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ON THE POLYCONVOLUTION FOR THE FOURIER COSINE AND FOURIER SINE TRANSFORMS

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ABSTRACT. In this paper we introduce the polyconvolution for the Fourier cosine and Fourier sine integral transforms. It is applied for solving integral equations and systems of integral equations.

1. INTRODUCTION

In 1941, R. V. Churchill introduced the convolution for the Fourier sine and Fourier cosine transforms (see [3])

$$\left(f \underset{1}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y) \left[g\left(|x-y|\right) - g(x+y)\right] dy, \; x > 0, \tag{1.1}$$

which satisfies the factorization equality

$$F_S \left(f \underset{1}{*} g\right)(y) = \left(F_S f\right)(y) \left(F_C g\right)(y), \; \forall y > 0, \tag{1.2}$$

where the integral Fourier sine transform takes the form (see [19])

$$(F_S f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(yx) dx,$$

and the integral Fourier cosine transform is (see [19])

$$(F_C f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(yx) dx.$$

Note that, (1.2) contains two integral transforms: Fourier sine and Fourier cosine. This is quite different from previous convolutions such as Fourier convolution, Laplace convolution, Mellin convolution, Fourier cosine convolution (see [19]), Fourier sine convolution, Hilbert convolution and Hankel convolution (see [4]). In the factorization equalities of these convolutions only one integral transform is

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involved. For example, the convolution of the functions f and g for the Fourier integral transforms is (see [19])

$$(1.3)\qquad \left(f \underset{F}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x-y)g(y)dy,\ x \in \mathbb{R},$$

which satisfies the following property

$$(1.4)\qquad F\left(f \underset{F}{*} g\right)(y) = (Ff)(y)\,(Fg)(y),\ \forall y \in \mathbb{R}.$$

The convolution of two functions f and g for the Laplace integral transform has the form (see [19])

$$(1.5)\qquad \left(f \underset{L}{*} g\right)(x) = \int\limits_{0}^{x} f(x-y)g(y)dy,\ x > 0$$

and the factorization property holds

$$(1.6)\qquad L\left(f \underset{L}{*} g\right)(y) = (Lf)(y)\,(Lg)(y),\ \forall y \in \mathbb{C}.$$

The convolution of two functions f and g for the Fourier cosine integral transforms has the form (see [19])

$$(1.7)\qquad \left(f \underset{F_C}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{\infty} f(y)\,\Big[g\left(|x-y|\right) + g\left(x+y\right)\Big]\,dy,\ x > 0,$$

which satisfies

$$(1.8)\qquad F_C\left(f \underset{F_C}{*} g\right)(y) = (F_C f)(y)\,(F_C g)(y),\ \forall y > 0.$$

The convolution with the weight function $\gamma_1(x) = \sin x$ for the Fourier sine integral transforms is defined as follows (see [4, 10])

$$(1.9)\qquad \Big(f \overset{\gamma_1}{\underset{F_S}{*}} g\Big)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_{0}^{\infty} f(y)\big[\operatorname{sign}(x+y-1)g(|x+y-1|) + \operatorname{sign}(x-y-1)g(|x-y-1|) - \operatorname{sign}(x-y+1)g(|x-y+1|) - g(x+y+1)\big]dy,\ x > 0$$

with

$$(1.10)\qquad F_S\left(f \overset{\gamma_1}{\underset{F_S}{*}} g\right)(y) = \sin y\,(F_S f)(y)\,(F_S g)(y),\ \forall y > 0.$$

Subsequently, S. B. Yakubovich and co-authors published a series of papers devoted to the generalized convolutions of several index integral transforms, such as integral transforms of Mellin type (see [22]), integral transforms of Kontorovich-Lebedev type (see [23]) and the G transforms (see [18]). We mention here the generalized convolution for the transform of Kontorovich-Lebedev type (see [23])

$$(f * g)(x) = \int_0^\infty \int_0^\infty \theta(x,y,z) f(y) g(z) dy dz, \ x > 0,$$

here

$$\theta(x,y,z) = \frac{1}{2} \int_0^\infty \int_0^\infty \int_0^\infty \exp\left[-\frac{1}{2}\left(\frac{uw}{v} + \frac{vw}{u} + \frac{uv}{w}\right)\right] \frac{k_3(xu)k_1(yv)k_2(zw)}{\sqrt{uvw}} dudvdw$$

and the following equality holds

$$I_{i\tau}^{k}(f * g) = \left(I_{i\tau}^{k_1} f\right)\left(I_{i\tau}^{k_2} g\right), \ \tau \in \mathbb{R},$$

where

$$I_{i\tau}^{k_j} f = \int_0^\infty I_{k_j}(\tau, u) f(u) du,$$

$$I_{k_j}(\tau, u) = \int_0^\infty K_{i\tau}(x) k_j(u,x) \frac{dx}{\sqrt{x}}, \ j = 1,2,3,$$

