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ON THE ASSOCIATED PRIMES AND THE SUPPORT OF GENERALIZED LOCAL COHOMOLOGY MODULES

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ABSTRACT. In this note, we prove the following results: (1) if $\dim(N) \leqslant 3$ then $\mathrm{Ass}(H^i_I(M,N))$ is finite for all $i \geqslant 0$; (2) Let $d = \dim(R)$ and $s(I,M) = \mathrm{depth}(M/I^nM)$ for some large n. If N has finite injective dimension then we have $H^i_I(M,N) = 0$ for all $i > d - s(I,M)$, $H^d_I(M,N)$ is Artinian, and $\mathrm{Supp}(H^{d-1}_I(M,N))$ is finite.

1. Introduction

Throughout this note, let $(R,\mathfrak{m})$ be a commutative Noetherian local ring of dimension d, I an proper ideal of R, M and N finitely generated R-modules. The j^{th} generalized local cohomology module $H^j_I(M,N)$ of two modules M and N with respect to I introduced by J. Herzog in [10], is defined as

$$H^j_I(M,N) = \varinjlim_n \mathrm{Ext}^j_R(M/I^nM, N).$$

It is clear that $H^j_I(R,N) = H^j_I(N)$. For local cohomology theory, one of the important problems is determining when the set of associated primes of $H^j_I(N)$ is finite, see [11]. There are number of works involving this problem, see for example [9], [12], [13], [14], etc. In [13], T. Marley has proved that $\mathrm{Supp}(H^{d-1}_I(N)) \subseteq \overline{A}^*(I) \cup \{\mathfrak{m}\}$, where $\overline{A}^*(I)$ is the stable value of $\mathrm{Ass}(R/\overline{I^n})$ for large n (here $\overline{I^n}$ is the integral closure of I^n); then, he has established the finiteness of the set of associated primes of local cohomology modules over a local ring of dimension three. These sugguest us to study similar properties for generalized local cohomology modules.

The main purpose of this note is to study the finiteness of the set of associated primes of generalized local cohomology modules for any finitely generated module of dimension less than or equal three, and we also prove the finiteness of the support of generalized local cohomology modules for the case of finite injective dimension.

The following two theorems are our main result.

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Theorem 1.1. *Assume that* $\dim(N) \leqslant 3$*, then* $\operatorname{Ass}(H_I^i(M,N))$ *is finite for all* $i \geqslant 0$*.*

The tools for proving Theorem 1.1 are the basic properties of weakly Laskerian modules (cf. [8]), and the fact that $\operatorname{Supp}(H_I^i(M,N)) \subseteq \bigcup_{j=0}^{i} \operatorname{Supp}(H_{I_M}^j(N))$, where I_M is the annihilator of R-module M/IM (cf. [6, Corollary 4.3]).

Recall that, there exists an integer n_0 such that $\operatorname{depth}(M/I^nM) = \operatorname{depth}(M/I^{n+1}M)$ for all $n \geqslant n_0$ (cf. [3, (2) Theorem]).

Theorem 1.2. *Let* $d = \dim(R)$*, and* $s(I,M)$ *be the stable value of* $\operatorname{depth}(M/I^nM)$*. Assume that* $\operatorname{injd}_R(N) < +\infty$*, then*
(a) $H_I^i(M,N) = 0$ *for all* $i > d - s(I,M)$*;*
(b) $H_I^d(M,N)$ *is Artinian;*
(c) $\operatorname{Supp}(H_I^{d-1}(M,N))$ *is finite.*

The techniques for proving Theorem 1.2 are basic properties of a finitely generated R-module of finite injective dimension, stability of $\operatorname{Ass}(M/I^nM)$, and local duality for a Cohen Macaulay homomorphic image of a Gorenstein local ring.

The note is divided into 4 sections. In Section 2, we summarize some known facts on generalized local cohomology modules. In Section 3, we first prove Theorem 1.1, then we show some its consequences. The last section devotes to prove Theorem 1.2.

2. Preliminaries

Let M, N be finitely generated R-modules, and I an ideal of R. We use I_M to denote the annihilator of the R-module M/IM. Let $\operatorname{pd}_R(M)$ (resp. $\operatorname{injd}_R(N)$) denote by the projective dimension of M (resp., the injective dimension of N).

