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UNIQUENESS THEOREMS AND UNIQUENESS POLYNOMIALS FOR p-ADIC HOLOMORPHIC CURVES

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Abstract. In this paper, using the techniques of value distribution theory in the p-adic case, we prove uniqueness theorems for nonconstant p-adic holomorphic curves and the existence of a class of strong uniqueness polynomials for algebraically nondegenerate p-adic holomorphic curves. It is shown that if X is a hypersurface defined by a polynomial in this class, then X is a unique range set for algebraically nondegenerate p-adic holomorphic curves.

1. Introduction

In 1926, Nevanlinna proved that two nonconstant meromorphic functions of one complex variable which attain the same five distinct values at the same points, must be identical.

It is observed that p-adic entire functions of one variable behave in many ways more like polynomials than like entire functions of one complex variable. In 1971, Adams and Straus [1] proved the following theorem.

Theorem A. *Let f, g be two nonconstant p-adic entire functions such that for two distinct (finite) values a, b we have $f(x) = a \Leftrightarrow g(x) = a$ and $f(x) = b \Leftrightarrow g(x) = b$. Then $f \equiv g$.*

For p-adic meromorphic functions, Adams and Straus [1] obtained the following result similar to Nevanlinna's.

Theorem B. *Let f, g be two nonconstant p-adic meromorphic functions such that for four distinct values a_1, a_2, a_3, a_4 we have $f(x) = a_i \Leftrightarrow g(x) = a_i, i = 1, 2, 3, 4$. Then $f \equiv g$.*

Ru [10] and Hu and Yang [5, Theorem 6.33] extended Theorem B to p-adic holomorphic curves. As a connection to the study of the uniqueness problem, many authors (see [5, 8]) introduced the following notions.

Definition C. A polynomial $P \in \mathbb{C}_p[z]$ is called a *uniqueness polynomial* (UPM for short) for p-adic meromorphic functions if $P(f) = P(g)$ then $f = g$ for all nonconstant meromorphic functions f, g on $\mathbb{C}_p$. Similarly, we say that a strong

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uniqueness polynomial (SUPM for short) for p-adic meromorphic functions if $P(f) = cP(g), c \neq 0$ implies $f = g$ for all nonconstant meromorphic functions f, g on $\mathbb{C}_p$ and nonzero $c \in \mathbb{C}_p$.

Similarly, a homogeneous polynomial P of variables $z_1, \dots, z_{n+1}$ is said to be a *uniqueness polynomial* (UPC for short) for p-adic holomorphic curves if the condition $P(\tilde{f}) = P(\tilde{g})$ implies $f = g$ for all algebraically nondegenerate holomorphic curves f and g from $\mathbb{C}_p$ to $\mathbb{P}^n(\mathbb{C}_p)$ with reduced representations $\tilde{f}$ and $\tilde{g}$, respectively. A homogeneous polynomial P of variables $z_1, \dots, z_{n+1}$ is said to be a *strong uniqueness polynomial* (SUPC for short) for p-adic holomorphic curves if $P(\tilde{f}) = cP(\tilde{g})$ implies $f = g$ for all algebraically nondegenerate holomorphic curves f and g from $\mathbb{C}_p$ to $\mathbb{P}^n(\mathbb{C}_p)$ with reduced representations $\tilde{f}$ and $\tilde{g}$, respectively, and nonzero $c \in \mathbb{C}_p$.

In recent years, several interesting results about UPM have been obtained (see [2, 3, 5, 8]).

In this paper, by using some arguments in [5, 8, 11, 12] we give some uniqueness theorems for nonconstant p-adic holomorphic curves and show the existence of a class of strong uniqueness polynomials for algebraically nondegenerate p-adic holomorphic curves. It is shown that if X is a hypersurface defined by a polynomial in this class, then X is a unique range set for algebraically nondegenerate p-adic holomorphic curves.

2. Uniqueness theorems for nonconstant p-adic holomorphic curves

Let f be a nonzero holomorphic function on $\mathbb{C}_p$. For every $a \in \mathbb{C}_p$, expanding f as $f = \sum P_i(z-a)$ with homogeneous polynomials P_i of degree i around a, we define

$$v_f(a) = \min\{i : P_i \not\equiv 0\}.$$

Let k be a positive integer or $+\infty$. Define the function $v_f^{\leqslant k}$ of $\mathbb{C}_p$ into $\mathbb{N}$ by

$$v_f^{\leqslant k}(z) = \begin{cases} 0 & \text{if } v_f(z) > k, \\ v_f(z) & \text{if } v_f(z) \leqslant k \end{cases}$$

and

$$n_f^{\leqslant k}(r) = \sum_{|z| \leqslant r} v_f^{\leqslant k}(z), \quad n_f^{\leqslant k}(a, r) = n_{f-a}^{\leqslant k}(r).$$

Fix a real number ρ, with $0 < \rho \leqslant r$. Define

$$N_f^{\leqslant k}(a, r) = \frac{1}{\ln p} \int_\rho^r \frac{n_f^{\leqslant k}(a, x)}{x} dx.$$

If $a = 0$, then set $N_f^{\leqslant k}(r) = N_f^{\leqslant k}(0, r)$.

