)=-\frac{1}{2\pi}\,\text{Im}(u\overline{v(u)}).$$

Therefore from (1.2) and (1.4) we have

$$(1.5)\qquad \text{Vol}(K_u)-\text{Vol}(K)=\text{Im}\,f(u)+\frac{1}{4}\,\text{Im}(u\overline{v(u)}),$$

$$(1.6)\qquad \text{CS}(K_u)-\text{CS}(K)\equiv-\text{Re}\,f(u)+\frac{\pi}{2}\,\text{Im}(r\,u+s\,v(u))\pmod{\pi^2\mathbb{Z}}.$$

Note that unfortunately we cannot express the Chern–Simons invariant only in terms of u.

2. Volume conjecture and its generalizations

In [23] Kashaev introduced link invariants parametrized by an integer N greater than one, by using the quantum dilogarithm. Moreover he observed in [24] that for the hyperbolic knots 4_1, 5_2, and 6_1 his invariants grow exponentially and the growth rates give the hyperbolic volumes of the knot complements. He also conjectured this would also hold for any hyperbolic knot in S^3.

In [48] J. Murakami and the author showed that Kashaev's invariant coincides with the N-colored Jones polynomial evaluated at the N-th root of unity and generalized his conjecture to general knots as follows.

Conjecture 2.1 (Volume Conjecture)**.** *For any knot K in S^3 we would have*

$$2\pi \lim_{N\to\infty} \frac{\log\left|J_N(K;\exp(2\pi\sqrt{-1}/N))\right|}{N} = \mathrm{Vol}(K), \tag{2.1}$$

where Vol(K) *is the simplicial volume* (*or Gromov norm*) *of the knot complement* $S^3 \setminus K$ (*see* [13] *and* [54, Chapter 6])*. Note that* Vol(K) *is normalized so that it coincides with the hyperbolic volume if K is a hyperbolic knot.*

This conjecture is so far proved for the following knots and links:

- torus knots by Kashaev and O. Tirkkonen [25]. Note that their simplicial volumes are zero since their complements contain no hyperbolic pieces. So in fact they proved that the left hand side of (2.1) converges to 0. See Theorem 4.1 and [9] for more precise asymptotic behaviors. See also [15, 16, 17, 18] for related topics.
- the figure-eight knot 4_1 by T. Ekholm (see for example [46, Section 3]),
- the hyperbolic knots 5_2, 6_1, and 6_2 by Y. Yokota,
- Whitehead doubles of torus knots of type $(2, b)$ by H. Zheng [60],
- twisted Whitehead links by Zheng [60],
- the Borromean rings by S. Garoufalidis and Lê [12, Theorem 12],
- Whitehead chains, which generalizes both the Borromean rings and twisted Whitehead links, by R. van der Veen [56].

Removing the absolute value sign of the left hand side of (2.1) we expect the Chern–Simons invariant as its imaginary part.

Conjecture 2.2 (Complexification of the Volume Conjecture, [49])**.** *For any knot K in S^3 we would have*

$$2\pi \lim_{N\to\infty} \frac{\log J_N(K;\exp(2\pi\sqrt{-1}/N))}{N} = \mathrm{Vol}(K) + \sqrt{-1}\,\mathrm{CS}(K), \tag{2.2}$$

where CS(K) *is the Chern–Simons invariant of the knot complement* [41] *if K is a hyperbolic knot. Note that we may regard the imaginary part of the left hand side as a definition of a* topological *Chern–Simons invariant.*

In [49], J. Murakami, M. Okamoto, T. Takata, Y. Yokota and the author used computer to confirm the Complexification of the Volume Conjecture for hyperbolic knots 6_3, 8_9, 8_{20} and for the Whitehead link up to several digits. (So we do not have rigorous proofs.)

Unfortunately it is observed in [9] that for torus knots except for the trefoil knot, the limit of (2.2) does knot exist. See Section 4 for details.

What happens if we replace $2\pi\sqrt{-1}$ with another complex number? Here is our conjecture which generalizes the complexification above.

Conjecture 2.3. *Let K be a knot in S^3. Then there exists a non-empty open set $\mathcal{O}$ of $\mathbb{C}$ such that for any $u \in \mathcal{O}$ the sequence*

$$\left\{\log\left(J_N\left(K;\exp\bigl((u+2\pi\sqrt{-1})/N\bigr)\right)\right)/N\right\}_{N=2,3,\dots}$$

converges and if we put

$$H(u) := (u+2\pi\sqrt{-1})\lim_{N\to\infty}\frac{\log J_N\left(K;\exp\bigl((u+2\pi\sqrt{-1})/N\bigr)\right)}{N}, \tag{2.3}$$

then $H(u)$ is analytic on $\mathcal{O}$. Moreover if we put

$$v(u) := 2\frac{d\,H(u)}{d\,u} - 2\pi\sqrt{-1},$$

then the function

$$h(u) := H(u) - \pi\sqrt{-1}\,u - \frac{1}{4}uv(u),$$

coincides with the f function appearing in (1.2) *and* (1.3) *in Subsection 1.3 up to a constant.*

If $H(0)$ exists, that is, if the limit of the right hand side of (2.3) *exists, then the function defined by*

$$\tilde{f}(u) := H(u) - \pi\sqrt{-1}\,u - \frac{1}{4}uv(u) - H(0)$$

coincides with the f function above modulo $\pi^2\mathbb{Z}$. Note that 0 may not be in $\mathcal{O}$.

Assuming Conjecture 2.3 above and the existence of $H(0)$, we have from (1.5)

$$\begin{aligned}\operatorname{Vol}(K_u) - \operatorname{Vol}(K) &= \operatorname{Im} H(u) - \operatorname{Im} H(0) - \pi \operatorname{Re} u - \frac{1}{4}\operatorname{Im}\bigl(uv(u)\bigr) + \frac{1}{4}\operatorname{Im}\bigl(u\overline{v(u)}\bigr) \\ &= \operatorname{Im} H(u) - \operatorname{Im} H(0) - \pi \operatorname{Re} u - \frac{1}{2}\operatorname{Re}(u)\operatorname{Im}\bigl(v(u)\bigr).\end{aligned}$$

From the Volume Conjecture (Conjecture 2.1) this is *almost* the same as the Parametrized Volume Conjecture [47, Conjecture 2.1], but note that we do not assume that the open set $\mathcal{O}$ contains 0.

In the following two sections we will show supporting evidence for Conjecture 2.3 above, giving the figure-eight knot and torus knots as examples.

3. Example 1 – figure-eight knot

In this section we describe how the colored Jones polynomials of the figure-eight knot E are related to representations of the fundamental group at $SL(2;\mathbb{C})$ and the corresponding volumes and Chern–Simons invariants. This gives evidence for Conjecture 2.3 if we put $\mathcal{O}$ to be a small open neighborhood of 0 minus purely imaginary numbers. Note that $H(0)$ exists in this case even though $0 \notin \mathcal{O}$.

3.1. **Representations of the fundamental group.** Here we follow R. Riley [53] to describe non-abelian representations of $\pi_1(S^3 \setminus E)$ at $SL(2;\mathbb{C})$, where E is the figure-eight knot. Let x and y be the generators of $\pi_1(S^3 \setminus E)$ indicated in Figure 5, where the basepoint of the fundamental group is above the paper. Then the other elements z and w indicated in Figure 6 can be expressed by x

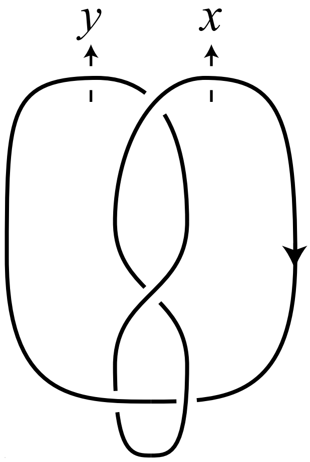


FIGURE 5. The two generators of $\pi_1(S^3 \setminus E)$

and y as follows.