First, we recall some known facts on generalized local cohomology modules that will be often used in the sequel.

Lemma 2.1. (cf. [6], [7]) (i) $H_I^0(M,N) \cong \Gamma_I(\operatorname{Hom}(M,N))$.
(ii) *If* N *is* I*-torsion, then* $H_I^j(M,N) \cong \operatorname{Ext}_R^j(M,N)$ *for all* $j \geqslant 0$*.*
(iii) *Let* $f : R \to S$ *be a flat ring homomorphism. Then, for any* $j \geqslant 0$*, we have*

$$H_I^j(M,N) \otimes_R S \cong H_{IS}^j(M \otimes_R S, N \otimes_R S).$$

(iv) *For any* $i \geqslant 0$*, we have* $\bigcup_{j=0}^{i} \operatorname{Supp}(H_I^j(M,N)) = \bigcup_{j=0}^{i} \operatorname{Supp}(H_{I_M}^j(N))$*.*

Following [14], a sequence $x_1, \ldots, x_n$ of elements in $\mathfrak{m}$ is said to be a *generalized regular sequence* of N if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(N/(x_1, \ldots, x_{i-1})N)$ satisfying $\dim(R/\mathfrak{p}) \geqslant 2$, for all $i = 1, \ldots, n$. Moreover, all generalized regular sequences of N in I have the same length. This common length is called *generalized depth* of N in I, and denoted by $\operatorname{gdepth}(I,N)$. This number can be used to characterize the infiniteness of the support of generalized local cohomology modules as follows.

Lemma 2.2. (cf. [6, Theorems 4.4 and 4.12]) *Let* $r = \operatorname{gdepth}(I_M, N)$ *and* $P_r = \bigcup_{j<r} \operatorname{Supp}(H_I^j(M,N))$. *Then*
(i) $r = \inf\{i \mid \operatorname{Supp}(H_I^i(M,N))$ *is not finite*$\}$;
(ii) $\operatorname{Ass}(H_I^r(M,N)) \bigcup P_r = \operatorname{Ass}(\operatorname{Ext}_R^r(M/IM,N)) \bigcup P_r$.

On the other hand, for any submodule T of an R-module K, we have the following inclusion

$$\operatorname{Ass}(K/T) \subseteq \operatorname{Ass}(K) \cup \operatorname{Supp}(T).$$

Indeed, let $\mathfrak{p} \in \operatorname{Ass}(K/T) \setminus \operatorname{Supp}(T)$, then $\mathfrak{p} = (T : m)_R$ for some $m \in K$. It yields that $\mathfrak{p}m \subseteq T$. So, since $T_\mathfrak{p} = 0$, we have $(\mathfrak{p}m)_\mathfrak{p} = 0$. From this, since $\mathfrak{p}m$ is a finitely generated R-module, it is easy to see that $\mathfrak{p} \subseteq \operatorname{ann}(rm)$ for some $r \notin \mathfrak{p}$. Moreover, let $x \in \operatorname{ann}(rm)$, then $rx \in \operatorname{ann}(m) \subseteq (T : m)_R = \mathfrak{p}$. So, as $r \notin \mathfrak{p}$, $x \in \mathfrak{p}$. Therefore, $\mathfrak{p} = \operatorname{ann}(rm)$, so that $\mathfrak{p} \in \operatorname{Ass}(K)$.

To finish this section, we recall the following known fact.

Lemma 2.3. (cf. [7, Theorems 3.1 and 5.4]) *Let* $d = \dim(R)$. *If* $\operatorname{pd}_R(M) < +\infty$, *then*
(i) $H_I^j(M,N) = 0$ *for all* $j > d$;
(ii) $\operatorname{Supp}(H_I^{d-1}(M,N))$ *and* $\operatorname{Supp}(H_I^d(M,N))$ *are finite.*

3. Proof of Theorem 1.1

We first recall that an R-module K is called *weakly Laskerian* if any quotient module of K has finitely many associated primes (cf. [8]). Note that, all Artinian modules, all finitely generated modules, and all modules with finite support are weakly Laskerian. Moreover, if $0 \to K_1 \to K_2 \to K_3 \to 0$ is an exact sequence, then K_2 is weakly Laskerian if and only if K_1 and K_3 are both weakly Laskerian.