For l a positive integer or $+\infty$, set

$$N_{l,f}^{\leqslant k}(a,r)=\frac{1}{\ln p}\int\limits_{\rho}^{r}\frac{n_{l,f}^{\leqslant k}(a,x)dx}{x},$$

where

$$n_{l,f}^{\leqslant k}(a,r)=\sum_{|z|\leqslant r}\min\left\{v_{f-a}^{\leqslant k}(z),l\right\}.$$

Define $v_f^{>k}$ from $\mathbb{C}_p$ to $\mathbb{N}$ by

$$v_f^{>k}(z)=\begin{cases}v_f(z) & \text{if} v_f(z)>k,\\ 0 & \text{if} v_f(z)\leqslant k\end{cases}$$

and

$$n_f^{>k}(r)=\sum_{|z|\leqslant r}v_f^{>k}(r),\ n_f^{>k}(a,r)=n_{f-a}^{>k}(r),$$

$$N_f^{>k}(a,r)=\frac{1}{\ln p}\int_{\rho}^{r}\frac{n_f^{>k}(a,x)}{x}dx,$$

$$N_f^{>k}(r)=N_f^{>k}(0,r),$$

$$N_{l,f}^{>k}(a,r)=\frac{1}{\ln p}\int_{\rho}^{r}\frac{n_{l,f}^{>k}(a,r)}{x}dx,$$

where

$$n_{l,f}^{>k}(a,r)=\sum_{|z|\leqslant r}\min\left\{v_{f-a}^{>k}(z),l\right\}.$$

Let f be a holomorphic curve from $\mathbb{C}_p$ to $\mathbb{P}^n(\mathbb{C}_p)=\mathbb{P}^n$ with reduced representation $\tilde{f}=(f_1,...,f_{n+1})$. Set

$$H_f(r)=\max_{1\leqslant i\leqslant n+1}H_{f_i}(r).$$

Let H be a hypersurface of $\mathbb{P}^n$ such that the image of f is not contained in H, and H is defined by the equation $F=0$.

Set

$$H_f(H,r)=H_{F\circ\tilde{f}}(r),\ N(H,r)=N_{F\circ\tilde{f}}(r),$$

$$N_{k,f}(H,r)=N_{k,F\circ\tilde{f}}(r),\ N_{l,f}^{\leqslant k}(H,r)=N_{l,F\circ\tilde{f}}^{\leqslant k}(r),\ N_{l,f}^{>k}(H,r)=N_{l,F\circ\tilde{f}}^{>k}(r).$$

$$E_f(H)=\left\{z\in\mathbb{C}_p : F\circ\tilde{f}(z)=0\ \text{ counting multiplicities}\right\},$$

$$\overline{E}_f(H)=\left\{z\in\mathbb{C}_p : F\circ\tilde{f}(z)=0\ \text{ ignoring multiplicities}\right\},$$

$$\overline{E}_f(H,\leqslant k)=\left\{z\in\mathbb{C}_p : F\circ\tilde{f}(z)=0\ \text{ ignoring multiplicities, } v_{F\circ\tilde{f}}(z)\leqslant k\right\},$$

$$\overline{E}_f(H,>k)=\left\{z\in\mathbb{C}_p : F\circ\tilde{f}(z)=0\ \text{ ignoring multiplicities, } v_{F\circ\tilde{f}}(z)>k\right\},$$

$$\overline{E}_f(H,\geqslant k)=\left\{z\in\mathbb{C}_p : F\circ\tilde{f}(z)=0\ \text{ ignoring multiplicities, } v_{F\circ\tilde{f}}(z)\geqslant k\right\}.$$

Notices that if f is a holomorphic curve from $\mathbb{C}_p$ to $\mathbb{P}^n$ and if $\tilde{f} = (f_1, ..., f_{n+1})$ and $\tilde{h} = (h_1, ..., h_{n+1})$ are two reduced representations of f, then there exists a constant c, $c \neq 0$, such that $f_i = ch_i$ for all i. Therefore, the above definitions are well-defined.

Theorem 2.1. [5] *Let $f : \mathbb{C}_p \longrightarrow \mathbb{P}^n$ be a m-linearly nondegenerate holomorphic curve from $\mathbb{C}_p$ to $\mathbb{P}^n$ and $H_1, \dots, H_q$ be hyperplanes of $\mathbb{P}^n$ in general position with $1 \leqslant m \leqslant n \leqslant 2n - m < q$ and $f(\mathbb{C}_p) \not\subset H_i$, $i = 1, \dots, q$. Then*

$$(q - 2n + m - 1)H_f(r) \leqslant \sum_{i=1}^{q} N_{m,f}(H_i, r) - \frac{m(n+1)}{2} \log r + O(1).$$

To prove this theorem, we need the following:

Lemma 2.1. *Let $f : \mathbb{C}_p \longrightarrow \mathbb{P}^n$ be a m-linearly nondegenerate holomorphic curve from $\mathbb{C}_p$ to $\mathbb{P}^n$, $k_1, \dots, k_q \in \mathbb{N}^*$ and $H_1, \dots, H_q$ be hyperplanes of $\mathbb{P}^n$ in general position with $1 \leqslant m \leqslant n \leqslant 2n - m < q$ and $f(\mathbb{C}_p) \not\subset H_i$, $i = 1, \dots, q$. Then*

$$\Big(\sum_{i=1}^{q} \frac{k_i - m + 1}{k_i + 1} - 2n + m - 1\Big) H_f(r) \leqslant \sum_{i=1}^{q} \frac{k_i}{k_i+1} N_{m,f}^{\leqslant k_i}(H_i, r) - \frac{m(n+1)}{2} \log r + O(1).$$