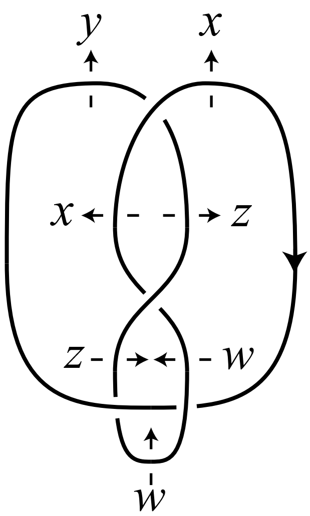


FIGURE 6. The other elements of $\pi_1(S^3 \setminus E)$

$$z = xyx^{-1} \quad \text{(from the top-most crossing)}, \tag{3.1}$$
$$w = z^{-1}xz \quad \text{(from the second top crossing)}.$$

Therefore we have

$$w = \left(xy^{-1}x^{-1}\right) x \left(xyx^{-1}\right) = xy^{-1}xyx^{-1} \tag{3.2}$$

Now from the bottom-left and the bottom-right crossings we have the following relations.

$$yw^{-1}y^{-1}z = 1,$$
$$yw^{-1}x^{-1}w = 1.$$

Using (3.1) and (3.2), these relations become a single relation

$$yxy^{-1}x^{-1}yx^{-1}y^{-1}xyx^{-1} = 1. \tag{3.3}$$

Therefore $\pi_1(S^3 \setminus E)$ has the two generators x and y with the single relation (3.3), that is:

$$\pi_1(S^3 \setminus E) = \langle x, y \mid \omega x = y\omega \rangle \tag{3.4}$$

with $\omega := xy^{-1}x^{-1}y$.

Let ρ be a non-abelian representation of $\pi_1(S^3 \setminus E)$ at $SL(2;\mathbb{C})$. From [53] we can assume up to conjugate that the images of $\rho(x)$ and $\rho(y)$ are given as follows.

$$\rho(x) := \begin{pmatrix} m^{1/2} & 1 \\ 0 & m^{-1/2} \end{pmatrix}, \quad \rho(y) := \begin{pmatrix} m^{1/2} & 0 \\ -d & m^{-1/2} \end{pmatrix}. \tag{3.5}$$

Then from the presentation (3.4), we have $\rho(\omega x) = \rho(y\omega)$. Since $\rho(\omega)$ is equal to

$$\begin{pmatrix} (d+1)^2 - dm & -m^{-1/2}(d-m+1) \\ dm^{-1/2}(d-m+1) & m^{-1}(m-d) \end{pmatrix},$$

we have

$$\rho(\omega x) = \begin{pmatrix} m^{1/2}((d+1)^2 - dm) & -m^{-1}(dm^2 - (d^2+2d+2)m + d + 1) \\ -d(m-d-1) & m^{-3/2}(-dm^2 + (d^2+d+1)m - d) \end{pmatrix},$$

and

$$\rho(y\omega) = \begin{pmatrix} m^{1/2}((d+1)^2 - dm) & m-d-1 \\ dm^{-1}(dm^2 - (d^2+2d+2)m + d + 1) & m^{-3/2}(-dm^2 + (d^2+d+1)m - d) \end{pmatrix}.$$

So d and m satisfy the equation

$$d^2 + d(3 - m - m^{-1}) + 3 - m - m^{-1} = 0, \tag{3.6}$$

and d becomes a function of m:

$$d = \frac{1}{2}\left(m + m^{-1} - 3 \pm \sqrt{(m + m^{-1} + 1)(m + m^{-1} - 3)}\right). \tag{3.7}$$

Therefore if we put $\xi := \operatorname{tr}\bigl(\rho(x)\bigr)$ and $\eta := \operatorname{tr}\bigl(\rho(xy)\bigr)$ as in Subsection 1.2, we have

$$\xi = m^{1/2} + m^{-1/2}$$
$$\eta = \operatorname{tr}\begin{pmatrix} m - d & m^{-1/2} \\ -dm^{-1/2} & m^{-1} \end{pmatrix} = m + m^{-1} - d.$$

Moreover since from (1.1) we calculate

$$\operatorname{tr}\bigl(\omega\bigr) = \eta^2 - \xi^2\eta + 2\xi^2 - 2$$
$$\operatorname{tr}\bigl(\omega^{(1)}\bigr) = \eta$$

we have

$$F(\xi, \eta) = \eta^2 - \eta + 2\xi^2 - \xi^2\eta - 1,$$

which also gives (3.6) (See Theorem 1.2.).

Now the longitude λ is (read off from the top right)

$$\lambda = wx^{-1}yz^{-1} = xy^{-1}xyx^{-1}x^{-1}yxy^{-1}x^{-1} = xy^{-1}xyx^{-2}yxy^{-1}x^{-1}.$$

From a direct calculation, we have

$$\rho(\lambda) = \begin{pmatrix} \ell(m)^{\pm 1} & (m^{1/2} + m^{-1/2})\sqrt{(m + m^{-1} + 1)(m + m^{-1} - 3)} \\ 0 & \ell(m)^{\mp 1} \end{pmatrix},$$

where

$$\ell(m) := \frac{\bigl(m^2 - m - 2 - m^{-1} + m^{-2}\bigr)}{2} + \frac{(m - m^{-1})}{2}\sqrt{(m + m^{-1} + 1)(m + m^{-1} - 3)}.$$

Note that $\ell(m)$ is a solution to the following equation.

$$\ell + (m^2 - m - 2 - m^{-1} + m^{-2}) + \ell^{-1} = 0, \tag{3.8}$$

which coincides with the A-polynomial of the figure-eight knot [6], replacing ℓ and m with $\mathfrak{l}$ and $\mathfrak{m}^2$, respectively.

Let $\rho_{m\pm}$ denote the representation (3.5) with d defined by (3.7).

3.2. Asymptotic behavior of the colored Jones polynomial.

Now we consider the colored Jones polynomial and its asymptotic behavior.

As described above, Habiro and Lê independently showed that for the figure-eight knot E, $J_N(E; q)$ can be expressed in a single summation as follows:

$$J_N(E; q) = \sum_{k=0}^{N-1} \prod_{j=1}^{k} \left(q^{(N-j)/2} - q^{-(N-j)/2}\right)\left(q^{(N+j)/2} - q^{-(N+j)/2}\right). \tag{3.9}$$

We put $q := \exp(\theta/N)$ and consider the asymptotic behavior of $J_N(E;\exp(\theta/N))$ for $N \to \infty$ fixing a complex parameter θ.

To state a formula describing the asymptotic behavior of $J_N(E;\exp(\theta/N))$ we prepare some functions. Put $\varphi(\theta) := \operatorname{arccosh}(\cosh(\theta) - 1/2)$, where we choose the branch of arccosh so that

$$\operatorname{arccosh}(x) = \log\left(x - \sqrt{-1}\sqrt{1-x^2}\right) + 2\pi\sqrt{-1}.$$

We also choose the branch cut of log as $(-\infty, 0)$. Note that we use $\sqrt{-1}\sqrt{1-x^2}$ instead of $\sqrt{x^2-1}$ to avoid the cut branch of the square root function, since later we will assume that x is near $1/2$. Let Li_2 be the dilogarithm function:

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-w)}{w}\,dw = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

with the branch cut $(1, \infty)$.

Now we can prove the following four theorems about the asymptotic behavior of the colored Jones polynomials of the figure-eight knot E.

(i). When θ is close to $2\pi\sqrt{-1}$.

Theorem 3.1 ([50])**.** *Let θ be a complex number near $2\pi\sqrt{-1}$. We also assume that θ is not purely imaginary except for $\theta = 2\pi\sqrt{-1}$. Then we have*

$$\theta \lim_{N\to\infty} \frac{\log J_N\bigl(E;\exp(\theta/N)\bigr)}{N} = H(\theta),$$

where we put

$$H(\theta) := \mathrm{Li}_2\left(e^{-\varphi(\theta)-\theta}\right) - \mathrm{Li}_2\left(e^{\varphi(\theta)-\theta}\right) + (\theta - 2\pi\sqrt{-1})\varphi(\theta).$$

(ii). When θ is real and $|\theta| \geqslant \operatorname{arccosh}(3/2)$.

