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PRODUCT PROPERTIES IN WEIGHTED PLURIPOTENTIAL THEORY

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Abstract. In this article we study product properties in weighted pluripotential theory.

1. Introduction

Pluripotential theory in recent years has seen several important developments. Many results of potential theory on the complex plane were successfully extended. Some authors have tried to extend results of normal pluripotential theory to the weighted pluripotential theory. The basic facts in weighted pluripontetial theory can be found in appendix B written by T. Bloom of the book of Saff-Totik. Further results were given in [5] where the authors have developed the weighted pluripotential theory to solve unweighted pluripotenital problems. More precisely, they introduced the concepts of weighted directional Chebyshev constant $\tau^{\omega}(E,\theta)$, weighted transfinite diameter $d^{\omega}(E)$ for an admissible weight function ω on a compact set E (see next section for specific definitions) and found out the relations with unweighted directional Chebyshev constant, unweighted transfinite diameter. The aim of the present paper is to study product-like properties for these quantities.

The paper is organized as follows. Beside the introduction the paper contains two sections. In Section 2 we recall some backgrounds of weighted pluripotential theory. The main results are presented in Section 3, where we study product properties of weighted directional Chebyshev constant, weighted transfinite diameter. We also give a simple proof of a theorem of T. Bloom and J.P. Calvi (see [4]).

2. Preliminaries

First we recall some elements of weighted pluripotential theory that will be used throughout the paper (see [2, 3, 5, 8, 10]). We also state and prove some auxilliary results.

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Let $\mathcal{L}(\mathbb{C}^n)$ denote the set of plurisubharmonic functions (PSH) in $\mathbb{C}^n$ of logarithmic growth

$$\mathcal{L} = \mathcal{L}(\mathbb{C}^n) = \left\{u \in \mathrm{PSH}(\mathbb{C}^n) : \ u(z) \leqslant \frac{1}{2}\log(1+|z|^2) + c_u\right\},$$

where $|z| = (|z_1|^2 + ... + |z_n|^2)^{\frac{1}{2}}$. Let

$$\mathcal{L}^+ = \mathcal{L}^+(\mathbb{C}^n) = \left\{u \in \mathrm{PSH}(\mathbb{C}^n) : \ \frac{1}{2}\log(1+|z|^2) - c_u \leqslant u(z) \leqslant \frac{1}{2}\log(1+|z|^2) + c_u\right\}.$$

For $u \in \mathcal{L}$ we define

$$\gamma_u(z) = \overline{\lim_{\lambda\in\mathbb{C},\ \lambda\to\infty}} \left[u(\lambda z) - \log|\lambda z|\right].$$

Let E be a compact non-pluripolar set in $\mathbb{C}^n$ and ω an admissible weight function on E, i.e. ω is a nonnegative, upper semicontinuous function on E and the set $\{z \in E : \ \omega(z) > 0\}$ is non-pluripolar. The weighted pluricomplex Green function is defined by

$$V_{E,Q} = \sup\{u(z) : \ u \in \mathcal{L},\ u \leqslant Q = -\log\omega \text{ on } E\}.$$

It is a lower semicontinuous function in $\mathbb{C}^n$. It is continuous if E is locally regular and Q continuous (see [10]). Its upper semicontinuous regularization

$$V^*_{E,Q}(z) = \overline{\lim_{w\to z}} V_{E,Q}(w),$$

belonging to $\mathcal{L}^+(\mathbb{C}^n)$. $(dd^cV^*_{E,Q})^n$ is a positive Borel measure of total mass $(2\pi)^n$ and with support in E, where $(dd^c)^n$ is the complex Monge-Ampère operator.

We define the weighted logarithmic capacity

$$c^\omega(E) = \exp(-\sup_{|z|=1} \gamma_{V^*_{E,Q}}).$$

Let ω be a weight admissible function on E and

$$\Sigma_n = \left\{\theta = (\theta_1, ..., \theta_n) \in \mathbf{R}^n : \ \sum_{j=1}^n \theta_j = 1,\ \theta_j \geqslant 0,\ \forall\ j = 1, ..., n\right\},$$

$$\Sigma_n^0 = \mathrm{Int}\Sigma = \{\theta \in \Sigma : \ \theta_j > 0,\ \forall\ j = 1, ..., n\},$$

$$S_n^0 = \left\{t = (t_1, ..., t_n) \in \mathbf{R}^n : \sum_{j=1}^n t_j < 1,\ t_j > 0,\ \forall\ j = 1, ..., n\right\}.$$

Then it is easy to compute that $\lambda_n(S_n^0) = \frac{1}{n!}$ and $\sigma_n(\Sigma_n^0) = \frac{\sqrt{n}}{(n-1)!}$.