Proof. Take $H_i \in \{H_1, \dots, H_q\}$ and $k_i \in \{k_1, \dots, k_q\}$, we have

$$\begin{aligned} N_{m,f}(H_i, r) &= N_{m,f}^{\leqslant k_i}(H_i, r) + N_{m,f}^{> k_i}(H_i, r) \\ &= \frac{k_i}{k_i+1} N_{m,f}^{\leqslant k_i}(H_i, r) + \frac{1}{k_i+1} N_{m,f}^{\leqslant k_i}(H_i, r) + N_{m,f}^{> k_i}(H_i, r) \\ &\leqslant \frac{k_i}{k_i+1} N_{m,f}^{\leqslant k_i}(H_i, r) + \frac{m}{k_i+1} N_{1,f}^{\leqslant k_i}(H_i, r) + N_{m,f}^{> k_i}(H_i, r) \\ &\leqslant \frac{k_i}{k_i+1} N_{m,f}^{\leqslant k_i}(H_i, r) + \frac{m}{k_i+1} N_f(H_i, r). \end{aligned}$$

By [7, Theorem 3.2]

$$N_f(H_i, r) + O(1) = H_f(H_i, r) \leqslant H_f(r) + O(1).$$

Therefore

$$N_{m,f}(H_i, r) \leqslant \frac{k_i}{k_i+1} N_{m,f}^{\leqslant k_i}(H_i, r) + \frac{m}{k_i+1} H_f(r) + O(1).$$

Thus

$$\sum_{i=1}^{q} N_{m,f}(H_i, r) \leqslant \sum_{i=1}^{q} \frac{k_i}{k_i+1} N_{m,f}^{\leqslant k_i}(H_i, r) + \sum_{i=1}^{q} \frac{m}{k_i+1} H_f(r) + O(1). \tag{2.1}$$

By Theorem 2.1 and (2.1), we obtain Lemma 2.1. □

Theorem 2.2. *Let f, g be two nonconstant holomorphic curves from $\mathbb{C}_p$ to $\mathbb{P}^n$, $k_1, \dots, k_q \in \mathbb{N}^*$ and $H_1, \dots, H_q$ be hyperplanes of $\mathbb{P}^n$ in general position such* *that* $f(\mathbb{C}_p) \not\subset H_i, g(\mathbb{C}_p) \not\subset H_i$, $i = 1, \ldots, q$. *Suppose that*

$$f(z) = g(z) \text{ with } z \in \bigcup_{i=1}^{q} \overline{E}_f(H_i, \leqslant k_i) \text{ and } z \in \bigcup_{i=1}^{q} \overline{E}_g(H_i, \leqslant k_i). \tag{2.2}$$

If

$$q \geqslant 2n^2 + n + 1 + \sum_{i=1}^{q} \frac{n}{k_i + 1}, \tag{2.3}$$

then $f \equiv g$.

Proof. Let $\tilde{f} = (f_1, ..., f_{n+1})$ and $\tilde{g} = (g_1, ..., g_{n+1})$ be reduced representations of f and g, respectively. Assume, on the contrary, that $f \not\equiv g$. Then there exist $h, l \in \{1, \ldots, n+1\}$, $h \neq l$, such that $f_h g_l - f_l g_h \not\equiv 0$. Because f, g are nonconstant, f is a k-linearly nondegenerate holomorphic curve and g is a m-linearly nondegenerate holomorphic curve for some k, m, $1 \leqslant k \leqslant n \leqslant 2n - k < q$, $1 \leqslant m \leqslant n \leqslant 2n - m < q$. By Lemma 2.1 and (2.2)

$$\begin{aligned}
&\Big(\sum_{i=1}^{q} \frac{k_i - k + 1}{k_i + 1} - 2n + k - 1\Big) H_f(r) \\
\leqslant\ & \sum_{i=1}^{q} \frac{k_i}{k_i + 1} N_{k,f}^{\leqslant k_i}(H_i, r) - \frac{k(n+1)}{2} \log r + O(1) \\
\leqslant\ & \sum_{i=1}^{q} \frac{k k_i}{k_i + 1} N_{1,f}^{\leqslant k_i}(H_i, r) - \frac{k(n+1)}{2} \log r \\
\leqslant\ & kn N_{f_h g_l - f_l g_h}(r) - \frac{k(n+1)}{2} \log r + O(1) \\
\leqslant\ & kn\big(H_f(r) + H_g(r)\big) - \frac{k(n+1)}{2} \log r + O(1).
\end{aligned}$$

So

$$\begin{aligned}
&\Big(q - 2n - 1 - \sum_{i=1}^{q} \frac{k}{k_i + 1} + k\Big) H_f(r) \\
\leqslant\ & kn(H_f(r) + H_g(r)) - \frac{k(n+1)}{2} \log r + O(1),
\end{aligned} \tag{2.4}$$

and similarly, we obtain

$$\begin{aligned}
&\Big(q - 2n - 1 - \sum_{i=1}^{q} \frac{m}{k_i + 1} + m\Big) H_g(r) \\
\leqslant\ & mn(H_f(r) + H_g(r)) - \frac{m(n+1)}{2} \log r + O(1).
\end{aligned} \tag{2.5}$$

Without loss of generality assume that $1 \leqslant m \leqslant k$. From (2.4) and (2.5) we have

$$\begin{aligned}&\Big(q-2n-1-\sum_{i=1}^{q}\frac{k}{k_i+1}+m\Big)H_f(r)\\ \leqslant& kn(H_f(r)+H_g(r))-\frac{k(n+1)}{2}\log r+O(1),\end{aligned}$$

and

$$\begin{aligned}&\Big(q-2n-1-\sum_{i=1}^{q}\frac{k}{k_i+1}+m\Big)H_g(r)\\ \leqslant& mn(H_f(r)+H_g(r))-\frac{m(n+1)}{2}\log r+O(1).\end{aligned}$$