Theorem 3.2 ([44, Theorem 8.1] (See also [46, Lemma 6.7]))**.** *Let θ be a real number with $|\theta| \geqslant \operatorname{arccosh}(3/2)$. Then we have*

$$\theta \lim_{N\to\infty} \frac{\log J_N\bigl(E;\exp(\theta/N)\bigr)}{N} = \tilde{H}(\theta),$$

where we put

$$\tilde{H}(\theta) := \mathrm{Li}_2\left(e^{-\tilde{\varphi}(\theta)-\theta}\right) - \mathrm{Li}_2\left(e^{\tilde{\varphi}(\theta)-\theta}\right) + \theta\tilde{\varphi}(\theta)$$

and $\tilde{\varphi}(\theta)$ is defined as $\varphi(\theta)$ by using the usual branch of arccosh *so that* $\operatorname{arccosh}(x) > 0$ *if* $|x| > 1$.

Note that $\tilde{H}(\theta) = 0$ if $\theta = \pm\operatorname{arccosh}(3/2)$.

(iii) When θ is close to 0.

Theorem 3.3 ([45],[11]). *Let θ be a complex number with $|2\cosh(\theta)-2|<1$ and $\operatorname{Im}(\theta)<\pi/3$, then the sequence $\{J_N\bigl(E;\exp(\theta/N)\bigr)\}_{N=2,3,\dots}$ converges and*

$$\lim_{N\to\infty} J_N\bigl(E;\exp(\theta/N)\bigr)=\frac{1}{\Delta\bigl(E;\exp(\theta)\bigr)},$$

where $\Delta(E;t)$ is the Alexander polynomial of the figure-eight knot E.

Here we normalize the Alexander polynomial for a knot K so that $\Delta(K;1)=1$ and that $\Delta(K;t^{-1})=\Delta(K;t)$.

Note that in this case

$$\lim_{N\to\infty}\frac{\log J_N\bigl(E;\exp(\theta/N)\bigr)}{N}=0.$$

(iv). When $\theta=\pm\operatorname{arccosh}(3/2)$.

From (ii) we know that when $\theta=\pm\operatorname{arccosh}(3/2)$ then the colored Jones polynomial does not grow exponentially. Since $\exp\bigl(\pm\operatorname{arccosh}(3/2)\bigr)=(3\pm\sqrt{5})/2$ is a zero of $\Delta(E;t)$, we cannot expect that the colored Jones polynomial converges. In fact in this case we can prove that it grows polynomially.

Theorem 3.4 ([19, Theorem 1.1]). *We have*

$$J_N(E;\exp(\pm\operatorname{arccosh}(3/2)/N))\underset{N\to\infty}{\sim}\frac{\Gamma(1/3)}{\bigl(3\operatorname{arccosh}(3/2)\bigr)^{2/3}}N^{2/3},$$

where Γ is the gamma function:

$$\Gamma(z):=\int_0^\infty t^{z-1}e^{-t}\,dt.$$

3.3. **Relation of the function H to a representation.** Now we want to relate the function H to a representation. It is convenient to put $u:=\theta-2\pi\sqrt{-1}$ so that the complete hyperbolic structure corresponds to $u=0$.

(i) When θ is close to $2\pi\sqrt{-1}$, that is, when u is close to 0.

We put

$$H(u):=\operatorname{Li}_2\left(e^{-\varphi(u)-u}\right)-\operatorname{Li}_2\left(e^{\varphi(u)-u}\right)+u\varphi(u). \tag{3.10}$$

Let us calculate the derivative $d\,H(u)/d\,u$. We have

$$
\begin{aligned}
&\frac{d\,H(u)}{d\,u}\\
&=-\frac{\log\left(1-e^{-\varphi(u)-u}\right)}{e^{-\varphi(u)-u}}\times e^{-\varphi(u)-u}\times\left(-\frac{d\,\varphi(u)}{d\,u}-1\right)\\
&\quad+\frac{\log\left(1-e^{\varphi(u)-u}\right)}{e^{\varphi(u)-u}}\times e^{\varphi(u)-u}\times\left(\frac{d\,\varphi(u)}{d\,u}-1\right)+\varphi(u)+u\times\frac{d\,\varphi(u)}{d\,u}\\
&=\frac{d\,\varphi(u)}{d\,u}\left\{\log\left(1-e^{-\varphi(u)-u}\right)+\log\left(1-e^{\varphi(u)-u}\right)+u\right\}\\
&\quad+\log\left(1-e^{-\varphi(u)-u}\right)-\log\left(1-e^{\varphi(u)-u}\right)+\varphi(u).
\end{aligned}
$$

Since $\varphi(u)$ satisfies $\cosh\bigl(\varphi(u)\bigr)=\cosh(u)-1/2$, we have

$$e^{\varphi(u)}+e^{\varphi(-u)}=e^{u}+e^{-u}-1$$

and so

$$\left(1-e^{-\varphi(u)-u}\right)\left(1-e^{\varphi(u)-u}\right)=e^{-u}\left(e^{u}-e^{-\varphi(u)}-e^{\varphi(u)}+e^{-u}\right)=e^{-u}.$$

Therefore we have

$$\frac{d\,H(u)}{d\,u}=2\log\left(1-e^{-\varphi(u)-u}\right)+\varphi(u)+u. \tag{3.11}$$

If we put

$$
\begin{aligned}
v(u):&=2\frac{d\,H(u)}{d\,u}-2\pi\sqrt{-1}\\
&=4\log\left(1-e^{-\varphi(u)-u}\right)+2\varphi(u)+2u-2\pi\sqrt{-1},
\end{aligned}
\tag{3.12}
$$

we have

$$\exp\left(\frac{v(u)}{2}\right)=-\exp\left(\log\left(e^{\varphi(u)+u}-2+e^{-\varphi(u)-u}\right)\right)=-e^{\varphi(u)+u}+2-e^{-\varphi(u)-u}.$$

We also put $m:=\exp(u)$. Then since

$$
\begin{aligned}
e^{\pm\varphi(u)}&=\cosh(u)-\frac{1}{2}\mp\sqrt{-1}\sqrt{1-\left(\cosh(u)-\frac{1}{2}\right)^{2}}\\
&=\frac{1}{2}\left(m+m^{-1}-1\right)\mp\frac{\sqrt{-1}}{2}\sqrt{(m+1+m^{-1})(3-m-m^{-1})},
\end{aligned}
$$

we have

$$
\begin{aligned}
\exp\left(\frac{v(u)}{2}\right)=&\frac{1}{2}\left(-m^{2}+m+2+m^{-1}-m^{-2}\right)\\
&-\frac{\sqrt{-1}\left(m-m^{-1}\right)}{2}\sqrt{(m+1+m^{-1})(3-m-m^{-1})}\\
=&-\ell(m).
\end{aligned}
$$

Therefore the representation $\rho_{m\pm}$ sends the longitude λ to

$$\begin{pmatrix} -e^{\pm v(u)/2} & * \\ 0 & -e^{\mp v(u)/2} \end{pmatrix}.$$

(ii) When θ is real and $|\theta| \geqslant \operatorname{arccosh}(3/2)$, that is, $\operatorname{Im} u = -2\pi\sqrt{-1}$ and $|\operatorname{Re} u| \geqslant \operatorname{arccosh}(3/2)$. In this case we put

$$\tilde{H}(u) := \operatorname{Li}_2\left(e^{-\tilde{\varphi}(u)-u}\right) - \operatorname{Li}_2\left(e^{\tilde{\varphi}(u)-u}\right) + (u + 2\pi\sqrt{-1})\tilde{\varphi}(u).$$

If we define $v(u)$ as in the previous case by using $\tilde{H}$ instead of H, then we see that u defines the same representations as (i).