here $k_3(x)$ and $k(x)$ are a pair of conjugate kernels (see [21]).
In 1998, V. A. Kakichev and Nguyen Xuan Thao proposed a constructive method of defining the generalized convolution for any integral transforms K_1, K_2 and K_3 with the weight function $\gamma(x)$ of functions f and g, for which we have the factorization property (see [6])

$$K_1\left(f \overset{\gamma}{*} g\right)(y) = \gamma(y)(K_2 f)(y)(K_3 g)(y).$$

After that, there have been some papers published on the generalized convolution for the Stieltjes, Hilbert and Fourier cosine-sine transforms (see [8]), the H-transforms (see [7]), the I-transforms (see [16]), the Fourier, Fourier cosine and sine transforms (see [14]), the Fourier cosine and sine transforms (see [11]) and the Kontorovich-Lebedev, Fourier sine and cosine transforms (see [17])... For example, the generalized convolution for the Fourier cosine and Fourier sine has been defined (see [11]) by the formula

$$(1.11) \quad \left(f \underset{2}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y) \left[\operatorname{sign}(y-x)\, g(|y-x|) + g(y+x)\right] dy, x > 0,$$

which satisfies the factorization equality

$$(1.12) \quad F_C\left(f \underset{2}{*} g\right)(y) = (F_S f)(y)(F_S g)(y), \ \forall y > 0.$$

The convolution with the weight function $\gamma_2(x) = \cos x$ for the Fourier cosine integral transform is defined as (see [13])

$$\left(f \overset{\gamma_2}{\underset{F_C}{*}} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(y)\big[g(x+y+1) + g(|x+y-1|) + g(|x-y+1|) + g(|x-y-1|)\big]dy,\ x>0, \tag{1.13}$$

which satisfies

$$F_C\left(f \overset{\gamma_2}{\underset{F_C}{*}} g\right)(y) = \cos y (F_C f)(y)(F_C g)(y),\ \forall y>0. \tag{1.14}$$

The generalized convolution with the weigth function $\gamma_1(x) = \sin x$ for the Fourier cosine and sine transforms has the form (see [12])

$$\left(f \overset{\gamma_1}{\underset{1}{*}} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(y)\big[g(|y-x-1|) - g(|y-x+1|) + g(|y+x-1|) - g(|y+x+1|)\big]dy,\ x>0 \tag{1.15}$$

and the factorization property holds

$$F_C\left(f \overset{\gamma_1}{\underset{1}{*}} g\right)(y) = \sin y (F_S f)(y)(F_C g)(y),\ \forall y>0. \tag{1.16}$$

The generalized convolution with the weight function $\gamma_1(x) = \sin x$ for the Fourier sine and Fourier cosine transforms of the functions f and g is defined by (see [15])

$$\left(f \overset{\gamma_1}{\underset{2}{*}} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(y)\big[g(|x+y-1|) + g(|x-y-1|) - g(x+y+1) - g(|x-y+1|)\big]dy,\ x>0 \tag{1.17}$$

which satisfies the factorization property

$$F_S\left(f \overset{\gamma_1}{\underset{2}{*}} g\right)(y) = \sin y (F_C f)(y)(F_C g)(y),\ \forall y>0. \tag{1.18}$$

In 1997, V.A. Kakichev proposed a constructive method of defining the polyconvolution for $n+1$ integral transforms K, K_1, K_2, ..., K_n with the weight function $\gamma(x)$ of functions f_1, f_2,..., f_n for which we have the factorization property (see [5])

$$K\left[*\left(f_1, f_2, ..., f_n\right)\right](y) = \gamma(y)\left(K_1 f_1\right)(y)\left(K_2 f_2\right)(y) ... \left(K_n f_n\right)(y).$$

The polyconvolution for the Hilbert, Stieltjes and Fourier cosine transforms was introduced by Nguyen Xuan Thao in 1999 (see [9]).

In this paper we define the polyconvolution of the Fourier cosine and Fourier sine integral transforms. We prove some properties of them and point out some relationships to several well-known convolutions and generalized convolutions. We also show that it doesn't have aliquote of zero. Finally, we apply this notion to solve some integral equations and systems of integral equations.

2. Polyconvolution for the Fourier cosine and Fourier sine transforms

Definition 2.1. The polyconvolution for the Fourier cosine and Fourier sine integral transforms of the functions f, g and h is defined by

$$(2.1)\quad *(f,g,h)(x) = \frac{1}{2\pi}\int_0^\infty\int_0^\infty f(u)g(v)\big[h(|x+u-v|)+h(|x-u+v|) -h(|x-u-v|)-h(x+u+v)\big]dudv,\ x>0.$$

Theorem 2.1. *Let f, g and h be functions in $L(\mathbb{R}_+)$, then the polyconvolution (2.1) for the Fourier cosine and Fourier sine transforms of the functions f, g and h makes sense and belongs to $L(\mathbb{R}_+)$ and the following factorization property holds*

$$(2.2)\quad F_C\big[*(f,g,h)\big](y) = (F_Sf)(y)(F_Sg)(y)(F_Ch)(y),\ \forall y>0.$$