Let $e_1(z), e_2(z), ...$ be listing of the monomials $\{e_i(z) = z^{\alpha(i)} = z_1^{\alpha_1}...z^{\alpha_n}\}$ indexed using a lexicographic order on the multiindices $\alpha(i) \in \mathbf{N}^n$, but with $\deg e_i = |\alpha(i)|$ nondecreasing. Set

$$P_i = P(\alpha(i)) = \left\{e_i(z) + \sum_{j<i} c_j e_j(z) : \ c_j \in \mathbb{C}\right\}$$

and

$$\tau_i^\omega(E) = \inf\left\{||\omega^{|\alpha(i)|}p||_E = \sup_E |\omega(z)^{|\alpha(i)|}p(z)| : \ p \in P_i\right\}^{\frac{1}{|\alpha(i)|}}$$

Zaharjuta's method shows that the limit

$$\tau^\omega(E,\theta) = \lim_{i\to\infty,\ \frac{\alpha(i)}{|\alpha(i)|}\to\theta} \tau_i^\omega(E)$$

exists for all $\theta \in \Sigma_n^0$ (see [11]). It is called the weighted directional Chebyshev constant for E with weight ω in the direction θ. T. Bloom and N. Levenberg proved an important result relating this notation: $\tau^\omega(E,\theta) = \tau^\omega(F,\theta)$ for all compact sets $E, F \subset \mathbb{C}^n$ with $F \subset E$ and $E\backslash F$ pluripolar. We next define the weighted transfinite diameter

$$d^\omega(E) = \exp\Big(\frac{1}{\text{meas}(\Sigma_0)}\int\limits_{\Sigma_0} \log \tau^\omega(E,\theta)d\sigma_n(\theta)\Big).$$

In the case $w = 1$ on E, weighted pluripotential theory becomes usual pluripotential theory. We will write $\tau(E,\theta)$ and $d(E)$ in place of $\tau^\omega(E,\theta)$ and $d^\omega(E)$.

Given a compact set $E \subset \mathbb{C}^n$ and a finite positive Borel measure μ on E, we say that (E,μ) satisfies Bernstein-Markov (B-M) inequality if, for every $\epsilon > 0$, there exists a constant $C = C(\epsilon)$ such that, for all holomorphic polynomials p in $\mathbb{C}^n$ we have

$$||p||_E \leqslant C(1+\epsilon)^{deg(p)}||p||_{L^2(\mu)}.$$

Let $\bar{B}(a,r)$ be the closed ball of radius r with center at a in $\mathbb{C}^n$ and λ_{2n} be Lebesgue measure in $\mathbb{C}^n$. It is known that $(\bar{B}(a,r), \lambda_{2n})$ satisfies Bernstein-Markov inequality, (see [8] or Theorem 4.1 in [6]). Hence if E is a finite union of closed balls in $\mathbf{C}^n$ then (E, λ_{2n}) also satisfies Bernstein-Markov inequality.

We first prove some auxiliary results.

Lemma 2.1. *Let E be a compact non-pluripolar set in $\mathbb{C}^n$ and ω an admissible weight function on E. For each M we set*

$$Z(M) = \{V_{E,Q} \leqslant M\},\ Z(M)^* = \overline{\{V_{E,Q}^* \leqslant M\}}.$$

Then

i) $\tau^\omega(E,\theta) = e^{-M}\tau(Z(M),\theta) = e^{-M}\tau(Z(M)^*,\theta)$

ii) $d^\omega(E) = e^{-M}d(Z(M)) = e^{-M}d(Z(M)^*)$

for all $M \geqslant M_0 = \sup_E V_{E,Q}$ and $\theta \in \Sigma_n^0$.

Proof. i) Let $R > 0$ be such that $E \subset \bar{B}(0,R) = \{z \in \mathbb{C}^n : \ |z| \leqslant R\}$. Define

$$\tilde{\omega} = \begin{cases} \omega \text{ on } E \\ 0 \text{ on } \bar{B}(0,R)\backslash E \end{cases}$$