To prove Theorem 1.1, we need the following lemma.

Lemma 3.1. *Assume that* $\dim(N) \leqslant 2$. *Then* $H_I^i(M,N)$ *is weakly Laskerian for all* $i \geqslant 0$, *therefore* $\operatorname{Ass}(H_I^i(M,N))$ *is finite for all* $i \geqslant 0$.

Proof. By Lemma 2.1(i), $H_I^0(M,N))$ is finitely generated. Let $i \geqslant 1$. We have an exact sequence $0 \to \Gamma_I(N) \to N \to \overline{N} \to 0$, where $\overline{N} = N/\Gamma_I(N)$. It induces the following exact sequence

$$H_I^i(M,\Gamma_I(N)) \xrightarrow{f_i} H_I^i(M,N) \xrightarrow{g_i} H_I^i(M,\overline{N}).$$

From this, we get an exact sequence

$$0 \to \operatorname{Im} f_i \to H_I^i(M,N) \to \operatorname{Im} g_i \to 0.$$

It is clear that $\operatorname{Im} f_i$ is a quotient module of $H_I^i(M,\Gamma_I(N))$. By Lemma 2.1(ii), $H_I^i(M,\Gamma_I(N))$ is finitely generated, so that $\operatorname{Im} f_i$ is weakly Laskerian. By using the above exact sequences, in order to prove the weakly Laskerianness of $H_I^i(M,N)$, we need only to show that $H_I^i(M,\overline{N})$ is weakly Laskerian. Therefore, we can assume that N is I-torsion-free of $\dim(N) \leqslant 2$.

The rest of our proof devotes to prove that $H_I^i(M,N)$ is weakly Laskerian for all $i \geqslant 1$ provided N is I-torsion-free of $\dim(N) \leqslant 2$. By the assumption of N, I contains a regular element of N, so that $H^0_{I_M}(N) = 0$. By [4, Theorem 7.1.6], $\mathrm{Supp}(H^2_{I_M}(N))$ is finite. By the Independence Theorem in [4], we have $H^1_{I_M}(N) \cong H^1_{I_M\overline{R}}(N)$ as R-modules, where $\overline{R} = R/\operatorname{ann}(N)$. Since $\dim(\overline{R}) \leqslant 2$, we get by [13, Corollaries 2.4, 2.5] that $\mathrm{Supp}_{\overline{R}}(H^1_{I_M\overline{R}}(N))$ is finite, so is $\mathrm{Supp}_{\overline{R}}(H^1_{I_M}(N))$. Note that $\mathrm{Supp}(H^1_{I_M}(N)) \subseteq \mathrm{Supp}(R/\operatorname{ann}(N))$ and

$$\mathrm{Supp}_{\overline{R}}(H^1_{I_M}(N)) = \{\mathfrak{p}/\operatorname{ann}(N) \mid \mathfrak{p} \in \mathrm{Supp}(H^1_{I_M}(N))\}.$$

Hence $\mathrm{Supp}(H^1_{I_M}(N))$ is finite. It follows that $\bigcup_{j=0}^2 \mathrm{Supp}(H^j_{I_M}(N))$ is a finite set. Finally, let $i \geqslant 1$. Then, in view of the Grothendieck's Vanishing Theorem, we get by Lemma 2.1(iii) that $\mathrm{Supp}(H_I^i(M,N)) \subseteq \bigcup_{j=0}^2 \mathrm{Supp}(H^j_{I_M}(N))$. Therefore, $\mathrm{Supp}(H_I^i(M,N))$ is a finite set, so that $H_I^i(M,N)$ is weakly Laskerian, as required. □

Proof of Theorem 1.1. By Lemma 2.1(i), $\mathrm{Ass}(H_I^0(M,N))$ is finite. Next, we prove that $\mathrm{Ass}(H_I^i(M,N))$ is finite for all $i \geqslant 1$. Indeed, for each $i \geqslant 1$, we have the following exact sequence

$$H_I^i(M,\Gamma_I(N)) \xrightarrow{f_i} H_I^i(M,N) \xrightarrow{g_i} H_I^i(M,\overline{N}),$$