Summing up these inequalities, we get

$$\Big(q-2n-1-\sum_{i=1}^{q}\frac{k}{k_i+1}+m-kn-mn\Big)\big(H_f(r)+H_g(r)\big)+\frac{(n+1)}{2}(k+m)\log r\leqslant O(1).$$

From this we obtain

$$q-2n-1-\sum_{i=0}^{q}\frac{k}{k_i+1}-kn-(n-1)m<0$$

Consider

$$\varphi(k,m)=k\Big(-\sum_{i=1}^{q}\frac{1}{k_i+1}-n\Big)+(1-n)m+q-2n-1.$$

Since $1 \leqslant m \leqslant k \leqslant n$ and $-\sum_{i=1}^{q}\frac{1}{k_i+1}-n<0$ and $1-n\leqslant 0$ and (2.3), $\varphi(k,m)\geqslant \varphi(n,n)=q-2n^2-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}\geqslant 0$. Thus, we obtain a contradiction.
So $f\equiv g$. □

Lemma 2.2. *Let f, g be two nonconstant holomorphic curves from $\mathbb{C}_p$ to $\mathbb{P}^n$, and X_i be hypersurfaces in $\mathbb{P}^n$ of degree d, Y_i be hypersurfaces of degree l, in general position, such that the images of f and g are not contained in X_i, Y_i, respectively, $i=1,\dots,n+1$.*

Suppose that $E_f(X_i)=E_g(Y_i)$, $i=1,\dots,n+1$. Then

$$dH_f(r)=lH_g(r)+O(1).$$

Proof. Let $\tilde f=(f_1,...,f_{n+1})$ and $\tilde g=(g_1,...,g_{n+1})$ be reduced representations of f and g, respectively. Suppose that X_i, Y_i are defined, respectively, by the following equations

$$\begin{aligned}P_i(x_1,\dots,x_{n+1})=0,\quad i=1,\dots,n+1,\\ Q_i(x_1,\dots,x_{n+1})=0,\quad i=1,\dots,n+1.\end{aligned}$$

From the hypothesis of general position and the Hilbert's Nullstellensatz [13] it follows that for any integer $k, 1 \leqslant k \leqslant n+1$ there is an integer $m_k \geqslant d$, $l_k \geqslant l$ such that

$$x_k^{m_k} = \sum_{i=1}^{n+1} a_{i_k}(x_1, \dots, x_{n+1}) P_i(x_1, \dots, x_{n+1}),$$
$$x_k^{l_k} = \sum_{i=1}^{n+1} b_{i_k}(x_1, \dots, x_{n+1}) Q_i(x_1, \dots, x_{n+1}),$$

where $a_{i_k}(x_1, \dots, x_{n+1})$, $b_{i_k}(x_1, \dots, x_{n+1})$, $1 \leqslant i \leqslant n+1$, $1 \leqslant k \leqslant n+1$, are homogeneous polynomials with coefficients in $\mathbb{C}_p$ of degree $m_k - d$, $l_k - l$, respectively. Therefore

$$f_k^{m_k} = \sum_{i=1}^{n+1} a_{i_k}(f_1, \dots, f_{n+1}) P_i(f_1, \dots, f_{n+1}), \quad k = 1, \dots, n+1,$$
$$g_k^{l_k} = \sum_{i=1}^{n+1} b_{i_k}(f_1, \dots, f_{n+1}) Q_i(f_1, \dots, f_{n+1}). \quad k = 1, \dots, n+1.$$

From this it follows that

$$H_{f_k^{m_k}}(r) = m_k H_{f_k}(r) \leqslant (m_k - d) H_f(r) + \max_{1 \leqslant i \leqslant n+1} H_{P_i \circ \tilde{f}}(r) + O(1),$$
$$H_{g_k^{l_k}}(r) = l_k H_{g_k}(r) \leqslant (l_k - l) H_g(r) + \max_{1 \leqslant i \leqslant n+1} H_{Q_i \circ \tilde{f}}(r) + O(1).$$

Hence

$$dH_f(r) \leqslant \max_{1 \leqslant i \leqslant n+1} H_{P_i \circ \tilde{f}}(r) + O(1),$$
$$lH_g(r) \leqslant \max_{1 \leqslant i \leqslant n+1} H_{Q_i \circ \tilde{f}}(r) + O(1). \tag{2.6}$$

On the other hand,

$$H_{P_i \circ \tilde{f}}(r) \leqslant dH_f(r) + O(1), \quad \text{for all } i = 1, \dots, n+1,$$
$$H_{Q_i \circ \tilde{f}}(r) \leqslant lH_g(r) + O(1), \quad \text{for all } i = 1, \dots, n+1. \tag{2.7}$$

By (2.6) and (2.7) we have

$$dH_f(r) = \max_{1 \leqslant i \leqslant n+1} H_{P_i \circ \tilde{f}}(r) + O(1),$$
$$lH_g(r) = \max_{1 \leqslant i \leqslant n+1} H_{Q_i \circ \tilde{f}}(r) + O(1). \tag{2.8}$$

