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SOME SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO k-SYMMETRIC POINTS

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ABSTRACT. In the present paper, we introduce two new subclasses $\mathcal{P}^{(k)}(\lambda, \alpha)$ and $\mathcal{Q}^{(k)}(\lambda, \alpha)$ of analytic functions with respect to k-symmetric points. Such results as integral representations, convolution properties and coefficient inequalities for these function classes are proven.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk $\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. Also let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of all functions which are of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqslant 0).$$

Let $\mathcal{T}(\lambda, \alpha)$ be the subclass of $\mathcal{T}$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\Re\left(\frac{\frac{zf'(z)}{f(z)}}{\lambda\frac{zf'(z)}{f(z)} + (1-\lambda)}\right) > \alpha \qquad (z \in \mathbb{U}),$$

for some α $(0 \leqslant \alpha < 1)$ and λ $(0 \leqslant \lambda < 1)$. And let $\mathcal{C}(\lambda, \alpha)$ be the subclass of $\mathcal{T}$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\Re\left(\frac{1 + \frac{zf''(z)}{f'(z)}}{1 + \lambda\frac{zf''(z)}{f'(z)}}\right) > \alpha \qquad (z \in \mathbb{U}),$$

for some α $(0 \leqslant \alpha < 1)$ and λ $(0 \leqslant \lambda < 1)$. The classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$ were first introduced and investigated by Altintas and Owa [1], then were studied by

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Aouf *et al.* [2]. They obtained such results as coefficient inequalities, distortion and covering theorems for these function classes.

Motivated by the classes $\mathcal{T}(\lambda,\alpha)$ and $\mathcal{C}(\lambda,\alpha)$, we now introduce and investigate the following two subclasses of $\mathcal{A}$ with respect to k-symmetric points, and obtain some interesting results.

Definition 1.1. A function $f \in \mathcal{A}$ is in the class $\mathcal{P}^{(k)}(\lambda,\alpha)$ if it satisfies the following inequality:

$$\Re\left(\frac{\frac{zf'(z)}{f_k(z)}}{\lambda\frac{zf'(z)}{f_k(z)}+(1-\lambda)}\right)>\alpha \qquad (z\in\mathbb{U}), \tag{1.2}$$

where $0 \leqslant \alpha < 1,\ 0 \leqslant \lambda < 1,\ k \geqslant 1$ is a fixed positive integer and $f_k(z)$ is defined by the following equation:

$$f_k(z)=\frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{-j}f(\varepsilon_k^j z) \qquad \left(\varepsilon_k=\exp\left(\frac{2\pi i}{k}\right);\ z\in\mathbb{U}\right). \tag{1.3}$$

And a function $f \in \mathcal{A}$ is in the class $\mathcal{Q}^{(k)}(\lambda,\alpha)$ if and only if $zf' \in \mathcal{P}^{(k)}(\lambda,\alpha)$.

For simplicity, we write

$$\mathcal{P}^{(k)}(\lambda,\alpha)\cap\mathcal{T}=:\mathcal{P}^{(k)}_{\mathcal{T}}(\lambda,\alpha) \quad \text{and} \quad \mathcal{Q}^{(k)}(\lambda,\alpha)\cap\mathcal{T}=:\mathcal{Q}^{(k)}_{\mathcal{T}}(\lambda,\alpha).$$

In the present paper, we aim at proving such results as integral representations, convolution properties and coefficient inequalities for the function classes $\mathcal{P}^{(k)}(\lambda,\alpha)$ and $\mathcal{Q}^{(k)}(\lambda,\alpha)$.

2. Main results

We first give some integral representations for the function classes $\mathcal{P}^{(k)}(\lambda,\alpha)$ and $\mathcal{Q}^{(k)}(\lambda,\alpha)$.