Then $\tilde{\omega}$ is an admissible weight function on $\bar{B}(0,R)$ satisfying

$$\tau^{\tilde{\omega}}(\bar{B}(0,R),\theta)=\tau^{\omega}(E,\theta)$$

for all $\theta\in\Sigma_n^0$. Thus by replacing (E,ω) by $(\bar{B}(0,R),\tilde{\omega})$ we may assume that E is locally regular. Let $\{\omega_j\}$ be a sequence of nonnegative continuous functions on E such that $\omega_j\searrow\omega$. Lemma 7.3 in [5] implies that

$$V_{E,Q_j}\nearrow V_{E,Q}$$

as $j\to\infty$. Hence

$$Z_j(M)=\{V_{E,Q_j}\leqslant M\}\searrow Z(M)$$

as $j\to\infty$. Using Proposition 3.4 in [5] we have

$$\tau^{\omega_j}(E,\theta)=e^{-M}\tau(Z_j(M),\theta).$$

Letting $j\to\infty$ and using Lemma 1.5 in [5] we have

$$\tau^{\omega}(E,\theta)=e^{-M}\tau(Z(M),\theta).$$

It is easy to check that $Z(M)^*\subset Z(M)$. From the Theorem in [2] we get that $Z(M)\backslash Z(M)^*$ is pluripolar. Using the result mentioning above we obtain $\tau(Z(M),\theta)=\tau(Z(M)^*,\theta)$.

ii) Use i). □

Lemma 2.2. *Let E be a compact set in $\mathbb{C}^n$ and ω an admissible weight function on E. Then $\{V_{E,Q}\leqslant M\}$ is a polynomially convex compact set for all $M\in\mathbb{R}$.*

Proof. As in the proof of Lemma 2.1 we can assume that $E=\bar{B}(0,R)$. Let $\{\omega_j\}$ be a sequence of nonnegative continuous functions on E such that $\omega_j\searrow\omega$. Lemma 7.3 in [5] implies that

$$V_{E,Q_j}\nearrow V_{E,Q}$$

as $j\to\infty$. Hence

$$\{V_{E,Q_j}\leqslant M\}\searrow\{V_{E,Q}\leqslant M\}$$

as $j\to\infty$. Since $\{V_{E,Q_j}\leqslant M\}$ is a polynomially convex compact set we get $\{V_{E,Q}\leqslant M\}$ is a polynomiall convex compact set. □

3. Some results on product properties in weighted pluripotential theory

In this section we first prove the product property of unweighted directional Chebyshev constant. Using it to give the integral proof of product formula for the multivariate transfinite diameter, the result is found by T. Bloom and J. P. Calvi (see [4]).

Proposition 3.1. *Let $E\subset\mathbb{C}^n$, $F\subset\mathbb{C}^m$ be compact sets. Then*

$$\tau(E\times F,(\theta_1,\theta_2))=[\tau(E,\frac{\theta_1}{|\theta_1|})]^{|\theta_1|}[\tau(F,\frac{\theta_2}{|\theta_2|})]^{|\theta_2|},\ (\theta_1,\theta_2)\in\Sigma_{n+m}^0.$$

Proof. We first suppose that E and F are finite unions of closed balls in $\mathbb{C}^n$ and $\mathbb{C}^m$. Then Lebesgue measures $\mu_1 = \lambda_{2n}$, $\mu_2 = \lambda_{2m}$ and $\mu = \lambda_{2n+2m}$ satisfy the Bernstein-Markov inequality on E, F and $E \times F$ respectively. We denote by p_α, q_{α_1} and r_{α_2} the monic orthogonals for the measures μ, μ_1 and μ_2 respectively. Recall that the three families are constructed using the graded lexicographic order in the corresponding space and Gram-Schmidt procedures. Then

$$p_\alpha(z) = q_{\alpha_1}(z_1) r_{\alpha_2}(z_2),\ ||p_\alpha||_\mu = ||q_{\alpha_1}||_{\mu_1} ||r_{\alpha_2}||_{\mu_2},$$

where $z = (z_1, z_2) \in \mathbb{C}^{n+m}, \alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^{n+m}$.

Let $(\theta_1, \theta_2) \in \Sigma^0_{n+m}$ then $\frac{(\alpha_1,\alpha_2)}{|\alpha_1|+|\alpha_2|} \to (\theta_1, \theta_2)$ gives us

$$\frac{\alpha_1}{|\alpha_1|} \to \frac{\theta_1}{|\theta_1|}, \frac{\alpha_2}{|\alpha_2|} \to \frac{\theta_2}{|\theta_2|}.$$