Since $E_f(X_i) = E_g(Y_i)$, for all $i = 1, \dots, n+1$, $\frac{P_i \circ \tilde{f}}{Q_i \circ \tilde{g}}$ is a non-vanishing analytic function. So $P_i \circ \tilde{f} = a_i Q_i \circ \tilde{g}$, $a_i \in \mathbb{C}_p$, $a_i \neq 0$, for all $i = 1, \dots, n+1$. Therefore

$$\max_{1 \leqslant i \leqslant n+1} H_{P_i \circ \tilde{f}}(r) = \max_{1 \leqslant i \leqslant n+1} H_{Q_j \circ \tilde{g}}(r) + O(1). \tag{2.9}$$

By (2.8) and (2.9) we have

$$dH_f(r) = lH_g(r) + O(1).$$

□

Theorem 2.3. *Let f, g be two nonconstant holomorphic curves from $\mathbb{C}_p$ to $\mathbb{P}^n$. Let X_i be hypersurfaces of degree d, H_j be hyperplanes in general position in $\mathbb{P}^n$, such that the image of f is not contained in X_i, H_j for $i = 1, \ldots, n+1$ and $j = 1, \ldots, q$. Suppose that $E_f(X_i) = E_g(X_i)$, $i = 1 \ldots, n+1$, and*

$$f(z) = g(z) \text{ with } z \in \bigcup_{i=1}^{q} \overline{E}_f(H_i, \leqslant k_i).$$

If

$$q \geqslant 2n^2 + n + 1 + \sum_{i=1}^{q} \frac{n}{k_i + 1},$$

then $f \equiv g$.

Proof. Let $\tilde{f} = (f_1, ..., f_{n+1})$ and $\tilde{g} = (g_1, ..., g_{n+1})$ be reduced representations of f and g, respectively. Assume, on the contrary, that $f \not\equiv g$. Then there exist $h, j \in \{1, \ldots, n+1\}$, $h \neq j$, such that $f_h g_j - f_j g_h \not\equiv 0$. Since $E_f(X_i) = E_g(X_i), i = 1, \ldots, n+1$, and by Lemma 2.2 we have $dH_f(r) = dH_g(r) + O(1)$. Because f is a nonconstant curve, f is a k-linearly nondegenerate holomorphic curve for some k, $1\ k \leqslant n \leqslant 2n - k < q$.

By Lemma 2.1

$$\begin{aligned}
&\Big(\sum_{i=1}^{q} \frac{k_i + 1 - k}{k_i + 1} - 2n + k - 1\Big) H_f(r) \\
&\leqslant \sum_{i=1}^{q} \frac{k_i}{k_i + 1} N_{k,f}^{\leqslant k_i}(H_i, r) - \frac{k(n+1)}{2} \log\, r + O(1) \\
&\leqslant \sum_{i=1}^{q} \frac{k k_i}{k_i + 1} N_{1,f}^{\leqslant k_i}(H_i, r) - \frac{k(n+1)}{2} \log r + O(1) \\
&\leqslant nk N_{f_h g_j - f_j g_h} - \frac{k(n+1)}{2} \log r + O(1) \\
&\leqslant nk\big(H_f(r) + H_g(r)\big) - \frac{k(n+1)}{2} \log r + O(1).
\end{aligned}$$

Since $H_f(r) = H_g(r) + O(1)$,

$$\Big(\sum_{i=1}^{q} \frac{k_i + 1 - k}{k_i + 1} - 2n + k - 1 - 2nk\Big) H_f(r) + \frac{n+1}{2} k \log r \leqslant O(1).$$

From this we have

$$k\Big(1 - \sum_{i=1}^{q} \frac{1}{k_i + 1} - 2n\Big) + q - 2n - 1 < 0.$$

Consider

$$\varphi(k) = k\Big(1 - \sum_{i=1}^{q} \frac{1}{k_i+1} - 2n\Big) + q - 2n - 1.$$

Since $1 - \sum\limits_{i=1}^{q} \dfrac{1}{k_i+1} - 2n < 0$,

$$\varphi(k) \geqslant \varphi(n) = q - 2n^2 - n - 1 - \sum_{i=1}^{q} \frac{n}{k_i+1} \geqslant 0.$$

We obtain a contradiction.

So $f \equiv g$. □

3. Uniqueness polynomials and unique range sets for algebraically nondegenerate p-adic holomorphic curves

For $n \in \mathbb{N}^*$, $n \geqslant 2m+8$, $m \geqslant 2$, $(m,n)=1$, consider a polynomial $P(z) = z^n - az^{n-m} + b$, where $a, b \in \mathbb{C}_p$ and $a \neq 0$, $b \neq 0$, and $\dfrac{a^n}{b^m} \neq \dfrac{n^n}{m^m(n-m)^{n-m}}$. Set

$$\tilde{P}(z_1, z_2) = z_1^n - az_1^{n-m}z_2^m + bz_2^n.$$

We define inductively

$$P_1(z_1, z_2) = \tilde{P}(z_1, z_2) = z_1^n - az_1^{n-m}z_2^m + bz_2^n,$$

$$P_s(z_1, z_2, \dots, z_{s+1}) = P_{s-1}\big(\widetilde{P}(z_1, z_2), \dots, \widetilde{P}(z_s, z_{s+1})\big), \quad s = 2, 3, \dots . (*)$$

Then, P_s is homogeneous and of degree n^s.