Theorem 2.1. *Let $f \in \mathcal{P}^{(k)}(\lambda,\alpha)$. Then*

$$f_k(z)=z\cdot\exp\left(\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{2(1-\alpha)\omega(t)}{t[1-\lambda-(1+\lambda-2\alpha\lambda)\omega(t)]}dt\right), \tag{2.1}$$

where $f_k(z)$ is defined by (1.3), *$\omega(z)$ is analytic in $\mathbb{U}$ with*

$$\omega(0)=0 \quad \textit{and} \quad |\omega(z)|<1 \quad (z\in\mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{P}^{(k)}(\lambda,\alpha)$. We know that the condition (1.2) can be written as follows:

$$\frac{\frac{zf'(z)}{f_k(z)}}{\lambda\frac{zf'(z)}{f_k(z)}+(1-\lambda)}\prec\frac{1+(1-2\alpha)z}{1-z} \qquad (z\in\mathbb{U}),$$

where " $\prec$ " stands for the subordination. It follows that

$$\frac{\frac{zf'(z)}{f_k(z)}}{\lambda\frac{zf'(z)}{f_k(z)}+(1-\lambda)}=\frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}\qquad(z\in\mathbb{U}),$$

where $\omega(z)$ is analytic in $\mathbb{U}$ with

$$\omega(0)=0\quad\text{and}\quad|\omega(z)|<1\quad(z\in\mathbb{U}).$$

This yields

$$\frac{zf'(z)}{f_k(z)}=\frac{(1-\lambda)[1+(1-2\alpha)\omega(z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(z)}. \tag{2.2}$$

Upon substituting z by $\varepsilon^\mu z$ $(\mu=0,1,2,\dots,k-1)$ in (2.2), we get

$$\frac{\varepsilon^\mu zf'(\varepsilon^\mu z)}{f_k(\varepsilon^\mu z)}=\frac{(1-\lambda)[1+(1-2\alpha)\omega(\varepsilon^\mu z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu z)}. \tag{2.3}$$

Noting that

$$f_k(\varepsilon^\mu z)=\varepsilon^\mu f_k(z)\qquad(z\in\mathbb{U}).$$

Thus, by letting $\mu=0,1,2,\dots,k-1$ in (2.3), successively, and summing the resulting equations, we have

$$\frac{zf_k'(z)}{f_k(z)}=\frac{1}{k}\sum_{\mu=0}^{k-1}\frac{(1-\lambda)[1+(1-2\alpha)\omega(\varepsilon^\mu z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu z)}. \tag{2.4}$$

We next find from (2.4) that

$$\frac{f_k'(z)}{f_k(z)}-\frac{1}{z}=\frac{1}{k}\sum_{\mu=0}^{k-1}\frac{2(1-\alpha)\omega(\varepsilon^\mu z)}{z[1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu z)]}. \tag{2.5}$$

Upon integrating (2.5), we have

$$\begin{aligned}\log\left(\frac{f_k(z)}{z}\right)&=\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^z\frac{2(1-\alpha)\omega(\varepsilon^\mu\zeta)}{\zeta[1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu\zeta)]}d\zeta\\&=\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{2(1-\alpha)\omega(t)}{t[1-\lambda-(1+\lambda-2\alpha\lambda)\omega(t)]}dt.\end{aligned} \tag{2.6}$$

From (2.6), we can easily get (2.1) asserted by Theorem 2.1. □

Corollary 2.1. *Let $f\in\mathcal{P}^{(k)}(\lambda,\alpha)$. Then*

$$\begin{aligned}f(z)=\int_0^z&\exp\left(\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu\zeta}\frac{2(1-\alpha)\omega(t)}{t[1-\lambda-(1+\lambda-2\alpha\lambda)\omega(t)]}dt\right)\\&\cdot\frac{(1-\lambda)[1+(1-2\alpha)\omega(\zeta)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\zeta)}d\zeta,\end{aligned} \tag{2.7}$$

where $\omega(z)$ is analytic in $\mathbb{U}$ with

$$\omega(0)=0\quad\textit{and}\quad|\omega(z)|<1\quad(z\in\mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{P}^{(k)}(\lambda, \alpha)$. It follows from (2.1) and (2.2) that

$$\begin{aligned} f'(z) =& \frac{f_k(z)}{z} \cdot \frac{(1-\lambda)[1+(1-2\alpha)\omega(z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(z)} \\ =& \exp\left(\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{2(1-\alpha)\omega(t)}{t[1-\lambda-(1+\lambda-2\alpha\lambda)\omega(t)]}dt\right) \\ & \cdot \frac{(1-\lambda)[1+(1-2\alpha)\omega(z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(z)}. \end{aligned} \tag{2.8}$$

Upon integrating (2.8), we can easily get (2.7). □

By similarly applying the method of proof of Theorem 2.1 and Corollary 2.1, we can get the following results for the class $\mathcal{Q}^{(k)}(\lambda, \alpha)$.