Proof. We first prove that $*(f,g,h)(x)\in L(\mathbb{R}_+)$. Indeed

$$\int_0^\infty |*(f,g,h)(x)|\,dx \leqslant \frac{1}{2\pi}\int_0^\infty |f(u)|\,du\int_0^\infty |g(v)|dv\int_0^\infty \Big[|h(u+v+x)| + +|h(|u-v-x|)|+|h(|u+v-x|)|+|h(|u-v+x|)|\big]\,dx.$$

It is easy to see that

$$\int_0^\infty \Big[|h(u+v+x)|+|h(|u-v-x|)|+|h(|u+v-x|)|+|h(|u-v+x|)|\Big]dx = 4\int_0^\infty |h(t)|\,dt.$$

Hence

$$\int_0^\infty |*(f,g,h)(x)|\,dx \leqslant \frac{2}{\pi}\int_0^\infty |f(u)|\,du\int_0^\infty |g(v)|dv\int_0^\infty |h(t)|\,dt < +\infty.$$

This means that $*(f,g,h)(x)\in L(\mathbb{R}_+)$. We now prove that the polyconvolution (2.1) satisfies the factorization equality (2.2). Indeed

$$(2.3)\quad \begin{aligned}&(F_Sf)(y)(F_Sg)(y)(F_Ch)(y)\\ &= \left(\sqrt{\frac{2}{\pi}}\right)^3\int_0^\infty du\int_0^\infty dv\int_0^\infty f(u)g(v)h(t)\sin(uy)\sin(vy)\cos(ty)dt\\ &= \frac{1}{\pi\sqrt{2\pi}}\int_0^\infty du\int_0^\infty dv\int_0^\infty f(u)g(v)h(t)\big[\cos y(t+u-v)\\ &\quad+\cos y(t-u+v)-\cos y(t+u+v)-\cos y(t-u-v)\big]dt.\end{aligned}$$

We have

$$\int_0^\infty du \int_0^\infty dv \int_0^\infty f(u)g(v)h(t)\cos y(t+u-v)dt \tag{2.4}$$
$$= \int_0^\infty du \int_0^\infty dv \int_{u-v}^\infty f(u)g(v)h(|x-u+v|)\cos(yx)dx.$$

Likewise, we obtain

$$\int_0^\infty du \int_0^\infty dv \int_0^\infty f(u)g(v)h(t)\cos y(t-u+v)dt \tag{2.5}$$
$$= \int_0^\infty du \int_0^\infty dv \int_{v-u}^\infty f(u)g(v)h(|x+u-v|)\cos(yx)dx,$$

$$\int_0^\infty du \int_0^\infty dv \int_0^\infty f(u)g(v)h(t)\cos y(t+u+v)dt \tag{2.6}$$
$$= \int_0^\infty du \int_0^\infty dv \int_{u+v}^\infty f(u)g(v)h(|x-u-v|)\cos(yx)dx$$

and

$$\int_0^\infty du \int_0^\infty dv \int_0^\infty f(u)g(v)h(t)\cos y(t-u-v)dt \tag{2.7}$$
$$= \int_0^\infty du \int_0^\infty dv \int_{-u-v}^\infty f(u)g(v)h(x+u+v)\cos(yx)dx.$$

On the other hand

$$\int_0^\infty du \int_0^\infty dv \int_{u-v}^0 f(u)g(v)h(|x-u+v|)\cos(yx)dx \tag{2.8}$$
$$= -\int_0^\infty du \int_0^\infty dv \int_{v-u}^0 f(u)g(v)h(|x+u-v|)\cos(yx)dx$$

and

$$\int_0^\infty du \int_0^\infty dv \int_{u+v}^0 f(u)g(v)h(|x-u-v|)\cos(yx)dx \tag{2.9}$$
$$= -\int_0^\infty du \int_0^\infty dv \int_{-u-v}^0 f(u)g(v)h(x+u+v)\cos(yx)dx.$$

From (2.3) $-$ (2.9) it follows that

$$(F_S f)(y)(F_S g)(y)(F_C h)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty *(f,g,h)(x)\cos(xy)dx$$
$$= F_C\big[\,*\,(f,g,h)\big](y).$$

The proof is complete. □

Remark 2.1. Formula (2.2) shows that $*(f,g,h) = *(g,f,h)$.