Lemma 3.1. *In the above hypothesis, $P(z)$ is a UPM .*

Proof. It follows immediately from [5,Theorem 3.21]. □

Theorem 3.1. *P_s defined by $(*)$ is a SUPC.*

Proof. Let f and g be algebraically nondegenerate holomorphic curves from $\mathbb{C}_p$ to $\mathbb{P}^s(\mathbb{C}_p)$ with reduced representations $\tilde{f} = (f_1, \dots, f_{s+1})$, $\tilde{g} = (g_1, \dots, g_{s+1})$ respectively .

We first prove Theorem 3.1 for $s = 1$.

Let $\tilde{f} = (f_1, f_2)$, $\tilde{g} = (g_1, g_2)$ such that

$$\widetilde{P}(f_1, f_2) = c\widetilde{P}(g_1, g_2)$$

for a nonzero $c \in\ C_p$. Therefore

$$f_1^n - af_1^{n-m}f_2^m + bf_2^n - cg_1^n + cag_1^{n-m}g_2^m - cbg_2^n = 0. \tag{3.1}$$

Consider the holomorphic curve F from $\mathbb{C}_p$ to $\mathbb{P}^2(\mathbb{C}_p)$ with reduced representation

$$\tilde{F} = \big(bf_2^n, f_1^{n-m}(f_1^m - af_2^m), bg_2^n\big).$$

Assume that F is linearly nondegenerate. Consider the following hyperplanes in general position in $\mathbb{P}^2(\mathbb{C}_p)$

$$H_1 : x_1 = 0;\quad H_2 : x_2 = 0;\quad H_3 : x_3 = 0;$$
$$H_4 : x_1 + x_2 - cx_3 = 0.$$

Using Theorem 2.1 and (3.1), and noting that

$$nH_f(r) \leqslant H_F(r) + O(1),$$

we have

$$\begin{aligned}
H_F(r) &\leqslant \sum_{i=1}^{4} N_{2,F}(H_i, r) - 3\log r + O(1)\\
&= N_{2,bf_2^n}(r) + N_{2,f_1^{n-m}(f_1^m - af_2^m)} + N_{2,bg_2^n}(r)\\
&\quad + N_{2,bf_2^n + f_1^{n-m}(f_1^m - af_2^m) - bcg_2^n}(r) - 3\log r + O(1)\\
&\leqslant 2N_{f_2}(r) + 2N_{f_1}(r) + N_{2,(f_1^m - af_2^m)}(r) + 2N_{g_2}(r)\\
&\quad + N_{2,g_1^{n-m}(g_1^m - ag_2^m)}(r) - 3\log r + O(1)\\
&\leqslant 4H_f(r) + 4H_g(r) + N_{f_1^m - af_2^m}(r) + N_{g_1^m - ag_2^m}(r) - 3\log r + O(1)\\
&\leqslant (m+4)\big(H_f(r) + H_g(r)\big) - 3\log r + O(1),
\end{aligned}$$

and

$$nH_f(r) \leqslant (m+4)\big(H_f(r) + H_g(r)\big) - 3\log r + O(1). \tag{3.2}$$

Consider the holomorphic curve G from $\mathbb{C}_p$ to $\mathbb{P}^2(\mathbb{C}_p)$ with reduced representation

$$\tilde{G} = \big(bg_2^n, g_1^{n-m}(g_1^m - ag_2^m), bf_2^n\big).$$

Assume that G is linearly nondegenerate. As in the proof of (3.2), we obtain

$$nH_g(r) \leqslant (m+4)\big(H_f(r) + H_g(r)\big) - 3\log r + O(1). \tag{3.3}$$

Summing up inequalities (3.2) and (3.3), we get

$$(n - 2m - 8)(H_f(r) + H_g(r)) + 6\log r \leqslant O(1).$$

We have $n \geqslant 2m + 8$, a contradiction.

So F or G is linearly degenerate. Without loss of generality, one can assume that F is linearly degenerate. Then there exist constants C_1, C_2, C_3 such that

$$(C_1, C_2, C_3) \neq (0, 0, 0)$$

and

$$C_1 bf_2^n + C_2 f_1^{n-m}(f_1^m - af_2^m) + C_3 bg_2^n = 0. \tag{3.4}$$

We consider the following possible cases:

Case 1. If $C_3 = 0$, then from (3.4) we have

$$C_1 bf_2^n + C_2 f_1^{n-m}(f_1^m - af_2^m) = 0. \tag{3.5}$$

Since f is nonconstant and $(C_1, C_2) \neq (0, 0)$, we obtain a contradiction. So $C_3 \neq 0$.

Case 2. If $C_2 = 0$, then from (3.4) we have

$$g_2^n = -\frac{C_1}{C_3} f_2^n.$$

From this and (3.1), it follows that

$$b\Big(1 + \frac{cC_1}{C_3}\Big) f_2^n + f_1^{n-m}(f_1^m - a f_2^m) = c g_1^{n-m}(g_1^m - a g_2^m).$$

Suppose that $1 + \dfrac{cC_1}{C_3} \neq 0$. Consider the holomorphic curve F_1 from $\mathbb{C}_p$ to $\mathbb{P}^1(\mathbb{C}_p)$ with reduced representation

$$\tilde{F}_1 = \Big(f_1^{n-m}(f_1^m - a f_2^m), b\Big(1 + \frac{cC_1}{C_3}\Big) f_2^n\Big).$$