where $\overline{N} = N/\Gamma_I(N)$. By Lemma 2.1(ii), $H_I^i(M,\Gamma_I(N))$ is finitely generated, so that $\mathrm{Ass}(\operatorname{Im} f_i)$ is finite. Thus, by the above exact sequence, in order to prove the finiteness of $\mathrm{Ass}(H_I^i(M,N))$, we need only to prove that $\mathrm{Ass}(H_I^i(M,\overline{N}))$ is finite. Therefore, we may assume that N is I-torsion-free with $\dim(N) \leqslant 3$. Thus, there exists $x \in I$ such that the sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$ is exact. It induces the following exact sequence

$$H_I^{i-1}(M,N/xN) \to (0:x)_{H_I^i(M,N)} \to 0.$$

Since $\dim(N/xN) \leqslant 2$, we get by Lemma 3.1 that $H_I^{i-1}(M,N/xN)$ is weakly Laskerian. Hence, by the above exact sequence, $(0:x)_{H_I^i(M,N)}$ is weakly Laskerian. This implies that $(0:I)_{H_I^i(M,N)}$ is weakly Laskerian, so $\mathrm{Ass}((0:I)_{H_I^i(M,N)})$ is a finite set. Note that $H_I^i(M,N)$ is I-torsion. Thus

$$\mathrm{Ass}(H_I^i(M,N)) = \mathrm{Ass}((0:I)_{H_I^i(M,N)})$$

is a finite set, as required. □

The following results are immediate consequences of Lemma 3.1 and Theorem 1.1.

Corollary 3.1. *If* $\dim(R) \leqslant 2$*, then* $H_I^i(M,N)$ *is weakly Laskerian for all* $i \geqslant 0$.

Corollary 3.2. *If* $\dim(R) \leqslant 3$*, then* $\mathrm{Ass}(H_I^i(M,N))$ *is finite for all* $i \geqslant 0$.

In the rest of this section, we study a further consequence of Lemma 3.1 and Theorem 1.1. Recall that N is called a *generalized f-module* if and only if any system of parameters of N is a generalized regular sequence of N (cf. [15]).

For a finitely generated R-module X, and an integer i, let $(\mathrm{Ass}_R(X))_i = \{\mathfrak{q} \in \mathrm{Ass}_R(X) \mid \dim R/\mathfrak{q} = i\}$. We denote by $\widehat{R}$ (resp. $\widehat{X}$) the $\mathfrak{m}$-adic completion of R (resp. X).

Over a four-dimensional local ring, with additional hypotheses, we can obtain a finiteness result on the set of associated primes of generalized local cohomology modules as follows.

Corollary 3.3. *Assume that* $\dim(R) = 4$ *and* $\mathrm{pd}_R(M) < +\infty$*. If* $(\mathrm{Ass}_{\widehat{R}}(\widehat{N}))_3 = \emptyset$ *and* $\mathrm{ht}(I) \geqslant 2$*, then* $\mathrm{Ass}(H^i_I(M,N))$ *is finite for all* $i \geqslant 0$*.*

Proof. By Theorem 1.1, the assertion is true for $\dim(N) \leqslant 3$. Thus, we may assume that $\dim(N) = 4$. In view of Lemma 2.1(i) and Lemma 2.3, it is enough to prove that $\mathrm{Ass}(H^i_I(M,N))$ is finite for $i = 1, 2$.

The case $i = 1$. By similar arguments as in the proof of Theorem 1.1, we can assume that N is I-torsion-free. Thus, $\mathrm{gdepth}(I_M, N) \geqslant 1$, so that, we get by Lemma 2.2 that $\mathrm{Ass}(H^1_I(M,N))$ is finite.

The case $i = 2$. By Lemma 2.1(iii) and [13, Lemma 2.1], we get $\mathrm{Ass}(H^2_I(M,N)) \subseteq \{\mathfrak{q} \cap R \mid \mathfrak{q} \in \mathrm{Ass}_{\widehat{R}}(H^2_{\widehat{I}}(\widehat{M},\widehat{N}))\}$. Thus, we can assume that R is complete. Let N_1 be the largest submodule of N such that $\dim(N_1) < 4$. By the hypothesis of N, we have $\dim(N_1) \leqslant 2$. Moreover, we obtain an exact sequence