Theorem 2.2 (Titchmarsch-type Theorem)**.** *Let f, g and h be functions in $L(e^x, \mathbb{R}_+)$. If $*(f,g,h) \equiv 0$ then either $f \equiv 0$ or $g \equiv 0$ or $h \equiv 0$.*

Proof. From the hypothesis and Theorem 2.1 we see that

$$(F_S f)(y)(F_S g)(y)(F_C h)(y) = 0, \ \forall y > 0. \tag{2.10}$$

As $(F_S f)(y)$, $(F_S g)(y)$ and $(F_C h)(y)$ are analytic, from (2.10) it can be concluded that $F_S f \equiv 0$ or $F_S g \equiv 0$ or $F_C h \equiv 0$ and so $f \equiv 0$ or $g \equiv 0$ or $h \equiv 0$. □

Theorem 2.3. *Let f, g and h be functions in $L(\mathbb{R}_+)$. The polyconvolution for the Fourier cosine and Fourier sine integral transforms relates to the known convolutions as follows*

a) $*(f,g,h) = -\Big[\Big(\mathrm{sign}v \ g(|v|)\Big) \underset{F}{*} h(|v|)\Big](u) \underset{F}{*} \big[\mathrm{sign}u \ f(|u|)\big]$

b) $*(f,g,h) = \big(f \underset{F_C}{*} (g \underset{1}{*} h)\big) - \sqrt{\tfrac{2}{\pi}}\big(f \underset{L}{*}(g \underset{1}{*} h)\big)$

c) $*(f,g,h) = f \underset{2}{*} \Big(g \underset{1}{*} h\Big)$.

Proof. Let us first prove the equality a). We have

$$*(f,g,h)(x) = \frac{1}{2\pi}\int_0^\infty f(u)du \int_0^\infty g(v)\big[h(|x+u-v|) - h(x+u+v) \tag{2.11}$$
$$-h(|x-u-v|) + h(|x-u+v|)\big]dv.$$

We see that

$$\begin{aligned}(2.12)\qquad &\int_0^\infty f(u)du\int_0^\infty g(v)\,\Big[h(|x+u-v|)-h(x+u+v)\Big]\,dv\\
&= \int_0^\infty f(u)du\int_{-\infty}^{\infty}\text{sign}v\ g(|v|)h(|x+u-v|)dv\\
&= \sqrt{2\pi}\int_0^\infty \Big[\Big(\text{sign}v\ g(|v|)\Big)\underset{F}{*}h(|v|)\Big]\,(x+u)f(u)du\\
&= -\sqrt{2\pi}\int_{-\infty}^{0}\ \Big[\Big(\text{sign}v\ g(|v|)\Big)\underset{F}{*}h(|v|)\Big]\,(x-t)\text{sign}t\ f(|t|)dt.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}(2.13)\qquad &\int_0^\infty f(u)du\int_0^\infty g(v)\,\Big[h(|x-u-v|)-h(|x-u+v|)\Big]\,dv\\
&= \sqrt{2\pi}\int_0^\infty \Big[\Big(\text{sign}v\ g(|v|)\Big)\underset{F}{*}h(|v|)\Big]\,(x-u)\text{sign}u\ f(|u|)du.\end{aligned}$$

Hence, thanks to (2.11) − (2.13) we get

$$\begin{aligned}*(f,g,h)(x) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\ \Big[\Big(\text{sign}v\ g(|v|)\Big)\underset{F}{*}h(|v|)\Big]\,(x-u)\text{sign}u\ f(|u|)du\\
&= -\Big\{\Big[\Big(\text{sign}v\ g(|v|)\Big)\underset{F}{*}h(|v|)\Big](u)\underset{F}{*}\big[\text{sign}u\ f(|u|)\big]\Big\}(x).\end{aligned}$$

The equality a) is proved.
On the other hand, we have

$$\begin{aligned}*(f,g,h)(x) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty f(u)(g\underset{1}{*}h)(x+u)du-\frac{1}{\sqrt{2\pi}}\int_0^x f(u)(g\underset{1}{*}h)(x-u)du\\
&\quad+\frac{1}{\sqrt{2\pi}}\int_x^\infty f(u)(g\underset{1}{*}h)(u-x)du\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u)\big[(g \underset{1}{*} h)(x+u) + (g \underset{1}{*} h)(|x-u|)\big]du \\
&\quad - \frac{2}{\sqrt{2\pi}} \int_0^x f(u)(g \underset{1}{*} h)(x-u)du \\
&= \big(f \underset{F_C}{*} (g \underset{1}{*} h)\big)(x) - \sqrt{\frac{2}{\pi}}\big(f \underset{L}{*} (g \underset{1}{*} h)\big)(x).
\end{aligned}
$$

The equality b) is proved. We next prove the equality c). From Theorem 2.1 we deduce that

$$F_C\left[*(f,g,h)\right](y) = (F_S f)(y)(F_S g)(y)(F_C h)(y).$$

Using the convolution formulas (1.2) and (1.12), we have

$$
\begin{aligned}
F_C\left[f \underset{2}{*} \left(g \underset{1}{*} h\right)\right](y) &= (F_S f)(y) F_S\left(g \underset{1}{*} h\right)(y) \\
&= (F_S f)(y)(F_S g)(y)(F_C h)(y).
\end{aligned}
$$

Therefore, $*(f,g,h) = f \underset{2}{*} \left(g \underset{1}{*} h\right)$, and the equality c) is proved.
The proof is complete. □