(iii) When θ is close to 0, that is u is close to $-2\pi\sqrt{-1}$.

In this case we may say that θ defines the abelian representation $\alpha_{\exp(\theta/2)}$ that sends the meridian element to the matrix

$$\begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix}.$$

This is because the complex value $\Delta(E;\exp(\theta))$ is the determinant of the Fox matrix corresponding to the map $\pi_1(S^3 \setminus E) \to H_1(S^3;\mathbb{Z}) \cong \mathbb{Z} \to \mathbb{C}^*$, where the first map is the abelianization, and the second map sends k to $\exp(k\theta)$. See for example [39, Chapter 11].

(iv) When $\theta = \pm\operatorname{arccosh}(3/2)$, that is, $u = \pm\operatorname{arccosh}(3/2) - 2\pi\sqrt{-1}$. Note that when $u = \pm\operatorname{arccosh}(3/2) - 2\pi\sqrt{-1}$, the corresponding representation is non-abelian from (ii), but 'attached' to an abelian one from (iii).

Let us study $\rho_{m\pm}$ defined in Subsection 3.1 more carefully when

$$m = \exp\bigl(\pm\operatorname{arccosh}(3/2)\bigr) = (3 \pm \sqrt{5})/2.$$

First note that $d = 0$, and so ρ_{m+} and ρ_{m-} coincide. Moreover the images of both x and y are upper triangle. Therefore these representations are reducible.

By the linear fractional transformation, the Lie group $SL(2;\mathbb{C})$ acts on $\mathbb{C}$ by $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot z := (pz + q)/(rz + s)$. For an upper triangle matrix this becomes an affine transformation. Therefore if $\theta = \pm\operatorname{arccosh}(3/2)$ the corresponding representations can be regarded as affine.

It is known that each affine representation of a knot group corresponds to a zero of the Alexander polynomial of the knot ([4, 8]). See also [27, Exercise 11.2].

Note also that $\operatorname{tr}\bigl(\rho_{m\pm}\bigr) = \operatorname{tr}\bigl(\alpha_{m^{\pm 1/2}}\bigr)$ in this case.

3.4. **Volume and the Chern–Simons invariant.** Now we know that u defines a representation (with some ambiguity) of $\pi_1(S^3 \setminus E)$ at $SL(2;\mathbb{C})$. In this subsection we calculate the corresponding volume and Chern–Simons invariant.

(i) When u is close to 0.

In this case the parameter u defines an (incomplete) hyperbolic structure on $S^3 \setminus E$, and as described in Subsection 1.3, if u and $v(u)$ satisfy $p\,u + q\,v(u) = 2\pi\sqrt{-1}$ with coprime integers p and q, then we can construct a closed three-manifold E_u. In [50] we proved that Conjecture 2.3 holds in this case. Note that since $H(0) = \mathrm{Vol}(E)\sqrt{-1}$, which is the original volume conjecture 2.1 for the figure-eight knot, $\tilde{f}(u)$ defined by $H(u)$ coincides with the f function appearing in (1.2) and (1.3).

Example 3.5. As an example let us consider the case where u is real, and study $H(u)$ and the corresponding representation.

Remark 3.6. In this example we do not mind whether the sequence

$$\log\left(J_N\left(E;\exp\left((u+2\pi\sqrt{-1})/N\right)\right)\right)/N$$

really converges or not.

First we assume that $-\operatorname{arccosh}(3/2) < u < \operatorname{arccosh}(3/2)$. Note that $\operatorname{arccosh}(3/2) = 0.9624236501\dots$.

Since $|x - \sqrt{-1}\sqrt{1-x^2}| = 1$ when $|x| \leqslant 1$ and $1/2 \leqslant \cosh(u) - 1/2 < 1$ for real u with $|u| \leqslant \operatorname{arccosh}(3/2)$, we see that $\varphi(u)$ is purely imaginary. Then from (3.10) we have

$$H(u) = \mathrm{Li}_2(e^{-\varphi(u)-u}) - \overline{\mathrm{Li}_2(e^{-\varphi(u)-u})} + u\varphi(u)$$

and so $H(u)$ is purely imaginary, where $\overline{z}$ is the complex conjugate of z. We can also see that $v(u)$ is also purely imaginary from (3.12).

Let us consider the corresponding volume function:

$$\mathrm{Vol}(E_u) := \operatorname{Im} H(u) - \pi\,\mathrm{Re}(u) - \frac{1}{2}\,\mathrm{Re}(u)\,\mathrm{Im}\big(v(u)\big).$$

See Figures 7 and 8 for graphs of $\operatorname{Im} H(u)$ and $\mathrm{Vol}(E_u)$ respectively. Note that

$$\mathrm{Vol}(E_0) = \operatorname{Im} H(0) = 2.0298832128\dots,$$

which is the volume of $S^3 \setminus E$ with the complete hyperbolic structure. Note also that

$$\operatorname{Im} H(\operatorname{arccosh}(3/2)) = 2\pi\operatorname{arccosh}(3/2) = 6.0470861377\dots$$

and

$$\mathrm{Vol}\left(E_{\pm\operatorname{arccosh}(3/2)}\right) = 0$$

since $\varphi(\operatorname{arccosh}(3/2)) = 2\pi\sqrt{-1}$.

If we choose u_q so that $v(u_q) = 2\pi\sqrt{-1}/q$ for a positive integer q, then E_{u_q} is the cone-manifold with cone-angle $2\pi/q$ whose underlying manifold is obtained

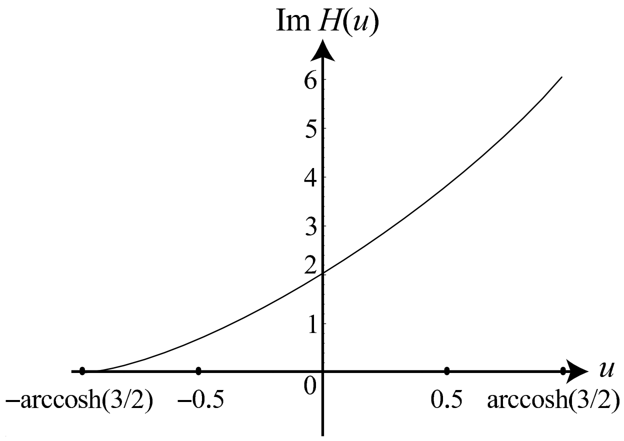


FIGURE 7. Graph of $\operatorname{Im} H(u)$ for $-\operatorname{arccosh}(3/2) \leqslant u \leqslant \operatorname{arccosh}(3/2)$

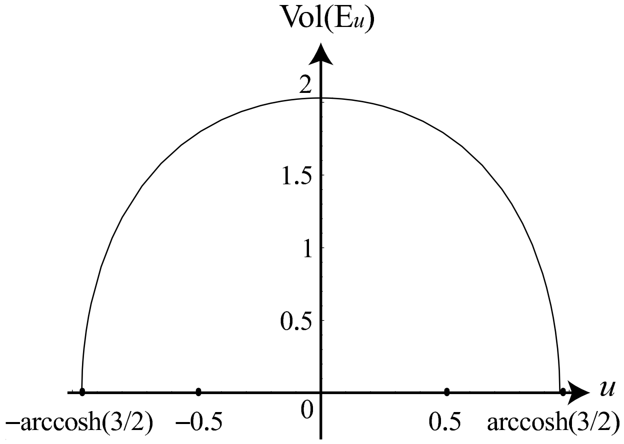


FIGURE 8. Graph of the volume function $\operatorname{Vol}(E_u)$ for $-\operatorname{arccosh}(3/2) \leqslant u \leqslant \operatorname{arccosh}(3/2)$

from the 0-surgery along the figure-eight knot (the singular set is the core of the surgery). This is because it corresponds to the generalized Dehn surgery of coefficient $(0, q)$, since $0 \times u_q + q \times v(u_q) = 2\pi\sqrt{-1}$ [54, Chapter 4]. Note that even though q is not an integer, $\operatorname{Vol}(E_{u_q})$ is still the volume of the corresponding incomplete hyperbolic manifold.