$$H^2_I(M,N_1) \xrightarrow{f} H^2_I(M,N) \xrightarrow{g} H^2_I(M,N/N_1).$$

Since $\dim(N_1) \leqslant 2$, we get by Lemma 3.1 that $H^2_I(M,N_1)$ is weakly Laskerian. So, $\mathrm{Ass}(\mathrm{Im}\, f)$ is finite. Thus, in order to prove the finiteness of $\mathrm{Ass}(H^2_I(M,N))$, we need only to show that $\mathrm{Ass}(H^2_I(M,N/N_1))$ is finite. Hence, we can suppose that N is unmixed. We denote by $K(N)$ the canonical module of N, and set $N^* = K(K(N))$. Since $\dim(N) = 4$, we get by [15, Corollary 3.5] that N^* is a generalized f-module with $\dim(N^*) = 4$; moreover, since N is unmixed, there exists an exact sequence $0 \to N \to N^* \to U \to 0$ for some finitely generated R-module U (cf. [16, 3.2.2 and 3.2.3]). It induces an exact sequence

$$H^1_I(M,N^*) \xrightarrow{\alpha} H^1_I(M,U) \xrightarrow{\beta} H^2_I(M,N) \xrightarrow{\gamma} H^2_I(M,N^*).$$

So, the sequence

$$0 \to \mathrm{Im}\,\beta \to H^2_I(M,N) \to \mathrm{Im}\,\gamma \to 0$$

is exact, where $\mathrm{Im}\,\beta \cong H^1_I(M,U)/\mathrm{Im}\,\alpha$. Moreover, as mentioned in Section 2, we have

$$\mathrm{Ass}(H^1_I(M,U)/\mathrm{Im}\,\alpha) \subseteq \mathrm{Ass}(H^1_I(M,U)) \cup \mathrm{Supp}(\mathrm{Im}\,\alpha).$$

By similar arguments as the case $i = 1$, we get that $\mathrm{Ass}(H^1_I(M,U))$ is a finite set. Thus, if we can show that $\mathrm{Supp}(\mathrm{Im}\,\alpha)$, $\mathrm{Ass}(\mathrm{Im}\,\gamma)$ are finite sets, then we get by the above exact sequence that $\mathrm{Ass}(H^2_I(M,N))$ is finite. Here, $\mathrm{Im}\,\alpha$ is a quotient of $H^1_I(M,N^*)$, and $\mathrm{Im}\,\gamma$ is a submodule of $H^2_I(M,N^*)$. Therefore, the rest of our proof devotes to prove the finiteness of $\mathrm{Supp}(H^1_I(M,N^*))$ and $\mathrm{Ass}(H^2_I(M,N^*))$. Indeed, since $\mathrm{ht}(I_M) \geqslant \mathrm{ht}(I) \geqslant 2$, so $\dim(N^*/I_M N^*) \leqslant 2$. It is clear that, if

$\dim(N^*/I_MN^*) \leqslant 1$ then $\operatorname{Supp}(H_I^2(M,N^*))$ is finite. If $\dim(N^*/I_MN^*) = 2$, then we can choose $x_1, x_2 \in I_M$ such that it is a part of system of parameters of N^*. Since N^* is a generalized f-module, so that x_1, x_2 is a generalized regular sequence of N^* in I_M. This implies that $\operatorname{gdepth}(I_M, N^*) = 2$. Therefore, by Lemma 2.2, we obtain that $\operatorname{Supp}(H_I^1(M,N^*))$ and $\operatorname{Ass}(H_I^2(M,N^*))$ are finite sets, as required. □

The following result is an immediate consequence of Corollary 3.3.

Corollary 3.4. *Assume that* $\dim(R) = 4$ *and* $\operatorname{pd}_R(M) < +\infty$. *If* N *is Cohen Macaulay and* $\operatorname{ht}(I) \geqslant 2$, *then* $\operatorname{Ass}(H_I^i(M,N))$ *is finite for all* $i \geqslant 0$.

4. Proof of Theorem 1.2

At first, we recall a known fact what follows: for a local ring R, if there exists a finitely generated R-module N such that $\operatorname{injd}_R(N) < +\infty$, then, by [5, Remark 9.6.4], R must be Cohen Macaulay; and, in this case, we get by [5, Theorem 3.1.17] that $\operatorname{injd}_R(N) = \dim(R)$. Moreover, as mentioned in Section 1, we use $s(I,M)$ to denote the stable value of $\operatorname{depth}(M/I^nM)$ provided n large.