Theorem 2.4. *In the space $L(\mathbb{R}_+)$, the polyconvolution for the Fourier cosine and Fourier sine integral transforms are neither commutative nor associative, and the following formulas hold*

a) $*\big(f,g,*(\varphi,\psi,h)\big) = *\big(\varphi,\psi,*(f,g,h)\big)$
b) $*\Big(\big(f \underset{1}{*} g\big),\varphi,\psi\Big) = *\Big(\big(\varphi \underset{1}{*} \psi\big),f,g\Big)$
c) $*\Big(f,g,\big(\varphi \underset{2}{*} \psi\big)\Big) = *\Big(\varphi,\psi,\big(f \underset{2}{*} g\big)\Big)$
d) $*\Big(f,g,\big(\varphi \overset{\gamma_1}{\underset{1}{*}} \psi\big)\Big) = *\Big(\varphi,g,\big(f \overset{\gamma_1}{\underset{1}{*}} \psi\big)\Big)$
e) $*\Big(\big(f \overset{\gamma_1}{\underset{2}{*}} g\big),\varphi,\psi\Big) = *\Big(\big(f \overset{\gamma_1}{\underset{2}{*}} \psi\big),\varphi,g\Big).$

Proof. The proof follows easily from formulas (2.1), (1.2), (1.12), (1.16) and (1.18). For example, we have

$$
\begin{aligned}
&F_C\big[*\big(f,g,*(\varphi,\psi,h)\big)\big] = (F_S f)(F_S g) F_C\big[*(\varphi,\psi,h)\big] \\
&= (F_S f)(F_S g)(F_S \varphi)(F_S \psi)(F_C h) = F_C\big[*\big(\varphi,\psi,*(f,g,h)\big)\big].
\end{aligned}
$$

Hence, $*\big(f,g,*(\varphi,\psi,h)\big) = *\big(\varphi,\psi,*(f,g,h)\big)$. The equality a) is proved. In the same way, one can verify the other parts. □

Remark 2.2. We can change the position of the functions either f,g,φ,ψ in formulas a) and c) or f,g,φ in formula d) or f,g,ψ in formula e).

3. Applications to integral equations and systems of integral equations

Theorem 3.1. *Consider the system of integral equations*

$$(3.1)\qquad \begin{aligned} f(x)+\lambda_1\int_0^\infty\int_0^\infty \varphi(u)\psi(v)\theta_1(x,u,v)dudv &= h(x)\\ \lambda_2\int_0^\infty \xi(u)\theta_2(x,u)du+g(x) &= k(x),\ \ x>0\end{aligned}$$

where

$$\theta_1(x,u,v)=\frac{1}{2\pi}\big[g(|x+u-v|)+g(|x-u+v|)-g(|x-u-v|)-g(x+u+v)\big],$$

$$\theta_2(x,u)=\frac{1}{2\sqrt{2\pi}}\big[f(|u-x-1|)-f(|u-x+1|)+f(|u+x-1|)-f(|u+x+1|)\big],$$

φ, ψ, ξ, h and k are functions of $L(\mathbb{R}_+)$, λ_1 and λ_2 are complex constants, f and g are the unknown functions. With the condition

$$1-\lambda_1\lambda_2F_C\Big[\Big(\varphi\underset{F_S}{\overset{\gamma_1}{*}}\psi\Big)\underset{2}{*}\xi\Big](x)\neq 0,\ \forall x>0$$

there exists a solution in $\mathrm{L}(\mathbb{R}_+)$ *of* (3.1) *which is given by*

$$f=h+\big(h\underset{F_C}{*}l\big)-\lambda_1\big[*(\varphi,\psi,k)\big]-\lambda_1\big[*(\varphi,\psi,k)\underset{F_C}{*}l\big]\in L(\mathbb{R}_+),$$

$$g=k+\big(k\underset{F_C}{*}l\big)-\lambda_2\Big(\xi\underset{1}{\overset{\gamma_1}{*}}h\Big)-\lambda_2\Big[\Big(\xi\underset{1}{\overset{\gamma_1}{*}}h\Big)\underset{F_C}{*}l\Big]\in L(\mathbb{R}_+),$$

here $l\in L(\mathbb{R}_+)$ and is determined by

$$F_Cl=\frac{\lambda_1\lambda_2F_C\Big[\Big(\varphi\underset{F_S}{\overset{\gamma_1}{*}}\psi\Big)\underset{2}{*}\xi\Big]}{1-\lambda_1\lambda_2F_C\Big[\Big(\varphi\underset{F_S}{\overset{\gamma_1}{*}}\psi\Big)\underset{2}{*}\xi\Big]}.$$

Proof. Using (1.15), (1.16) and Theorem 2.1 we obtain a linear system

$$\begin{aligned} F_Cf+\lambda_1(F_S\varphi)(F_S\psi)(F_Cg)&=F_Ch\\ \lambda_2\gamma_1(F_S\xi)(F_Cf)+F_Cg&=F_Ck.\end{aligned}$$