From (2.7) in [4], we have

$$\begin{aligned}
\tau(E \times F, (\theta_1, \theta_2)) &= \lim_{\frac{(\alpha_1,\alpha_2)}{|\alpha_1|+|\alpha_2|} \to (\theta_1,\theta_2)} ||p_\alpha||_\mu^{\frac{1}{|\alpha|}} \\
&= \lim_{\frac{\alpha_1}{|\alpha_1|+|\alpha_2|} \to \theta_1, \frac{\alpha_2}{|\alpha_1|+|\alpha_2|} \to \theta_2} (||q_{\alpha_1}||_{\mu_1} ||r_{\alpha_2}||_{\mu_2})^{\frac{1}{|\alpha_1|+|\alpha_2|}} \\
&= \lim_{\frac{\alpha_1}{|\alpha_1|} \to \frac{\theta_1}{|\theta_1|}, \frac{\alpha_2}{|\alpha_2|} \to \frac{\theta_2}{|\theta_2|}} (||q_{\alpha_1}||_{\mu_1}^{\frac{1}{|\alpha_1|}})^{\frac{|\alpha_1|}{|\alpha_1|+|\alpha_2|}} (||r_{\alpha_2}||_{\mu_2}^{\frac{1}{|\alpha_2|}})^{\frac{|\alpha_2|}{|\alpha_1|+|\alpha_2|}} \\
&= [\tau(E, \frac{\theta_1}{|\theta_1|})]^{|\theta_1|} [\tau(F, \frac{\theta_2}{|\theta_2|})]^{|\theta_2|}.
\end{aligned}$$

In the general case, by the compactness of E and F we can choose E_k and F_k, which are finite unions of closed balls in $\mathbb{C}^n$ and $\mathbb{C}^m$, such that $E_k \searrow E$ and $F_k \searrow F$. Applying to the specific case and Lemma 1.5 in [5] then letting $k \to \infty$ we have the formula. □

Theorem 3.1. *Let $E \subset \mathbb{C}^n$ and $F \subset \mathbb{C}^m$ be compact sets. Then*

$$d(E \times F) = d^{\frac{n}{n+m}}(E).d^{\frac{m}{n+m}}(F).$$

Proof. Using Proposition 3.1 and integral formula of transfinite diameter we have

$$\log d(E \times F)$$

$$= \frac{1}{\sigma_{n+m}(\Sigma^0_{n+m})} \int_{\Sigma^0_{n+m}} [|\theta_1| \log \tau(E, \frac{\theta_1}{|\theta_1|}) + |\theta_2| \log \tau(F, \frac{\theta_2}{|\theta_2|})] d\sigma_{n+m}(\theta_1, \theta_2).$$

The theorem will be proved if we have

$$I_1 = \frac{1}{\sigma_{n+m}(\Sigma^0_{n+m})} \int_{\Sigma^0_{n+m}} |\theta_1| \log \tau(E, \frac{\theta_1}{|\theta_1|}) d\sigma_{n+m}(\theta_1, \theta_2) = \frac{n}{n+m} \log d(E).$$

Using surface integral formula and the Fubini theorem we deduce

$$I_1 = \frac{1}{\frac{\sqrt{n+m}}{(n+m-1)!}}\sqrt{n+m}\int\limits_{S^0_{n+m-1}} |\theta_1|\log\tau(E,\frac{\theta_1}{|\theta_1|})d\lambda_{n+m-1}(\theta_1,\theta_2^{'})),\ (\theta_2=(\theta_2^{'},\theta_2^{(m)}))$$

$$= (n+m-1)!\int\limits_{S^0_n}|\theta_1|\log\tau(E,\frac{\theta_1}{|\theta_1|})d\lambda_n(\theta_1)\int\limits_{|\theta_2^{'}|\leqslant 1-|\theta_1|,\theta_2^{'}\in\mathbf{R}_+^{m-1}} d\lambda_{m-1}(\theta_2^{'}) \quad .$$

$$= \frac{(n+m-1)!}{(m-1)!}\int\limits_{S^0_n}|\theta_1|(1-|\theta_1|)^{m-1}\log\tau(E,\frac{\theta_1}{|\theta_1|})d\lambda_n(\theta_1)$$

We now change coordinate system of the last integral

$$\Phi : (0,1)\times S^0_{n-1}\to S^0_n$$

$$\Phi(t,\eta) = (t.\eta,(1-|\eta|)t),\ \eta=(\eta_1,\dots,\eta_{n-1})$$

The Jacobian of changing coordinate is $J(\Phi) = (-1)^{n-1}t^{n-1}$. Hence we have

$$I_1 = \frac{(n+m-1)!}{(m-1)!}\int\limits_{S^0_{n-1}}(\int\limits_0^1 t^n(1-t)^{m-1}dt)\log\tau(E,(\eta,1-|\eta|))d\lambda_{n-1}(\eta)$$