Proof of Theorem 1.2 (a), (b). (a) Since $\operatorname{injd}_R(N) < +\infty$, we get by [5, Exercise 3.1.24] that $\operatorname{Ext}_R^i(M/I^nM,N) = 0$ for all $i > d - \operatorname{depth}(M/I^nM)$. It follows that, for each $i > d - s(I,M)$, we have $\operatorname{Ext}_R^i(M/I^nM,N) = 0$ for all $n \gg 0$. Note that $H_I^i(M,N) = \varinjlim_n \operatorname{Ext}_R^i(M/I^nM,N)$. Therefore, by basic properties of direct limits, $H_I^i(M,N) = 0$ for all $i > d - s(I,M)$, as required.

(b) Since $\operatorname{injd}_R(N) < +\infty$, R is Cohen Macaulay, so is $\widehat{R}$. It is clear that $H_I^d(M,N)$ is Artinian if and only if so is $H_{\widehat{I}}^d(\widehat{M},\widehat{N})$. Moreover, by [5, Proposition 3.1.14], $\operatorname{injd}_R(N) < \infty$ if and only if $\operatorname{injd}_{\widehat{R}}(\widehat{N}) < \infty$. Therefore, we may assume that R is a complete Cohen Macaulay local ring. So, by [5, Corollary 3.3.8], R admits the canonical module ω_R.

We first claim that $H_I^d(M,\omega_R)$ is Artinian. Indeed, by local duality for a Cohen Macaulay homomorphic image of a Gorenstein local ring, for each $n > 0$, we obtain the following isomorphism

$$\operatorname{Ext}_R^d(M/I^nM,\omega_R) \cong \operatorname{Hom}(H_{\mathfrak{m}}^0(M/I^nM), E(k)),$$

where $E(k)$ is the injective hull of $k = R/\mathfrak{m}$ as R-module. For each $n > 0$, there exists an integer $t_n > 0$ such that

$$H_{\mathfrak{m}}^0(M/I^nM) \cong \operatorname{Hom}(R/\mathfrak{m}^{t_n}, M/I^nM)$$

and

$$\bigcap_{i>0} \mathfrak{m}^i(0:I^n)_{D(M)} = \mathfrak{m}^{t_n}(0:I^n)_{D(M)},$$

where $D(M)$ is the Matlis duality of M. Thus, we have the following isomorphisms

$$\begin{aligned}\mathrm{Ext}^d_R(M/I^nM,\omega_R) &\cong \mathrm{Hom}(\mathrm{Hom}(R/\mathfrak{m}^{t_n}, M/I^nM), E(k))\\ &\cong R/\mathfrak{m}^{t_n}\otimes_R \mathrm{Hom}(M/I^nM, E(k))\\ &\cong R/\mathfrak{m}^{t_n}\otimes_R (0:I^n)_{D(M)}\\ &\cong \big((0:I^n)_{D(M)}\big)/\big(\mathfrak{m}^{t_n}(0:I^n)_{D(M)}\big)\\ &\cong \big((0:I^n)_{D(M)}\big)/\big(\bigcap_{i>0}\mathfrak{m}^i(0:I^n)_{D(M)}\big).\end{aligned}$$

Note that $\big\{\mathfrak{m}^{t_n}(0:I^n)_{D(M)} = \bigcap_{i>0}\mathfrak{m}^i(0:I^n)_{D(M)}, h^n_m\big\}$ forms a direct system of R-modules, where

$$h^n_m : \bigcap_{i>0}\mathfrak{m}^i(0:I^n)_{D(M)} \to \bigcap_{i>0}\mathfrak{m}^i(0:I^m)_{D(M)}$$

is the inclusion homomorphism provided $n\leqslant m$. Since passage to direct limits preserves exactness, therefore we get an isomorphism

$$\begin{aligned}H^d_I(M,\omega_R) &\cong \Gamma_I(D(M))/\big(\sum_{n>0}\mathfrak{m}^{t_n}(0:I^n)_{D(M)}\big)\\ &= D(M)/\big(\sum_{n>0}\mathfrak{m}^{t_n}(0:I^n)_{D(M)}\big).\end{aligned}$$

It follows that $H^d_I(M,\omega_R)$ is Artinian, thus the claim is proved.