We have

$$\Delta=\begin{vmatrix}1 & \lambda_1(F_S\varphi)(F_S\psi)\\ \lambda_2\gamma_1F_S\xi & 1\end{vmatrix}=1-\lambda_1\lambda_2F_C\Big[\Big(\varphi\underset{F_S}{\overset{\gamma_1}{*}}\psi\Big)\underset{2}{*}\xi\Big],$$

$$\frac{1}{\Delta}=1+\frac{\lambda_1\lambda_2F_C\Big[\Big(\varphi\underset{F_S}{\overset{\gamma_1}{*}}\psi\Big)\underset{2}{*}\xi\Big]}{1-\lambda_1\lambda_2F_C\Big[\Big(\varphi\underset{F_S}{\overset{\gamma_1}{*}}\psi\Big)\underset{2}{*}\xi\Big]}$$

and

$$\Delta_f = \begin{vmatrix} F_C h & \lambda_1 (F_S\varphi)(F_S\psi) \\ F_C k & 1 \end{vmatrix} = F_C h - \lambda_1 F_C\big[\ast(\varphi,\psi,k)\big],$$

$$\Delta_g = \begin{vmatrix} 1 & F_C h \\ \lambda_2\gamma_1 F_S\xi & F_C k \end{vmatrix} = F_C k - \lambda_2 F_C\Big(\xi \overset{\gamma_1}{\underset{1}{\ast}} h\Big).$$

By the Wiener-Levi theorem (see [1]) there exists a function $l \in L(\mathbb{R}_+)$ such that

$$F_C l = \frac{\lambda_1\lambda_2 F_C\Big[\Big(\varphi \overset{\gamma_1}{\underset{F_S}{\ast}} \psi\Big) \underset{2}{\ast} \xi\Big]}{1 - \lambda_1\lambda_2 F_C\Big[\Big(\varphi \overset{\gamma_1}{\underset{F_S}{\ast}} \psi\Big) \underset{2}{\ast} \xi\Big]}.$$

Therefore

$$\begin{aligned} F_C f &= (1 + F_C l)\Big\{F_C h - \lambda_1 F_C\big[\ast(\varphi,\psi,k)\big]\Big\} \\ &= F_C h + F_C\big(h \underset{F_C}{\ast} l\big) - \lambda_1 F_C\big[\ast(\varphi,\psi,k)\big] - \lambda_1 F_C\big[\ast(\varphi,\psi,k) \underset{F_C}{\ast} l\big]. \end{aligned}$$

This means that

$$f = h + \big(h \underset{F_C}{\ast} l\big) - \lambda_1\big[\ast(\varphi,\psi,k)\big] - \lambda_1\big[\ast(\varphi,\psi,k) \underset{F_C}{\ast} l\big] \in L(\mathbb{R}_+).$$

We conclude similarly that

$$\begin{aligned} F_C g &= (1 + F_C l)\Big[F_C k - \lambda_2 F_C\Big(\xi \overset{\gamma_1}{\underset{1}{\ast}} h\Big)\Big] \\ &= F_C k + F_C\big(k \underset{F_C}{\ast} l\big) - \lambda_2 F_C\Big(\xi \overset{\gamma_1}{\underset{1}{\ast}} h\Big) - \lambda_2 F_C\Big[\Big(\xi \overset{\gamma_1}{\underset{1}{\ast}} h\Big) \underset{F_C}{\ast} l\Big]. \end{aligned}$$

Consequently,

$$g = k + \big(k \underset{F_C}{\ast} l\big) - \lambda_2\Big(\xi \overset{\gamma_1}{\underset{1}{\ast}} h\Big) - \lambda_2\Big[\Big(\xi \overset{\gamma_1}{\underset{1}{\ast}} h\Big) \underset{F_C}{\ast} l\Big] \in L(\mathbb{R}_+).$$

□

Theorem 3.2. *Consider the system of integral equations*

$$\text{(3.2)} \qquad \begin{aligned} f(x) + \lambda_1 \int_0^\infty \int_0^\infty \varphi(u)\psi(v)\theta_1(x,u,v)\,du\,dv &= h(x) \\ \lambda_2 \int_0^\infty \theta_3(x,u) f(u)\,du + g(x) &= k(x), \ \ x > 0 \end{aligned}$$

where

$$\theta_3(x,u) = \frac{1}{2\sqrt{2\pi}}\big[\xi(x+u+1) + \xi(|x+u-1|) + \xi(x-u+1|) + \xi(|x-u-1|)\big],$$