Corollary 2.2. *Let $f \in \mathcal{Q}^{(k)}(\lambda, \alpha)$. Then*

$$f_k(z) = \int_0^z \exp\left(\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta}\frac{2(1-\alpha)\omega(t)}{t[1-\lambda-(1+\lambda-2\alpha\lambda)\omega(t)]}dt\right)d\zeta,$$

where $f_k(z)$ is defined by (1.3), $\omega(z)$ is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Corollary 2.3. *Let $f \in \mathcal{Q}^{(k)}(\lambda, \alpha)$. Then*

$$\begin{aligned} f(z) =& \int_0^z \frac{1}{\xi}\int_0^\xi \exp\left(\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta}\frac{2(1-\alpha)\omega(t)}{t[1-\lambda-(1+\lambda-2\alpha\lambda)\omega(t)]}dt\right) \\ & \cdot \frac{(1-\lambda)[1+(1-2\alpha)\omega(\zeta)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\zeta)}d\zeta d\xi, \end{aligned}$$

where $\omega(z)$ is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Let $f,\ g \in \mathcal{A}$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n.$$

Then the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n c_n z^n =: (g * f)(z).$$

We now provide some convolution properties for the classes $\mathcal{P}^{(k)}(\lambda, \alpha)$ and $\mathcal{Q}^{(k)}(\lambda, \alpha)$, Here we choose to omit the details of proof.

Corollary 2.4. *A function* $f \in \mathcal{P}^{(k)}(\lambda, \alpha)$ *if and only if*

$$\frac{1}{z}\Big\{f * \Big\{\frac{z}{(1-z)^2}\Big\{\Big(1-e^{i\theta}\Big) - \lambda\Big[1+(1-2\alpha)e^{i\theta}\Big]\Big\} - (1-\lambda)\Big[1+(1-2\alpha)e^{i\theta}\Big]h(z)\Big\}\Big\} \neq 0$$

for all $z \in \mathbb{U}$ *and* $0 \leqslant \theta < 2\pi$*, where* $h(z)$ *is given by*

$$h(z) = \frac{1}{k}\sum_{v=0}^{k-1}\frac{z}{1-\varepsilon^v z} \qquad (z \in \mathbb{U}). \tag{2.9}$$

Corollary 2.5. *A function* $f \in \mathcal{Q}^{(k)}(\lambda, \alpha)$ *if and only if*

$$\frac{1}{z}\Big\{f * \Big\{z\Big\{\frac{z}{(1-z)^2}\Big\{\Big(1-e^{i\theta}\Big) - \lambda\Big[1+(1-2\alpha)e^{i\theta}\Big]\Big\} - (1-\lambda)\Big[1+(1-2\alpha)e^{i\theta}\Big]h(z)\Big\}'\Big\}\Big\} \neq 0$$

for all $z \in \mathbb{U}$ *and* $0 \leqslant \theta < 2\pi$*, where* $h(z)$ *is given by* (2.9).

In the following we provide the coefficient sufficient conditions for functions belonging to the classes $\mathcal{P}^{(k)}(\lambda, \alpha)$ and $\mathcal{Q}^{(k)}(\lambda, \alpha)$.

Theorem 2.2. *Let* $0 \leqslant \alpha < 1$ *and* $0 \leqslant \lambda < 1$*. If*

$$\sum_{n=1}^{\infty}[(1-\lambda\alpha)(nk+1) - \alpha(1-\lambda)]|a_{nk+1}| + \sum_{\substack{n=2\\(n\neq lk+1)}}^{\infty}(1-\lambda\alpha)n|a_n| \leqslant 1-\alpha, \tag{2.10}$$

then $f \in \mathcal{P}^{(k)}(\lambda, \alpha)$.

Proof. It suffices to show that

$$\left|\frac{\frac{zf'(z)}{f_k(z)}}{\lambda\frac{zf'(z)}{f_k(z)} + (1-\lambda)} - 1\right| < 1-\alpha.$$