$$= \frac{(n+m-1)!}{(m-1)!}\frac{n!(m-1)!}{(n+m)!}\int\limits_{S^0_{n-1}}\log\tau(E,(\eta,1-|\eta|))d\lambda_{n-1}(\eta)$$

$$= \frac{n}{n+m}\frac{1}{\frac{\sqrt{n}}{(n-1)!}}\sqrt{n}\int\limits_{S^0_{n-1}}\log\tau(E,(\eta,1-|\eta|))d\lambda_{n-1}(\eta) \quad .$$

$$= \frac{n}{n+m}\frac{1}{\sigma_n(\Sigma^0_n)}\int\limits_{\Sigma^0_n}\log\tau(E,\eta)d\sigma_n(\eta)$$

$$= \frac{n}{n+m}\log d(E).$$

□

A natural problem is to generalize the above theorem in the case of weighted transfinite diameter. In this case, we only obtain inequalies. However we give some conditions in order to have equalities.

Proposition 3.2. *Let $E_1\subset\mathbb{C}^{n_1}$, $E_2\subset\mathbb{C}^{n_2}$ be compact non-pluripolar sets and ω_1,ω_2 admissible weight functions on E_1, E_2 respectively. Then*

i) $V_{E_1\times E_2,\min(Q_1,Q_2)}\leqslant\max(V_{E_1,Q_1},V_{E_2,Q_2})\leqslant V_{E_1\times E_2,\max(Q_1,Q_2)}$ *in* $\mathbb{C}^{n_1+n_2}$

ii) $V^*_{E_1\times E_2,\min(Q_1,Q_2)}\leqslant\max(V^*_{E_1,Q_1},V^*_{E_2,Q_2})\leqslant V^*_{E_1\times E_2,\max(Q_1,Q_2)}$ *in* $\mathbb{C}^{n_1+n_2}$

where $Q_1=-\log\omega_1$, $Q_2=-\log\omega_2$.

We need the following

Lemma 3.1. *Let E^j be compact non-pluripolar sets in $\mathbb{C}^n$ such that $E^j \searrow E$ and ω an admissible weight function on E^1. Then $V_{E^j,Q} \nearrow V_{E,Q}$.*

Proof. We set

$$Q_j = \begin{cases} Q \text{ on } E^j \\ +\infty \text{ on } E^1 \backslash E^j \end{cases}$$

Using Lemma 7.3 in [5] we get

$$\lim_{j\to\infty} V_{E^j,Q} = \lim_{j\to\infty} V_{E^1,Q_j} = V_{E,Q}$$

□

Proof of Proposition 3.2. i) It is easy to check that $V_{E_1,Q_1} \leqslant V_{E_1\times E_2,\max(Q_1,Q_2)}$ and $V_{E_2,Q_2} \leqslant V_{E_1\times E_2,\max(Q_1,Q_2)}$ in $\mathbb{C}^{n_1+n_2}$. Hence

$$\max(V_{E_1,Q_1}, V_{E_2,Q_2}) \leqslant V_{E_1\times E_2,\max(Q_1,Q_2)}.$$

By Lemma 7.3 in [5] we can assume that ω_1, ω_2 are positive continuous functions on E_1, E_2. From Tietze's theorem we can extend ω_1, ω_2 to positive continuous functions in $\mathbb{C}^{n_1}, \mathbb{C}^{n_2}$. Let E_1^j, E_2^j be locally regular compact sets such that $E_1^j \searrow E_1$ and $E_2^j \searrow E_1$. Using Lemma 3.1 we can assume that E_1, E_2 are locally regular compact sets. From the definition of weighted pluricomplex Green function we have

$$V_{E_1\times E_2,\min(Q_1,Q_2)} \leqslant \max(V_{E_1,Q_1}, V_{E_2,Q_2}) \text{ in } (E_1 \times \mathbb{C}^{n_2}) \cup (\mathbb{C}^{n_1} \times E_2).$$

For all $a > 1$ there exists $R_0 > 0$ such that $a \max(V_{E_1,Q_1}, V_{E_2,Q_2}) > V_{E_1\times E_2,\min(Q_1,Q_2)}$ in $\mathbb{C}^{n_1+n_2} \backslash B(0,R)$, for all $R \geqslant R_0$. Moreover using Theorem 2.1.11 in [1] with remark on support of Monge-Ampère measures we get

$$\int_{\{a\max(V_{E_1,Q_1},V_{E_2,Q_2})<V_{E_1\times E_2,\min(Q_1,Q_2)}\}} (dd^c \max(V_{E_1,Q_1}, V_{E_2,Q_2}))^{n_1+n_2}$$

$$\leqslant \int_{\{(\mathbb{C}^{n_1}\backslash E_1)\times(\mathbb{C}^{n_2}\backslash E_2)\}} (dd^c \max(V_{E_1,Q_1}, V_{E_2,Q_2}))^{n_1+n_2} = 0.$$