Since f is nonconstant, so is F_1. Consider three points $(1,0),(0,1)$ $(1,1)$ of $\mathbb{P}^1(\mathbb{C}_p)$. Using Theorem 2.1 and noting that

$$nH_f(r) \leqslant H_{F_1}(r) + O(1),$$

we have

$$\begin{aligned}
&nH_f(r)\\
&\leqslant H_{F_1}(r) + O(1)\\
&\leqslant N_{1,f_1^{n-m}(f_1^m - af_2^m)}(r) + N_{1,f_2^n}(r) + N_{1,g_1^{n-m}(g_1^m - ag_2^m)} - \log r + O(1)\\
&\leqslant N_{1,f_1}(r) + N_{f_1^m - af_2^m}(r) + N_{1,f_2}(r) + N_{1,g_1}(r) + N_{g_1^m - ag_2^m}(r) - \log r + O(1)\\
&\leqslant N_{f_1}(r) + N_{f_2}(r) + N_{f_1^m - af_2^m}(r) + N_{g_1}(r) + N_{g_1^m - ag_2^m}(r) - \log r + O(1)\\
&\leqslant (m+2)H_f(r) + (m+1)H_g(r) - \log r + O(1).
\end{aligned}$$

So

$$nH_f(r) \leqslant (m+2)H_f(r) + (m+1)H_g(r) - \log r + O(1). \tag{3.6}$$

On the other hand, $C_1 \neq 0$ and

$$f_2^n = -\frac{C_3}{C_1} g_2^n.$$

From this and (3.1), it follows that

$$b\Big(c + \frac{C_3}{C_1}\Big) g_2^n + c g_1^{n-m}(g_1^m - a g_2^m) = f_1^{n-m}(f_1^m - a f_2^m).$$

If $c + \frac{C_3}{C_1} = 0$, then $f_2^n = c g_2^n$.

Consider $c + \frac{C_3}{C_1} \neq 0$ and the holomorphic curve G_1 from $\mathbb{C}_p$ to $\mathbb{P}^1(\mathbb{C}_p)$ with reduced representation

$$\tilde{G}_1 = \left(g_1^{n-m}(g_1^m - a g_2^m),\ b\left(c + \frac{C_3}{C_1}\right) g_2^n\right).$$

Since g is nonconstant, so is G_1.

Similarly, we obtain

$$nH_g(r) \leqslant (m+2)H_g(r) + (m+1)H_f(r) - \log r + O(1). \tag{3.7}$$

From (3.6) and (3.7) we have

$$n\big(H_f(r) + H_g(r)\big) \leqslant (2m+3)\big(H_f(r) + H_g(r)\big) - 2\log r + O(1).$$

So

$$(n-2m-3)\big(H_f(r) + H_g(r)\big) + 2\log r \leqslant O(1).$$

We have a contradiction for $n \geqslant 2m+3$. So $1 + \dfrac{cC_1}{C_3} = 0$ and $c + \dfrac{C_1}{C_3} = 0$. Therefore $f_2^n = cg_2^n$.

Case 3. If $C_1 = 0,\ C_2 \neq 0$. From (3.4) we have

$$C_2 f_1^{n-m}(f_1^m - af_2^m) = -C_3 b g_2^n.$$

So

$$C_2\Big(\frac{f_1}{f_2}\Big)^n - C_2 a\Big(\frac{f_1}{f_2}\Big)^{n-m} = -C_3 b\Big(\frac{g_2}{f_2}\Big)^n. \tag{3.8}$$

Since $n \geqslant 2m+8, m \geqslant 2$, the equation

$$C_2 z^n - C_2 a z^{n-m} = 0$$

has at least 3 distinct roots $z_1 = 0, z_2, z_3$. Since f is nonconstant, so is $\frac{g_2}{f_2}$. From (3.8), for each $i = 1, 2, 3$, all the zeros of $\dfrac{f_1}{f_2} - z_i$ have multiplicities $\geqslant n$. By [9,Theorem 3.10], $3\Big(1 - \dfrac{1}{n}\Big) < 2$. Therefore $n < 3$. From $n \geqslant 2m+8$, we obtain a contradiction.

Case 4. $C_1 \neq 0, C_2 \neq 0,\ C_3 \neq 0$.

Consider the holomorphic curve F_2 from $\mathbb{C}_p$ to $\mathbb{P}^1(\mathbb{C}_n)$ with reduced representation

$$\tilde{F}_2 = \big(C_2 f_1^{n-m}(f_1^m - af_2^m), C_1 b f_2^n\big).$$

Since f is nonconstant, so is F_2.

Consider three points $(1,0), (0,1)\ (1,1)$ of $\mathbb{P}^1(\mathbb{C}_p)$. Using Theorem 2.1 and noting that

$$nH_f(r) \leqslant H_{F_2}(r) + O(1),$$

we have

$$\begin{aligned} nH_f(r) \leqslant & H_{F_2}(r) + O(1) \leqslant N_{1,C_2 f_1^{n-m}(f_1^m - af_2^m)}(r) + N_{1,C_1 b f_2^n}(r) \\ & + N_{1,C_2 f_1^{n-m}(f_1^m - af_2^m) + C_1 b f_2^n} - \log r + O(1) \\ \leqslant & N_{f_1}(r) + N_{f_2}(r) + N_{f_1^m - af_2^m} + N_{g_2}(r) - \log r + O(1) \\ \leqslant & (m+2)H_f(r) + H_g(r) - \log r + O(1). \end{aligned}$$