See [20] for geometric structures of the E_{u_q}. In that paper, H. Hilden, M. Lozano, and J.M. Montesinos-Amilibia also observed that when $q = 1$, that is, when $u = \pm \operatorname{arccosh}(3/2)$, the corresponding manifold is just the 0-surgery and has a Sol-geometry. They also calculated $\operatorname{Vol}(E_{u_q})$ with $q = 2, 3, \dots, 81$ [20, p. 559].

Next we consider the case where u is real and $|u| > \operatorname{arccosh}(3/2)$. In this case $\operatorname{Im} v(u) = 2\pi\sqrt{-1}$ since $\operatorname{Im} \varphi(u) = 2\pi\sqrt{-1}$.

If we put $p = -\operatorname{Re} v(u)/u$, $q = 1$, $s = 0$ and $r = -1$, then $p\,u + q\,v(u) = 2\pi\sqrt{-1}$ and $ps - qr = 1$. (For a while we do not mind whether p is an integer or not. See Remark 3.7 below.) Assuming that this would give a genuine manifold, we have from (1.4) and (2.3)

$$\operatorname{Vol}(E_u) + \sqrt{-1}\,\operatorname{CS}(M_u) - \left\{\operatorname{Vol}(E) + \sqrt{-1}\,\operatorname{CS}(E)\right\} \equiv \frac{H(u)}{\sqrt{-1}} - \frac{H(0)}{\sqrt{-1}} - \frac{3}{2}\pi u - \frac{uv(u)}{4\sqrt{-1}} \pmod{\pi^2\sqrt{-1}\mathbb{Z}}.$$

Since $H(0) = \sqrt{-1}\operatorname{Vol}(E)$ and $\operatorname{CS}(E) = 0$ from the amphicheirality of E (that is, the figure-eight knot is equivalent to its mirror image), we have

$$\operatorname{Vol}(E_u) = \operatorname{Im} H(u) - 2\pi u = 0$$

and

$$\operatorname{CS}(E_u) \equiv -\operatorname{Re} H(u) + \frac{1}{4} u \operatorname{Re} v(u) \pmod{\pi^2\mathbb{Z}},$$

since $\operatorname{Im}\varphi(u) = 2\pi$ if $|u| > \operatorname{arccosh}(3/2)$.

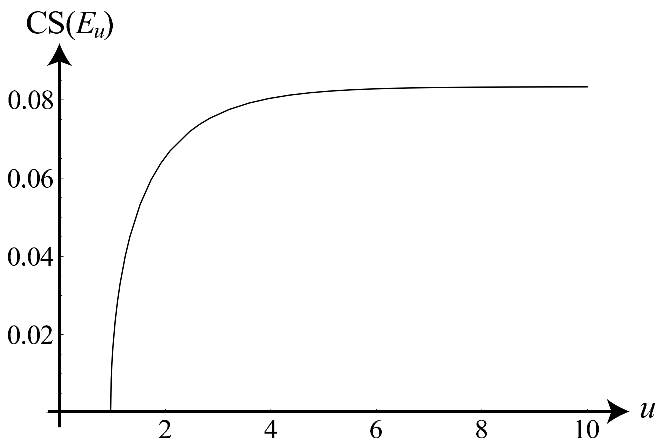


FIGURE 9. Graph of the $\operatorname{CS}(M_u)/(2\pi^2)$ for $\leqslant \operatorname{arccosh}(3/2) < u < 10$

Remark 3.7. Here we show some calculations by using Mathematica and SnapPea. See Figure 9.

If e^u satisfies the equation

$$x^6 - 4x^4 - 7x^3 - 4x^2 + 1 = 0$$

($u = 0.9839865622\ldots$), then $u + v(u) = 2\pi\sqrt{-1}$ and so this corresponds to the $(1,1)$-surgery along E. Moreover we have $\dfrac{\operatorname{CS}(E_u)}{2\pi\sqrt{-1}} = 0.01190476190\ldots = \dfrac{1}{84}$, which coincides with the calculation by SnapPea. Note that E_u in this case is the Brieskorn homology sphere $\Sigma(2,3,7)$, which is a Seifert fibered space over the sphere with three singular fibers. (See for example [31, Problem 1.77].)

If $u = \log\left(\dfrac{1+\sqrt{5}}{2} + \sqrt{\dfrac{\sqrt{5}+1}{2}}\right) = 1.061275062\ldots$, then $2u + v(u) = 2\pi\sqrt{-1}$ and so this corresponds to the $(2,1)$-surgery along E. We also have $\dfrac{\operatorname{CS}(E_u)}{2\pi^2} = 0.025 = \dfrac{1}{40}$, which coincides with the calculation by SnapPea. In this case E_u is the Brieskorn homology sphere $\Sigma(2,4,5)$.

If $u = \log\left(\frac{1}{2} + \sqrt{2} + \frac{1}{2}\sqrt{5+4\sqrt{2}}\right) = 1.265948638\ldots$, then $3u + v(u) = 2\pi\sqrt{-1}$, and so this corresponds to the $(3,1)$-surgery along E. We also have $\dfrac{\operatorname{CS}(E_u)}{2\pi^2} = 0.041666666667\ldots = \dfrac{1}{24}$, which coincides with the calculation by SnapPea. In this case E_u is $\Sigma(3,3,4)$.

Note that from Figure 9 it seems that

$$\lim_{u\to\infty} \frac{\operatorname{CS}(E_u)}{2\pi^2} = \frac{1}{12},$$

which would be the Chern–Simons invariant of E_∞ that corresponds to the $(4,1)$-surgery of the figure-eight knot.

(ii) When θ is real and $|\theta| > \operatorname{arccosh}(3/2)$.

In this case $\tilde{H}(u - 2\pi\sqrt{-1})$ is real and so $\operatorname{Im} v(u) = -2\pi$. Therefore formulas for $\operatorname{Vol}(E_u)$ and $\operatorname{CS}(E_u)$ similar to the case where u is real and $|u| > \operatorname{arccosh}(3/2)$ hold.

Remark 3.8. Note that

$$\operatorname{Re} H(u) = \operatorname{Re} \tilde{H}(u - 2\pi\sqrt{-1})$$

but $\operatorname{Im} H(u)$ and $\operatorname{Im} \tilde{H}(u - 2\pi\sqrt{-1})$ are different. In fact $\operatorname{Im} \tilde{H}(u - 2\pi\sqrt{-1}) = 0$ but $\operatorname{Im} H(u) = 2\pi u$ for $-\operatorname{arccosh}(3/2) < u < \operatorname{arccosh}(3/2)$. This means that if $|\operatorname{Re} u| \geqslant \operatorname{arccosh}(3/2)$, then the cases where $\operatorname{Im} u = 0$ and $\operatorname{Im} u = -2\pi\sqrt{-1}$ give different limits but the same Vol and CS. As observed in Example 3.5, this would correspond to cone-manifold whose underlying manifold is the 0-surgery along E.

4. Example 2 – Torus knots

In this section we study torus knots. Putting

$$\mathcal{O} := \{u \in \mathbb{C} \mid |u + 2\pi\sqrt{-1}| > 2\pi/(ab), \operatorname{Re}(u) < 0\},$$

we can show that Conjecture 2.3 holds for the torus knot $T(a,b)$. Note that $0 \notin \mathcal{O}$ and that $H(0)$ does not exist from Theorem 4.1 below.

In particular we will explicitly give representations of the trefoil knot $T := T(2,3)$ (Figure 3) and the cinquefoil knot $C := T(2,5)$ (Figure 12), and consider their relations to the asymptotic behaviors of their colored Jones polynomials.