By comparison principle (see [8]), we get $a\max(V_{E_1,Q_1}, V_{E_2,Q_2}) \geqslant V_{E_1\times E_2,\min(Q_1,Q_2)}$ on $\bar{B}(0,R)$. So this inequality holds in $\mathbb{C}^{n_1+n_2}$. Let $a \to 1$ we have

$$V_{E_1\times E_2,\min(Q_1,Q_2)} \leqslant \max(V_{E_1,Q_1}, V_{E_2,Q_2}).$$

ii) Use i). □

Theorem 3.2. *Let $E_1 \subset \mathbb{C}^{n_1}$, $E_2 \subset \mathbb{C}^{n_2}$ be compact non-pluripolar sets and ω_1, ω_2 admissible weight functions on E_1, E_2 respectively. Then*

$$c^{\min(\omega_1,\omega_2)}(E_1 \times E_2) \leqslant \min(c^{\omega_1}(E_1), c^{\omega_2}(E_2)) \leqslant c^{\max(\omega_1,\omega_2)}(E_1 \times E_2).$$

Proof. From Proposition 3.2 we have

$$\gamma_{V^*_{E_1\times E_2,\min(Q_1,Q_2)}} \leqslant \max(\gamma_{V^*_{E_1,Q_1}} + \log\frac{|z_1|}{|z|}, \gamma_{V^*_{E_2,Q_2}} + \log\frac{|z_2|}{|z|}) \leqslant \gamma_{V^*_{E_1\times E_2,\max(Q_1,Q_2)}}.$$

Hence

$$\begin{aligned}\sup_{|(z_1,z_2)|=1} \gamma_{V^*_{E_1\times E_2,\min(Q_1,Q_2)}} &\leqslant \max(\sup_{|z_1|=1} \gamma_{V^*_{E_1,Q_1}}, \sup_{|z_2|=1} \gamma_{V^*_{E_2,Q_2}})\\ &\leqslant \sup_{|(z_1,z_2)|=1} \gamma_{V^*_{E_1\times E_2,\max(Q_1,Q_2)}}\end{aligned}$$

Therefore

$$c^{\min(\omega_1,\omega_2)}(E_1\times E_2) \leqslant \min(c^{\omega_1}(E_1), c^{\omega_2}(E_2)) \leqslant c^{\max(\omega_1,\omega_2)}(E_1\times E_2).$$

□

Theorem 3.3. *Let $E_1\subset\mathbb{C}^{n_1}$, $E_2\subset\mathbb{C}^{n_2}$ be compact sets and ω_1,ω_2 nonnegative upper semicontinuous functions on E_1, E_2 respectively. Then*

i) $\tau^{\min(\omega_1,\omega_2)}(E_1\times E_2,(\theta_1,\theta_2)) \leqslant [\tau^{\omega_1}(E_1,\frac{\theta_1}{|\theta_1|})]^{|\theta_1|}[\tau^{\omega_2}(E_2,\frac{\theta_2}{|\theta_2|})]^{|\theta_2|} \leqslant \tau^{\max(\omega_1,\omega_2)}(E_1\times E_2,(\theta_1,\theta_2))$, *for all* $(\theta_1,\theta_2)\in\Sigma^0_{n_1+n_2}$.

ii) $d^{\min(\omega_1,\omega_2)}(E_1\times E_2) \leqslant [d^{\omega_1}(E_1)]^{\frac{n_1}{n_1+n_2}}[d^{\omega_2}(E_2)]^{\frac{n_2}{n_1+n_2}} \leqslant d^{\max(\omega_1,\omega_2)}(E_1\times E_2)$.