φ, ψ, ξ, h and k are functions of $L(\mathbb{R}_+)$, λ_1 and λ_2 are complex constants, f and g are the unknown functions. With the condition

$$1 - \lambda_1\lambda_2 F_C\Big[\Big(\varphi \underset{2}{\ast} \psi\Big) \overset{\gamma_2}{\underset{F_C}{\ast}} \xi\Big](x) \neq 0, \ \forall x > 0$$

there exists a solution in $\mathrm{L}(\mathbb{R}_+)$ *of* (3.2) *which is given by*

$$f = h + l \underset{F_C}{*} h - \lambda_1 \Big[*(\varphi, \psi, k) + *(\varphi, \psi, k) \underset{F_C}{*} l \Big] \in L(\mathbb{R}_+),$$

$$g = k + l \underset{F_C}{*} k - \lambda_2 \Big[\xi \overset{\gamma_2}{\underset{F_C}{*}} h + \Big(\xi \overset{\gamma_2}{\underset{F_C}{*}} h \Big) \underset{F_C}{*} l \Big] \in L(\mathbb{R}_+),$$

here $l \in L(\mathbb{R}_+)$ *and is determined by*

$$F_C l = \frac{\lambda_1 \lambda_2 F_C \Big[\Big(\varphi \underset{2}{*} \psi \Big) \overset{\gamma_2}{\underset{F_C}{*}} \xi \Big]}{1 - \lambda_1 \lambda_2 F_C \Big[\Big(\varphi \underset{2}{*} \psi \Big) \overset{\gamma_2}{\underset{F_C}{*}} \xi \Big]}.$$

Proof. By (1.13), (1.14) and Theorem 2.1 we obtain a linear system

$$F_C f + \lambda_1 (F_S \varphi)(F_S \psi)(F_C g) = F_C h$$
$$\lambda_2 \gamma_2 (F_C \xi)(F_C f) + F_C g = F_C k.$$

We see that

$$\Delta = \begin{vmatrix} 1 & \lambda_1 (F_S \varphi)(F_S \psi) \\ \lambda_2 \gamma_2 F_C \xi & 1 \end{vmatrix} = 1 - \lambda_1 \lambda_2 F_C \Big[\Big(\varphi \underset{2}{*} \psi \Big) \overset{\gamma_2}{\underset{F_C}{*}} \xi \Big],$$

$$\frac{1}{\Delta} = 1 + \frac{\lambda_1 \lambda_2 F_C \Big[\Big(\varphi \underset{2}{*} \psi \Big) \overset{\gamma_2}{\underset{F_C}{*}} \xi \Big]}{1 - \lambda_1 \lambda_2 F_C \Big[\Big(\varphi \underset{2}{*} \psi \Big) \overset{\gamma_2}{\underset{F_C}{*}} \xi \Big]}$$

and

$$\Delta_f = \begin{vmatrix} F_C h & \lambda_1 (F_S \varphi)(F_S \psi) \\ F_C k & 1 \end{vmatrix} = F_C h - \lambda_1 F_C [*(\varphi, \psi, k)],$$

$$\Delta_g = \begin{vmatrix} 1 & F_C h \\ \lambda_2 \gamma_2 F_C \xi & F_C k \end{vmatrix} = F_C k - \lambda_2 F_C \Big(\xi \overset{\gamma_2}{\underset{F_C}{*}} h \Big).$$

According to Wiener-Levi's theorem (see [1]), there exists a function $l \in L(\mathbb{R}_+)$ such that

$$F_C l = \frac{\lambda_1 \lambda_2 F_C \Big[\Big(\varphi \underset{2}{*} \psi \Big) \overset{\gamma_2}{\underset{F_C}{*}} \xi \Big]}{1 - \lambda_1 \lambda_2 F_C \Big[\Big(\varphi \underset{2}{*} \psi \Big) \overset{\gamma_2}{\underset{F_C}{*}} \xi \Big]}.$$

Hence

$$F_C f = (1 + F_C l) \{ F_C h - \lambda_1 F_C [*(\varphi, \psi, k)] \}$$
$$= F_C h + F_C \Big(l \underset{F_C}{*} h \Big) - \lambda_1 F_C [*(\varphi, \psi, k)] - \lambda_1 F_C \Big[*(\varphi, \psi, k) \underset{F_C}{*} l \Big],$$

we obtain

$$f = h + l \underset{F_C}{*} h - \lambda_1 \Big[*(\varphi, \psi, k) + *(\varphi, \psi, k) \underset{F_C}{*} l \Big] \in L(\mathbb{R}_+).$$

Likewise, we have

$$\begin{aligned} F_C g &= (1+F_C l)\Big[F_C k - \lambda_2 F_C\Big(\xi \overset{\gamma_2}{\underset{F_C}{*}} h\Big)\Big] \\ &= F_C k + F_C\left(l \underset{F_C}{*} k\right) - \lambda_2 F_C\Big(\xi \overset{\gamma_2}{\underset{F_C}{*}} h\Big) - \lambda_2 F_C\Big[\Big(\xi \overset{\gamma_2}{\underset{F_C}{*}} h\Big) \underset{F_C}{*} l\Big]. \end{aligned}$$

Therefore,

$$g = k + l \underset{F_C}{*} k - \lambda_2\Big[\xi \overset{\gamma_2}{\underset{F_C}{*}} h + \Big(\xi \overset{\gamma_2}{\underset{F_C}{*}} h\Big) \underset{F_C}{*} l\Big] \in L(\mathbb{R}_+).$$

□

An analysis similar to that of the proof of Theorem 3.2, with the use of (1.7), (1.8) and Theorem 2.1, gives us the following results.