4.1. **Representations of the fundamental group of the trefoil knot.** We choose the generators x and y for $\pi_1(S^3 \setminus T)$ as in Figure 10. Note that we take these generators so that their linking numbers with the knot are one. Then the

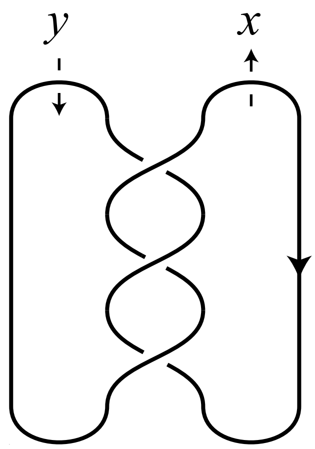


FIGURE 10. The two generators of $\pi_1(S^3 \setminus T)$

element z (see Figure 11) can be expressed in terms of x and y as follows:

$$z = xyx^{-1}. \tag{4.1}$$

From the second crossing and from the third, we have the following relations.

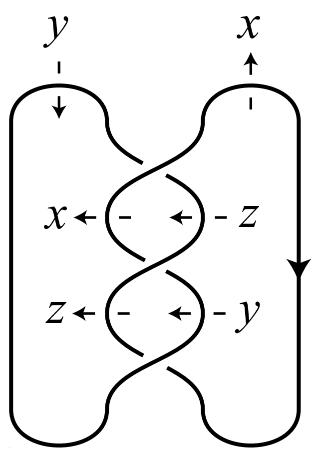


FIGURE 11. The other element of $\pi_1(S^3 \setminus T)$

$$yzx^{-1}z^{-1} = 1,$$
$$yzy^{-1}z^{-1} = 1.$$

Using (4.1), these relation are equivalent to the relation

$$yxy = xyx \tag{4.2}$$

Therefore $\pi_1(S^3 \setminus T)$ has the following presentation.

$$\pi_1(S^3 \setminus T) = \langle x, y \mid \omega x = y\omega \rangle$$

with $\omega := xy$.

Let ρ be a non-abelian representation of $\pi_1(S^3 \setminus T)$ at $SL(2;\mathbb{C})$. By [53], we can assume that ρ sends x and y to

$$\begin{pmatrix} m^{1/2} & 1 \\ 0 & m^{-1/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} m^{1/2} & 0 \\ -d & m^{-1/2} \end{pmatrix},$$

respectively. Since

$$\rho(\omega) = \begin{pmatrix} m-d & m^{-1/2} \\ -dm^{-1/2} & m^{-1} \end{pmatrix},$$

we have

$$\rho(y\omega) = \begin{pmatrix} m^{1/2}(m-d) & 1 \\ -d(m+m^{-1}-d) & m^{-3/2}-dm^{-1/2} \end{pmatrix}$$

and

$$\rho(\omega x) = \begin{pmatrix} m^{1/2}(m-d) & m+m^{-1}-d \\ -d & m^{-3/2}-dm^{-1/2} \end{pmatrix}.$$

Therefore d should equal $m + m^{-1} - 1$ and m uniquely defines a representation.

Putting $\xi := \operatorname{tr}\bigl(\rho(x)\bigr)$ and $\eta := \operatorname{tr}\bigl(\rho(xy)\bigr)$, we have

$$\xi = m^{1/2} + m^{-1/2}$$
$$\eta = m + m^{-1} - d$$

and

$$F(\xi, \eta) = P_\omega(\xi, \eta) = \eta - 1.$$

This confirms Theorem 1.2.

Now reading off from the top-right, the longitude of λ is

$$\lambda = yxzx^{-3} = yx^2yx^{-4}.$$

(Note that we add x^{-3} so that the longitude λ has linking number zero with the knot.) So its image by ρ is

$$\rho(\lambda) = \begin{pmatrix} -m^{-3} & \dfrac{m^3 - m^{-3}}{m^{1/2} - m^{-1/2}} \\ 0 & -m^3 \end{pmatrix}.$$

We also put $g := yzx = yxy$ and $h := xy$. Then we see that $g^2 = h^3$. So we have another presentation of $\pi_1(S^3 \setminus T)$:

$$\langle g, h \mid g^2 = h^3 \rangle.$$

We also compute

$$\rho(g) = \begin{pmatrix} m^{1/2} - m^{-1/2} & 1 \\ 1 - m - m^{-1} & -m^{1/2} + m^{-1/2} \end{pmatrix}$$

and

$$\rho(h) = \begin{pmatrix} 1 - m^{-1} & m^{-1/2} \\ -m^{1/2} + m^{-1/2} - m^{-3/2} & m^{-1} \end{pmatrix}.$$

Therefore we have $\operatorname{tr}\bigl(\rho(g)\bigr) = 0$ and $\operatorname{tr}\bigl(\rho(h)\bigr) = 1$, and so this representation belongs to the same component as $\rho_{1,1}$ in Theorem 1.4 in the $SL(2;\mathbb{C})$-character variety.

4.2. **Representations of the fundamental group of the cinquefoil knot.** Our next example is the cinquefoil knot (or the double overhand knot) $C = T(2,5)$. The elements z, w and v indicated in Figure 12 are presented in terms

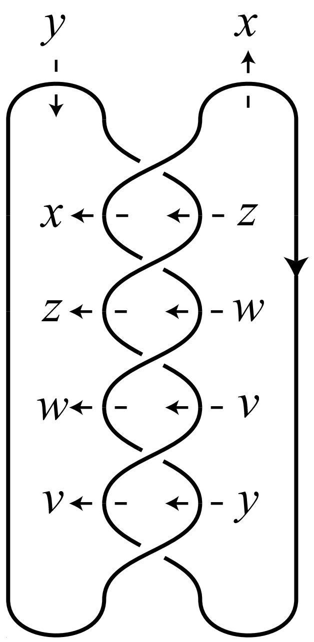


FIGURE 12. Cinquefoil knot and elements of $\pi_1(S^3 \setminus C)$

of x and y:

$$\begin{aligned} z &= xyx^{-1}, \\ w &= zxz^{-1} = xyxy^{-1}x^{-1}, \\ v &= wzw^{-1} = xyxyx^{-1}y^{-1}x^{-1}. \end{aligned}$$

The last two crossings give the same relation

$$yxyxyx^{-1}y^{-1}x^{-1}y^{-1}x^{-1} = 1.$$

Therefore we have a presentation of $\pi_1(S^3 \setminus C)$.

$$\pi_1(S^3 \setminus C) = \langle x, y \mid \omega x = y\omega \rangle$$

with $\omega = xyxy$. Then the longitude λ is

$$\lambda = yxyxy^{-1}xyxyx^{-7}.$$

Let ρ be a non-abelian representation of $\pi_1(S^3 \setminus C)$ at $SL(2, \mathbb{C})$. Then we may assume that

$$\rho(x) = \begin{pmatrix} m^{1/2} & 1 \\ 0 & m^{-1/2} \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} m^{1/2} & 0 \\ -d & m^{-1/2} \end{pmatrix}.$$

Similar calculations show that d and m satisfy the following equation.

$$d^2 - (2m + 2m^{-1} - 1)d + m^2 - m + 1 - m^{-1} + m^{-2}. \tag{4.3}$$

If we put $\xi := \operatorname{tr}\bigl(\rho(x)\bigr)$ and $\eta : \operatorname{tr}\bigl(xy\bigr)$, then we have

$$\xi = m^{1/2} + m^{-1/2},$$
$$\eta = m + m^{-1} - d$$

and

$$F(\xi, \eta) = P_{xyxy}(\xi, \eta) - P_{xy} + 1 = \eta^2 - \eta - 1,$$

which coincides (4.3) and this confirms Theorem 1.2.