Proof. i) We first assume that E_1, E_2 are not pluripolar and ω_1,ω_2 are admissible weight functions. Set $M_{\max} = \sup\limits_{E_1\times E_2} V_{E_1\times E_2,\max(Q_1,Q_2)}$ and let $M\geqslant M_{\max}$. Using Lemma 2.1. we have

$$\tau^{\min(\omega_1,\omega_2)}(E_1\times E_2,(\theta_1,\theta_2)) = e^{-M}\tau(Z_{\min}(M),(\theta_1,\theta_2))\ \forall\ (\theta_1,\theta_2)\in\Sigma^0_{n_1+n_2},$$
$$\tau^{\max(\omega_1,\omega_2)}(E_1\times E_2,(\theta_1,\theta_2)) = e^{-M}\tau(Z_{\max}(M),(\theta_1,\theta_2))\ \forall\ (\theta_1,\theta_2)\in\Sigma^0_{n_1+n_2},$$
$$\tau^{\omega_j}(E_j,\theta_j) = e^{-M}\tau(Z_j(M),\theta_j)\ \forall\ \theta_j\in\Sigma^0_{n_j},$$

where

$$Z_{\min}(M) = \{V_{E_1\times E_2,\max(Q_1,Q_2)}\leqslant M\},$$
$$Z_{\max}(M) = \{V_{E_1\times E_2,\min(Q_1,Q_2)}\leqslant M\},$$
$$Z_j(M) = \{V_{E_j,Q_j}\leqslant M\},$$

for all $j=1,2$. From Proposition 3.2 we get

$$Z_{\min}(M)\subset Z_1(M)\times Z_2(M)\subset Z_{\max}(M).$$

Moreover from Proposition 3.1 we obtain the inequalities. In the general case, we set

$$E_j^k = \{z_1\in\mathbb{C}^{n_j} : \mathrm{dist}(z_j,E_j)\leqslant\frac{1}{k}\},\ \ \omega_j^k = \begin{cases}\max(\omega_j,\frac{1}{k}) \text{ on } E_j\\ \frac{1}{k} \text{ on } E_j^k\backslash E_j\end{cases}$$

Using the above case and then letting $k\to\infty$ we obtain i).

ii) Arguing as in the proof of i) by using Lemma 2.1, Theorem 3.1 and Proposition 3.2 one gets ii). □

Theorem 3.4. *Let $E_1 \subset \mathbb{C}^{n_1}$, $E_2 \subset \mathbb{C}^{n_2}$ be compact non-pluripolar sets and ω_1, ω_2 admissible weight functions on E_1, E_2 respectively. Then the following are equivalent*

i) $d^{\min(\omega_1,\omega_1)}(E_1 \times E_2) = [d^{\omega_1}(E_1)]^{\frac{n_1}{n_1+n_2}}[d^{\omega_2}(E_2)]^{\frac{n_2}{n_1+n_2}}$.

ii) $\tau^{\min(\omega_1,\omega_1)}(E_1 \times E_2, (\theta_1, \theta_2)) = [\tau^{\omega_1}(E_1, \frac{\theta_1}{|\theta_1|})]^{|\theta_1|}[\tau^{\omega_2}(E_2, \frac{\theta_2}{|\theta_2|})]^{|\theta_2|}$, $\forall\ (\theta_1, \theta_2) \in \Sigma^0_{n_1+n_2}$.

iii) $V^*_{E_1\times E_2,\max(Q_1,Q_2)} = \max(V^*_{E_1,Q_1}, V^*_{E_2,Q_2})$ *on* $\mathbb{C}^{n_1+n_2}\backslash B(0, R)$ *for some* $R > 0$.

iv) $d(\{V_{E_1\times E_2,\max(Q_1,Q_2)} \leqslant M\}) = d(\{\max(V_{E_1,Q_1}, V_{E_2,Q_2}) \leqslant M\})$, $\forall\ M \geqslant M_{\max}$.

v) $\tau(\{V_{E_1\times E_2,\max(Q_1,Q_2)} \leqslant M\}, \theta) = \tau(\{\max(V_{E_1,Q_1}, V_{E_2,Q_2}) \leqslant M\}, \theta)$, $\forall\ M \geqslant M_{\max}$, $\theta \in \Sigma_0^{n_1+n_2}$,

where $M_{\max} = \sup\limits_{E_1\times E_2} V_{E_1\times E_2,\max(Q_1,Q_2)}$.

Proof. Arguing as in the proof of Theorem 3.3 we have i)$\Leftrightarrow$ iv), ii)$\Leftrightarrow$ v) and iii)$\Rightarrow$ i). Since

$\tau(\{V_{E_1\times E_2,\max(Q_1,Q_2)} \leqslant M\}, \theta) \leqslant \tau(\{\max(V_{E_1,Q_1}, V_{E_2,Q_2}) \leqslant M\}, \theta),\ \forall\ \theta \in \Sigma^0_{n_1+n_2}$

and by the fact that dircetional Chebyshev constants are continuous in θ, we get iv)$\Leftrightarrow$ v).