By noting that for $|z| = r < 1$, we have

$$\begin{aligned}\left|\frac{\frac{zf'(z)}{f_k(z)}}{\lambda\frac{zf'(z)}{f_k(z)} + (1-\lambda)} - 1\right| &= \left|\frac{\sum_{n=2}^{\infty}(1-\lambda)(n-b_n)a_n z^{n-1}}{1+\sum_{n=2}^{\infty}[\lambda n + (1-\lambda)b_n]a_n z^{n-1}}\right| \\ &\leqslant \frac{\sum_{n=2}^{\infty}(1-\lambda)(n-b_n)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}[\lambda n + (1-\lambda)b_n]|a_n||z|^{n-1}} \\ &\leqslant \frac{\sum_{n=2}^{\infty}(1-\lambda)(n-b_n)|a_n|}{1-\sum_{n=2}^{\infty}[\lambda n + (1-\lambda)b_n]|a_n|},\end{aligned} \tag{2.11}$$

where

$$b_n = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu} = \begin{cases} 1, & n = lk+1 \quad (l \in \mathbb{N}), \\ 0, & n \neq lk+1 \quad (l \in \mathbb{N}). \end{cases} \tag{2.12}$$

The last expression of (2.11) is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty}[(1-\lambda\alpha)n - \alpha(1-\lambda)b_n]|a_n| \leqslant 1-\alpha. \tag{2.13}$$

By substituting (2.12) into (2.13), we can get (2.10), hence f satisfies the condition (1.2), that is, $f \in \mathcal{P}^{(k)}(\lambda, \alpha)$. The proof of Theorem 2.2 is thus completed. □

By similarly applying the method of proof of Theorem 2.2, we can get the following result for the class $\mathcal{Q}^{(k)}(\lambda, \alpha)$.

Corollary 2.6. *Let* $0 \leqslant \alpha < 1$ *and* $0 \leqslant \lambda < 1$. *If*

$$\sum_{n=1}^{\infty}(nk+1)[(1-\lambda\alpha)(nk+1) - \alpha(1-\lambda)]|a_{nk+1}| + \sum_{\substack{n=2 \\ (n\neq lk+1)}}^{\infty} (1-\lambda\alpha)n^2|a_n| \leqslant 1-\alpha,$$

then $f(z) \in \mathcal{Q}^{(k)}(\lambda, \alpha)$.

Finally, we provide the necessary and sufficient coefficient conditions for functions belonging to the classes $\mathcal{P}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$ and $\mathcal{Q}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$.

Theorem 2.3. *Let* $0 \leqslant \alpha < 1$, $0 \leqslant \lambda < 1$ *and* $f \in \mathcal{T}$. *Then* $f \in \mathcal{P}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}[(1-\lambda\alpha)(nk+1) - \alpha(1-\lambda)]a_{nk+1} + \sum_{\substack{n=2 \\ (n\neq lk+1)}}^{\infty} (1-\lambda\alpha)na_n \leqslant 1-\alpha. \tag{2.14}$$

Proof. In view of Theorem 2.2, we need only to prove the necessity. Suppose that $f \in \mathcal{P}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$. Then from (1.2), we can get

$$\Re\left(\frac{zf'(z)}{\lambda zf'(z) + (1-\lambda)f_k(z)}\right) > \alpha,$$

that is,

$$\Re\left(\frac{1-\sum_{n=2}^{\infty} na_n z^{n-1}}{1-\sum_{n=2}^{\infty}[\lambda n + (1-\lambda)b_n]a_n z^{n-1}}\right) > \alpha, \tag{2.15}$$

where b_n is given by (2.12). By letting $z \to 1^-$ through real values in (2.15), we can get

$$\frac{1-\sum_{n=2}^{\infty} na_n}{1-\sum_{n=2}^{\infty}[\lambda n + (1-\lambda)b_n]a_n} \geqslant \alpha,$$

or equivalently,

$$\sum_{n=2}^{\infty}[(1-\lambda\alpha)n-\alpha(1-\lambda)b_n]a_n \leqslant 1-\alpha. \tag{2.16}$$

Upon substituting (2.12) into (2.16), we can easily get (2.14). This completes the proof of Theorem 2.3. □

By similarly applying the method of proof of Theorem 2.3, we can get the following result for the class $\mathcal{Q}_{\mathcal{T}}^{(k)}(\lambda,\alpha)$.

Corollary 2.7. *Let* $0 \leqslant \alpha < 1,\ 0 \leqslant \lambda < 1$ *and* $f \in \mathcal{T}$. *Then* $f \in \mathcal{Q}_{\mathcal{T}}^{(k)}(\lambda,\alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}(nk+1)[(1-\lambda\alpha)(nk+1)-\alpha(1-\lambda)]a_{nk+1}+\sum_{\substack{n=2\\(n\neq lk+1)}}^{\infty}(1-\lambda\alpha)n^2a_n \leqslant 1-\alpha.$$

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