So we have

$$d = m + m^{-1} - \frac{1 \pm \sqrt{5}}{2}$$

and

$$\rho(\lambda) = \begin{pmatrix} -m^{-5} & \dfrac{m^5 - m^{-5}}{m^{1/2} - m^{-1/2}} \\ 0 & -m^5 \end{pmatrix}$$

for both choices of d, where ρ is given as before. We put $d_\pm := m + m^{-1} - \dfrac{1 \pm \sqrt{5}}{2}$ and denote the corresponding representation by $\rho_\pm$. Note that if we put $g := yvwzx = yxyxy$ and $h := xy$ then $g^2 = h^5$, and so we have another presentation of $\pi_1(S^3 \setminus C)$:

$$\langle g, h \mid g^2 = h^5 \rangle.$$

We also calculate

$$\rho_\pm(g) = \begin{pmatrix} \dfrac{1 \pm \sqrt{5}}{2}(m^{1/2} - m^{-1/2}) & \dfrac{1 \pm \sqrt{5}}{2} \\ -\dfrac{1 \pm \sqrt{5}}{2}(m - 1 + m^{-1}) + 1 & -\dfrac{1 \pm \sqrt{5}}{2}(m^{1/2} - m^{-1/2}) \end{pmatrix},$$

$$\rho_\pm(h) = \begin{pmatrix} \dfrac{1 \pm \sqrt{5}}{2} - m^{-1} & m^{-1/2} \\ -m^{-3/2} - m^{1/2} + \dfrac{1 \pm \sqrt{5}}{2}m^{-1/2} & m^{-1} \end{pmatrix}.$$

Therefore we have $\operatorname{tr}\bigl(\rho_{\pm}(g)\bigr)=0$, $\operatorname{tr}\bigl(\rho_{+}(h)\bigr)=\cos(\pi/5)$ and $\operatorname{tr}\bigl(\rho_{-}(h)\bigr)=\cos(3\pi/5)$. This shows that ρ_+ and ρ_- belong to the same component as $\rho_{1,1}$ and $\rho_{1,3}$ respectively in Theorem 1.4.

4.3. **Asymptotic behavior of the colored Jones polynomial.** First of all, the volume conjecture for torus knots is proved by Kashaev and Tirkkonen [25]. In fact J. Dubois and Kashaev [9] proved a stronger result.

Theorem 4.1 ([25])**.** *Let $T(a,b)$ be the torus knot of type (a,b). Then we have the following asymptotic equality for large N.*

$$\begin{aligned}&J_N\bigl(T(a,b);\exp(2\pi\sqrt{-1}/N)\bigr)\\&=\exp\left(\left(ab-\frac{a}{b}-\frac{b}{a}\right)\frac{\pi\sqrt{-1}}{2N}\right)\\&\times\left\{\frac{\exp\left(\frac{\pi\sqrt{-1}}{4}\right)}{4\sqrt{2ab}}N^{3/2}\exp\left(\frac{-abN\pi\sqrt{-1}}{2}\right)Z_N\bigl(T(a,b)\bigr)\right.\\&\qquad\left.+\frac{1}{4}\sum_{k=1}^{\infty}\frac{a_k\bigl(T(a,b)\bigr)}{k!}\left(\frac{\pi\sqrt{-1}}{2abN}\right)^{k-1}\right\},\end{aligned}$$

where

$$Z_N\bigl(T(a,b)\bigr):=\sum_{j=1}^{ab-1}(-1)^{(N-1)j}j^2\frac{4}{ab}\sin\left(\frac{j\pi}{a}\right)\sin\left(\frac{j\pi}{b}\right)\exp\left(\frac{-j^2N\pi\sqrt{-1}}{2ab}\right).$$

and $a_k\left(T(a,b)\right)$ is a finite type invariant for every k. See [9] *for a topological interpretation of each term of $Z_N\bigl(T(a,b)\bigr)$.*

Observe that the limit of (2.1) always exists but that that of (2.2) does not unless $(a,b)=(2,3)$ or $(a,b)=(3,2)$.

Next we review the results in [43] and [19].

Theorem 4.2. *Let θ be a complex number with $|\theta|>2\pi/(ab)$.*

- *If* $\operatorname{Re}(\theta)>0$*, then we have*
$$\lim_{N\to\infty}J_N\bigl(T(a,b);\exp(\theta/N)\bigr)=\frac{1}{\Delta\bigl(T(a,b);\exp(\theta)\bigr)}.$$
- *If* $\operatorname{Re}(\theta)<0$*, then we have*
$$\lim_{N\to\infty}\frac{\log J_N\bigl(T(a,b);\exp(\theta/N)\bigr)}{N}=\left(1-\frac{\pi\sqrt{-1}}{ab\theta}-\frac{ab\theta}{4\pi\sqrt{-1}}\right)\pi\sqrt{-1}.$$

Remark 4.3. In [43], the author only proved the case where $\operatorname{Im}(\theta)>0$. But taking the complex conjugate we have a similar formula for the other case. This was pointed out by A. Gibson.

Putting $u:=\theta-2\pi\sqrt{-1}$, we have

Corollary 4.4. *Let u be a complex number with $|u+2\pi\sqrt{-1}| > 2\pi/(ab)$.*

- *If* $\operatorname{Re}(u) > 0$*, then we have*

$$\lim_{N\to\infty} J_N\bigl(T(a,b);(u+2\pi\sqrt{-1})/N\bigr) = \frac{1}{\Delta\bigl(T(a,b);\exp(u)\bigr)}.$$

- *If* $\operatorname{Re}(u) < 0$*, then we have*

$$(u+2\pi\sqrt{-1}) \lim_{N\to\infty} \frac{\log J_N\bigl(T(a,b);\exp((u+2\pi\sqrt{-1})/N)\bigr)}{N} = \frac{-1}{4ab}\left\{ab(u+2\pi\sqrt{-1}) - 2\pi\sqrt{-1}\right\}^2.$$

For small θ, Garoufalidis and Lê proved the following.

Theorem 4.5 ([11, Theorem 1])**.** *For any knot K, there exists a neighborhood U of* 0 *such that if $\theta \in U$ then*

$$\lim_{N\to\infty} J_N\bigl(K;\exp(\theta/N)\bigr) = \frac{1}{\Delta(K;\exp(\theta))}.$$

As in the case of the figure-eight knot when $\exp(\theta)$ is a zero of the Alexander polynomial, we can prove that the colored Jones polynomial grows polynomially.

Theorem 4.6 ([19, Theorem 1.2])**.** *We have*

$$J_N\left(T(a,b);\exp\bigl(\pm 2\pi\sqrt{-1}/(abN)\bigr)\right) \underset{N\to\infty}{\sim} e^{\mp\pi\sqrt{-1}/4}\frac{\sin(\pi/a)\sin(\pi/b)}{\sqrt{2}\sin(\pi/(ab))}N^{1/2}.$$

4.4. **Relation of the function H to a representation.** If $|\theta|$ is small (Theorem 4.5), or $|\theta| > 2\pi/(ab)$ and $\operatorname{Re}(\theta) > 0$ (Theorem 4.2), then we can associate an abelian representation as in Subsection 3.3 (iii).

If $|\theta| > 2\pi/(ab)$ and $\operatorname{Re}(\theta) < 0$, that is, if $|u+2\pi\sqrt{-1}| > 2\pi/(ab)$ and $\operatorname{Re}(u) < 0$, we put

$$H(u) := \frac{-1}{4ab}\left\{ab(u+2\pi\sqrt{-1}) - 2\pi\sqrt{-1}\right\}^2$$

and consider its relation to representations at $SL(2;\mathbb{C})$. Note that $H(0)$ does not exist, that is, the limit of (2.2) does not exist except for the case of the trefoil from Theorem 4.1.

The functions v and h defined in Conjecture 2.3 become

$$v(u) = -ab(u+2\pi\sqrt{-1}), \tag{4.4}$$

and

$$h(u) = -2\pi^2 + \frac{\pi^2}{ab} + ab\pi^2 - \frac{1}{2}abu\pi\sqrt{-1}. \tag{4.5}$$

To consider which representation is associated with u, we will recall representations of the trefoil knot and the cinquefoil knot described in Subsections 4.1 and 4.2.