It remains to prove iv) $\Rightarrow$ iii). From Theorem 0.3 in [5] and Lemma 2.2 the set

$$\{\max(V_{E_1,Q_1}, V_{E_2,Q_2}) \leqslant M\}\backslash\{V_{E_1\times E_2,\max(Q_1,Q_2)} \leqslant M\}$$

is pluripolar for all $M \geqslant M_{\max}$. Hence from Proposition 3.3 we have

$$V^*_{E_1\times E_2,\max(Q_1,Q_2)} = \max(V^*_{E_1,Q_1}, V^*_{E_2,Q_2}) \text{ in } \mathbb{C}^{n_1+n_2}\backslash B(0, R)$$

for some $R > 0$. □

Remark 3.1. 1) In Theorem 3.4 we can replace $\min(\omega_1, \omega_2)$ by $\max(\omega_1, \omega_2)$ in statements i), ii) and get a similar theorem.

2) Let $E \subset \mathbb{C}^n$ be a compact set and $w > 0$ upper semicontinuous on E. Then $d^\omega(E) = 0$ if and only if E is pluripolar.

Indeed, We assume that $d^\omega(E) = 0$ but E is not pluripolar. Then $V^*_{E,Q} \in \mathcal{L}^+(\mathbb{C}^n)$. From Proposition 3.1 we have

$$d^\omega(E) = e^{-M}d(Z(M)^*) > 0.$$

For the reverse conclusion, let $M' = \sup_E \omega(z)$, it is easy to see that

$$\tau^\omega_i(E) \leqslant M'\tau_i(E), \forall i \geqslant 1.$$

Hence $\tau^\omega(E, \theta) \leqslant M'\tau(E, \theta) = 0, \forall\ \theta \in \Sigma^0_n$ and $d^\omega(E) \leqslant M'd(E) = 0$ (see [9]).

The facts above show that if either E_1 or E_2 is pluripolar two statements i) and ii) in Theorem 3.4 hold.

3) We recall a new result in [7]: Let $u \in \mathcal{L}(\mathbb{C}^n)$ be such that $u \leqslant V^*_{E,Q}$ and $u(z) = V^*_{E,Q}(z)$ when z is large enough . Then $u = V^*_{E,Q}$ on $\mathbb{C}^n \backslash \hat{E}$, where $\hat{E}$ is the polynomially convex hull of E. It means that all statements in Theorem 3.4 are equivalent to

$$V^*_{E_1 \times E_2, \max(Q_1,Q_2)} = \max(V^*_{E_1,Q_1}, V^*_{E_2,Q_2}) \text{ in } \mathbb{C}^{n_1+n_2} \backslash (\widehat{E_1 \times E_2}).$$

Example 3.1. *Let $\bar{B}(0, R_1)$ and $\bar{B}(0, R_2)$ be closed discs in $\mathbb{C}$. Take continuous functions Q_1, Q_2 on $\bar{B}(0, R_1)$, $\bar{B}(0, R_2)$ respectively such that $a \leqslant Q_1 < b, Q_1 = a$ on $\{|z| = R_1\}$ and $c \leqslant Q_2 < d, Q_2 = c$ on $\{|z| = R_2\}$. Then by maximal principle and definition of the weighted Green funcion we have*

$$V_{E_1,Q_1} = a + \log_+ \frac{|z|}{R_1},\ V_{E_2,Q_2} = c + \log_+ \frac{|z|}{R_2},$$

where $\log_+ x = \max(\log x, 0), x \geqslant 0$.

We first choose $a = c = 1$ then $\max(Q_1, Q_2) \geqslant 1$ on $\bar{B}(0, R_1) \times \bar{B}(0, R_2)$ and $\max(Q_1, Q_2) = 1$ on $\{|z_1| = R_1\} \times \{|z_2| = R_2\}$, the Silov boundary of $\bar{B}(0, R_1) \times \bar{B}(0, R_2)$. We have

$$V_{\bar{B}(0,R_1)\times\bar{B}(0,R_2),\max(Q_1,Q_2)} = 1 + V_{\bar{B}(0,R_1)\times\bar{B}(0,R_2)} = 1 + \max(\log_+ \frac{|z_1|}{R_1}, \log_+ \frac{|z_2|}{R_2}).$$

Therefore the equality in iii) of Theorem 3.4 holds, so do the other statements.

If we choose $a = 1, b = c = 2$ using the above argument we get

$$V_{\bar{B}(0,R_1)\times\bar{B}(0,R_2),\max(Q_1,Q_2)} = 2 + \max(\log_+ \frac{|z_1|}{R_1}, \log_+ \frac{|z_2|}{R_2}).$$

Hence the equality in iii) of Theorem 3.4 does not hold, the other statements don't too.

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