Finally, we apply the claim to prove the Artinianness of $H^d_I(M,N)$. Indeed, since R admits the canonical module, we get by [1, Theorem A] that there exists an exact sequence $0\to U\to\Omega\to N\to 0$, where U is a finitely generated R-module with $\mathrm{injd}_R(U) < +\infty$, and Ω is a maximal Cohen-Macaulay module. Since $\mathrm{injd}_R(N) < \infty$ and $\mathrm{injd}_R(U) < \infty$, we get by the above exact sequence that $\mathrm{injd}_R(\Omega) < \infty$. Hence, by [5, Exercise 3.3.28(a)], we obtain that $\Omega = \omega_R^n$ for some integer $n > 0$. So, we get an exact sequence $0\to U\to\omega^n_R\to N\to 0$. Therefore, in view of (a), we get an exact sequence

$$H^d_I(M,\omega^n_R)\to H^d_I(M,N)\to 0.$$

Since $H^d_I(M,-)$ is an additive functor, we get by the claim that $H^d_I(M,\omega^n_R)$ is Artinian. It follows by the above exact sequence that $H^d_I(M,N)$ is Artinian, as required. □

To prove Theorem 1.2(c), for an R-module K, and an integer $i\geqslant 0$, we denote

$$\mathrm{Supp}^i(K) = \{\mathfrak{p}\in\mathrm{Supp}(K)\mid \mathrm{ht}(\mathfrak{p}) = i\}.$$

Proof of theorem 1.2(c). We first claim a stronger result that, if $\mathrm{injd}_R(N) < +\infty$ then $\mathrm{Supp}^i(H^i_I(M,N))$ is finite for all $i\geqslant 0$. Indeed, for any $i\geqslant 0$, let $\mathfrak{p}\in\mathrm{Supp}^i(H^i_I(M,N))$, then $\dim(R_\mathfrak{p}) = i$ and $H^i_I(M,N)_\mathfrak{p}\neq 0$. So, since $\mathrm{injd}_{R_\mathfrak{p}}(N_\mathfrak{p}) < +\infty$, we get by (a) that $i\leqslant \dim(R_\mathfrak{p}) - s(I_\mathfrak{p}, M_\mathfrak{p})$. It yields that $s(I_\mathfrak{p}, M_\mathfrak{p}) = 0$, i.e, $\operatorname{depth}(M_{\mathfrak{p}}/(I_{\mathfrak{p}})^n M_{\mathfrak{p}}) = 0$ for some large n. Hence, $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass}((M/I^nM)_{\mathfrak{p}})$, so that $\mathfrak{p} \in \operatorname{Ass}(M/I^nM)$. Therefore,

$$\operatorname{Supp}^i(H_I^i(M,N)) \subseteq \bigcup_{n>0} \operatorname{Ass}(M/I^nM).$$

On the other hand, by [2], the set $\bigcup_{n>0} \operatorname{Ass}(M/I^nM)$ is finite, so we get by the above inclusion that $\operatorname{Supp}^i(H_I^i(M,N))$ is a finite set for all $i \geqslant 0$, so the claim is proved.

Finally, we apply the claim to prove the finiteness of $\operatorname{Supp}(H_I^{d-1}(M,N))$. Keep in mind that $d = \dim(R)$ and $\operatorname{injd}_R(N) < +\infty$. Now, let $\mathfrak{p} \in \operatorname{Supp}(H_I^{d-1}(M,N))$, then $H_I^{d-1}(M,N)_{\mathfrak{p}} \neq 0$. This implies by (a) that $d-1 \leqslant \dim(R_{\mathfrak{p}}) - s(I_{\mathfrak{p}}, M_{\mathfrak{p}})$, thus $\dim(R_{\mathfrak{p}}) \geqslant d-1$. It yields that $\mathfrak{p} = \mathfrak{m}$ or $\operatorname{ht}(\mathfrak{p}) = d-1$. Thus, $\mathfrak{p} \in \operatorname{Supp}^{d-1}(H_I^{d-1}(M, N)) \cup \{\mathfrak{m}\}$. From this, we get by the claim that Supp $(H_I^{d-1}(M,N))$ is a finite set, as required. □

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