So

$$nH_f(r) \leqslant (m+2)H_f(r) + H_g(r) - \log r + O(1). \tag{3.9}$$

By (3.4),

$$f_1^{n-m}(f_1^m - af_2^m) = -\frac{bC_3}{C_2}g_2^n - \frac{bC_1}{C_2}f_2^n.$$

From this and (3.1), we obtain

$$A_1bg_2^n + A_2g_1^{n-m}(g_1^m - ag_2^m) + A_3bf_2^n = 0.$$

If one of the numbers A_1, A_2, A_3 is equal to 0, then we can use the arguments similarly to the one above. If $A_1 \neq 0$ and $A_2 \neq 0$ and $A_3 \neq 0$, then consider the holomorphic curve G_2 from $\mathbb{C}_p$ to $\mathbb{P}^1(\mathbb{C}_p)$ with reduced representation

$$\tilde{G}_2 = \big(A_2g_1^{n-m}(g_1^m - ag_2^m),\ A_1bg_2^n\big).$$

Since g is nonconstant, so is G_2. Similarly, to the relation (3.9) for F_2, we obtain the following inequality for G_2 :

$$nH_g(r) \leqslant (m+2)H_g(r) + H_f(r) - \log r + O(1). \tag{3.10}$$

From (3.9) and (3.10) we obtain

$$n\big(H_f(r) + H_g(r)\big) \leqslant (m+3)\big(H_f(r) + H_g(r)\big) - 2\log r + O(1).$$

So

$$(n-m-3)\big(H_f(r) + H_g(r)\big) + 2\log r \leqslant O(1).$$

From this we have $n < m+3$, a contradiction.

Thus, the above cases give us $f_2^n = cg_2^n$. Since $f_2^n = cg_2^n$ and $\widetilde{P}(f_1, f_2) = c\widetilde{P}(g_1, g_2)$, we have $P\Big(\frac{f_1}{f_2}\Big) = P\Big(\frac{g_1}{g_2}\Big)$. Because $n \geqslant 2m+8$, $m \geqslant 2$, $(m,n) = 1$, by Lemma 3.1 we obtain $f = g$.

We now continue to prove Theorem 3.1. Consider

$$P_s(f_1, \dots, f_{s+1}) = cP_s(g_1, \dots, g_{s+1}).$$

Set

$$Q_k = P_{k-1}(f_1, f_2, ..., f_k), M_{k+1} = P_{k-1}(f_2, ..., f_{k+1}),$$

and

$$R_k = P_{k-1}(g_1, g_2, ..., g_k), N_{k+1} = P_{k-1}(g_2, ..., g_{k+1}),$$

$$Q_{k,k+1} = \frac{Q_k}{M_{k+1}}, R_{k,k+1} = \frac{R_k}{N_{k+1}},$$

with $k = 2, ..., s$.
Then

$$Q_s^n - aQ_s^{n-m}M_{s+1}^m + bM_{s+1}^n - cR_s^n + caR_s^{n-m}N_{s+1}^m - cbN_{s+1}^n = 0. \tag{3.11}$$

Note that each common zero of

$$bM_{s+1}^n, Q_s^n - aQ_s^{n-m}M_{s+1}^m, bN_{s+1}^n$$

or of

$$bN_{s+1}^n, R_s^n - aR_s^{n-m}N_{s+1}^m, bM_{s+1}^n$$

is also a common zero of

$$bM_{s+1}^n, Q_s^n - aQ_s^{n-m}M_{s+1}^m, bN_{s+1}^n, R_s^n - aR_s^{n-m}N_{s+1}^m,$$

and hence is a common zero of

$$Q_s, M_{s+1}, R_s, N_{s+1}.$$

Then we can omit those common zeros.

From this, and since f, g are two algebraically nondegenerate holomorphic curves from $\mathbb{C}_p$ to $\mathbb{P}^s(\mathbb{C}_p)$, it follows that the results in case $s = 1$ also hold in this case.

So

$$Q_{s,s+1} = R_{s,s+1}.$$

Hence

$$P_{s-1}(f_1, \dots, f_s) = c_{s-1}P_{s-1}(g_1, \dots, g_s)$$

and

$$P_{s-1}(f_2, \dots, f_{s+1}) = c_{s-1}P_{s-1}(g_2, \dots, g_{s+1}).$$

Similarly

$$Q_{k,k+1} = R_{k,k+1},$$

with $k = 2, ..., s-1$.

From

$$Q_{2,3} = R_{2,3}$$

we have

$$P_1(f_1, f_2) = c_1P_1(g_1, g_2),$$

and

$$P_1(f_2, f_3) = c_1P_1(g_2, g_3).$$

Therefore

$$\frac{f_1}{g_1} = \frac{f_2}{g_2},$$

and

$$\frac{f_2}{g_2} = \frac{f_3}{g_3}.$$

Similarly

$$\frac{f_i}{g_i} = \frac{f_{i+1}}{g_{i+1}},\ i = 1, 2, ..., s-1,$$

$$\frac{f_i}{g_i} = \frac{f_{i+1}}{g_{i+1}},\ i = 2, ..., s.$$

Thus $f = g$. □

Theorem 3.2. *Let f, g be two algebraically nondegenerate holomorphic curves from $\mathbb{C}_p$ to $\mathbb{P}^s$. Let X be a hypersurface of $\mathbb{P}^s$, defined by $P_s = 0$. Suppose that $E_f(X) = E_g(X)$. Then $f \equiv g$.*

Proof. Since $E_f(X) = E_g(X)$, $P_s \circ f = cP_s \circ g$, $c \in \mathbb{C}_p$, $c \neq 0$. By Theorem 3.1, $f \equiv g$. □

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