4.4.1. *Relation of the function H to a representation – trefoil knot.* Put $m := \exp(u)$ and consider the representation ρ described in Subsection 4.1. Then the meridian $\mu := x$ is sent to the matrix

$$\begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix},$$

and the longitude λ is to

$$\begin{pmatrix} -e^{-3u} & \dfrac{e^{3u}-e^{-3u}}{e^{u/2}-e^{-u/2}} \\ 0 & -e^{3u} \end{pmatrix} = \begin{pmatrix} -e^{v(u)/2} & * \\ 0 & -e^{-v(u)/2} \end{pmatrix}.$$

Therefore the situation here is the same as the case of the figure-eight knot in Subsection 3.3.

4.4.2. *Relation of the function H to a representation – cinquefoil knot.* Similarly we put $m := \exp(u)$ and consider the representations $\rho_{\pm}$ described in Subsection 4.2. Then the meridian $\mu := x$ and the longitude λ are sent to

$$\begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix}$$

and

$$\begin{pmatrix} -e^{v(u)/2} & * \\ 0 & -e^{-v(u)/2} \end{pmatrix}$$

respectively. Therefore both $\rho_{\pm}$ are candidates of such a representation so far.

4.5. **Volume and the Chern–Simons invariant.** The Chern–Simons function of a torus knot for non-abelian part of the $SL(2;\mathbb{C})$-character variety is calculated in [9, Proposition 7]. Let $\rho_{k,l}$ be the representation described in Theorem 1.4. Then J. Dubois and Kashaev proved the following formula [9, Proposition 7]:

$$\mathrm{cs}_{T(a,b)}([\rho_{k,l}]) = \left[\frac{u}{4\pi\sqrt{-1}}, \frac{1}{2} - \frac{abu}{4\pi\sqrt{-1}}; \exp\left(2\pi\sqrt{-1}\left(\frac{(lad+\varepsilon kbc)^2}{4ab} - \frac{u}{8\pi\sqrt{-1}}\right)\right)\right],$$

where $\varepsilon = \pm$ and the right hand side does not depend of the choice of ε.

We want to express the right hand side by using the same basis as in (1.3). From Subsection 1.3, this is equivalent to

$$
\begin{aligned}
&\operatorname{cs}_{T(a,b)}\left([\rho_{k,l}]\right) \\
=&\Bigl[\frac{u}{4\pi\sqrt{-1}}, \frac{1}{2}-\frac{abu}{4\pi\sqrt{-1}}-\frac{ab+1}{2}; \\
&\qquad \exp\left(2\pi\sqrt{-1}\left(\frac{(lad+\varepsilon kbc)^2}{4ab}-\frac{u}{8\pi\sqrt{-1}}+\left(\frac{ab+1}{2}\right)\frac{u}{4\pi\sqrt{-1}}\right)\right)\Bigr] \\
=&\left[\frac{u}{4\pi\sqrt{-1}}, \frac{-ab(u+2\pi\sqrt{-1})}{4\pi\sqrt{-1}}; \exp\left(\frac{\pi\sqrt{-1}(lad+\varepsilon kbc)^2}{2ab}+\frac{abu}{4}\right)\right] \\
=&\left[\frac{u}{4\pi\sqrt{-1}}, \frac{v(u)}{4\pi\sqrt{-1}}; \exp\left(\frac{\pi\sqrt{-1}(lad+\varepsilon kbc)^2}{2ab}+\frac{abu}{4}\right)\right].
\end{aligned}
\tag{4.6}
$$

from (4.4).

Remark 4.7. A careful reader may notice that we subtract a half integer $(ab+1)/2$ from the second term if ab is even, which is not allowed in $SL(2;\mathbb{C})$ theory. We may be using $PSL(2;\mathbb{C})$ theory indeed (see [33, p. 543] and Remark 1.6).

Now we will compare this formula with the h function derived from H in (4.5).

4.5.1. *Volume and the Chern–Simons invariant of a representation – trefoil knot.* Putting $m := \exp(u)$, $a := 2$, $b := 3$, $c := -1$, $d := -1$ and $k = l = 1$ in (4.6), we have

$$
\operatorname{cs}_T\bigl([\rho]\bigr) = \left[\frac{u}{4\pi\sqrt{-1}}, \frac{3(u+2\pi\sqrt{-1})}{2\pi\sqrt{-1}}; \exp\left(\frac{\pi\sqrt{-1}}{12}+\frac{3u}{2}\right)\right].
$$

Therefore the corresponding f function introduced in Subsection 1.3 is

$$
\begin{aligned}
f(u) :=& \frac{2\pi}{\sqrt{-1}} \times \left(\frac{\pi\sqrt{-1}}{12}+\frac{3u}{2}\right) \\
=& \frac{\pi^2}{6} - 3u\pi\sqrt{-1} \pmod{\pi^2\mathbb{Z}}.
\end{aligned}
$$

The corresponding h function defined by the H function is:

$$
h(u) = \frac{\pi^2}{6} - 3u\pi\sqrt{-1} + 4\pi^2
$$

from (4.5). Therefore $f(u)$ and $h(u)$ coincide modulo $\pi^2\mathbb{Z}$.

4.5.2. *Volume and the Chern–Simons invariant of a representation – cinquefoil knot.* Putting $m = \exp(u)$, $a := 2$, $b := 5$, $c := -1$, $d := -2$, we have

$$
\operatorname{cs}_C\left([\rho_+]\right) = \left[\frac{u}{4\pi\sqrt{-1}}, \frac{-5(u+2\pi\sqrt{-1})}{2\pi\sqrt{-1}}; \exp\left(\frac{\pi\sqrt{-1}}{20}+\frac{5u}{2}\right)\right]
$$

and

$$
\operatorname{cs}_C\left([\rho_-]\right) = \left[\frac{u}{4\pi\sqrt{-1}}, \frac{-5(u+2\pi\sqrt{-1})}{2\pi\sqrt{-1}}; \exp\left(\frac{9\pi\sqrt{-1}}{20}+\frac{5u}{2}\right)\right],
$$

where we put $k = l = 1$ for ρ_+ and $k = 1$ and $l = 3$ for ρ_-. So the corresponding f functions are

$$f(u) = \frac{\pi^2}{10} - 5u\pi\sqrt{-1} \pmod{\pi^2\mathbb{Z}},$$

and

$$f(u) = \frac{9\pi^2}{10} - 5u\pi\sqrt{-1} \pmod{\pi^2\mathbb{Z}},$$

respectively. On the other hand the h function defined by the H function is:

$$\frac{\pi^2}{10} - 5u\pi\sqrt{-1} + 8\pi^2.$$

Therefore if we choose ρ_+, h and f coincide modulo $\pi^2\mathbb{Z}$.

4.5.3. *Volume and the Chern–Simons invariant of a representation – general torus knot.* Now for a general torus knot $T(a,b)$, let us consider the representation $\rho_{1,1}$ parametrized by $(k,l) = (1,1)$. Then from (4.6) we have

$$\operatorname{cs}_{T(a,b)}([\rho_{1,1}]) = \left[\frac{u}{4\pi\sqrt{-1}}, \frac{-ab(u+2\pi\sqrt{-1})}{4\pi\sqrt{-1}}; \exp\left(\frac{\pi\sqrt{-1}}{2ab} + \frac{abu}{4}\right)\right],$$

since $ad - bc = 1$. So we have

$$f(u) = \frac{\pi^2}{ab} - \frac{1}{2}abu\pi\sqrt{-1} \pmod{\pi^2\mathbb{Z}}$$

and

$$h(u) = \frac{\pi^2}{ab} - \frac{1}{2}abu\pi\sqrt{-1} + (ab-2)\pi^2.$$

Therefore for a general torus knot h and f coincide modulo $\pi^2\mathbb{Z}$ if we choose $\rho_{1,1}$.

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