Theorem 3.3. *Consider the system of integral equations*

$$\begin{aligned} &f(x) + \lambda_1 \int_0^\infty \int_0^\infty \varphi(u)\psi(v)\theta_1(x,u,v)dudv = h(x) \\ &\frac{\lambda_2}{\sqrt{2\pi}} \int_0^\infty \big[\xi(|x-u|) + \xi(x+u)\big] f(u)du + g(x) = k(x),\ x > 0, \end{aligned} \tag{3.3}$$

where φ, ψ, ξ, h and k are functions of $L(\mathbb{R}_+)$, λ_1 and λ_2 are complex constants, f and g are the unknown functions. With the condition

$$1 - \lambda_1\lambda_2 F_C\big[* (\varphi,\psi,\xi)\big](x) \neq 0,\ \forall x > 0$$

there exists a solution in $\mathrm{L}(\mathbb{R}_+)$ *of* (3.3) *which is given by*

$$\begin{aligned} f &= h + l \underset{F_C}{*} h - \lambda_1\Big[* (\varphi,\psi,k) + *(\varphi,\psi,k) \underset{F_C}{*} l\Big] \in L(\mathbb{R}_+), \\ g &= k + l \underset{F_C}{*} k - \lambda_2\Big[\xi \underset{F_C}{*} h + \Big(\xi \underset{F_C}{*} h\Big) \underset{F_C}{*} l\Big] \in L(\mathbb{R}_+), \end{aligned}$$

here $l \in L(\mathbb{R}_+)$ and is determined by

$$F_C l = \frac{\lambda_1\lambda_2 F_C\big[* (\varphi,\psi,\xi)\big]}{1 - \lambda_1\lambda_2 F_C\big[* (\varphi,\psi,\xi)\big]}.$$

Theorem 3.4. *Consider the system of integral equations*

$$\begin{aligned} &f(x) + \lambda_1 \int_0^\infty \int_0^\infty \varphi_1(u)\psi_1(v)\theta_1(x,u,v)dudv = h(x) \\ &\lambda_2 \int_0^\infty \int_0^\infty \varphi_2(u)\psi_2(v)\theta_4(x,u,v)dudv + g(x) = k(x),\ x > 0, \end{aligned} \tag{3.4}$$

where

$$\theta_4(x,u,v) = \frac{1}{2\pi}\big[f(|x+u-v|) + f(|x-u+v|) - f(|x-u-v|) - f(x+u+v)\big],$$

φ_1, ψ_1, φ_2, ψ_2, h and k are functions of $L(\mathbb{R}_+)$, λ_1 and λ_2 are complex constants, f and g are the unknown functions. With the condition

$$1-\lambda_1\lambda_2 F_C\Big[*\big(\varphi_1,\psi_1,(\varphi_2 \underset{2}{*} \psi_2)\big)\Big](x)\neq 0,\ \forall x>0,$$

there exists a solution in $\mathrm{L}(\mathbb{R}_+)$ *of* (3.4) *which is given by*

$$f = h + l \underset{F_C}{*} h - \lambda_1\Big[*(\varphi_1,\psi_1,k) + *(\varphi_1,\psi_1,k) \underset{F_C}{*} l\Big] \in L(\mathbb{R}_+),$$

$$g = k + l \underset{F_C}{*} k - \lambda_2\Big[*(\varphi_2,\psi_2,h) + *(\varphi_2,\psi_2,h) \underset{F_C}{*} l\Big] \in L(\mathbb{R}_+),$$

here $l\in L(\mathbb{R}_+)$ *and is determined by*

$$F_C l = \frac{\lambda_1\lambda_2 F_C\Big[*\big(\varphi_1,\psi_1,(\varphi_2 \underset{2}{*} \psi_2)\big)\Big]}{1-\lambda_1\lambda_2 F_C\Big[*\big(\varphi_1,\psi_1,(\varphi_2 \underset{2}{*} \psi_2)\big)\Big]}.$$

Proof. From Theorem 2.1 we obtain a linear system of algebraic equations

$$F_C f + \lambda_1(F_S\varphi_1)(F_S\psi_1)(F_C g) = F_C h$$
$$\lambda_2(F_S\varphi_2)(F_S\psi_2)(F_C f) + F_C g = F_C k.$$

We show that

$$\Delta = \begin{vmatrix} 1 & \lambda_1(F_S\varphi_1)(F_S\psi_1) \\ \lambda_2(F_S\varphi_2)(F_S\psi_2) & 1\end{vmatrix} = 1-\lambda_1\lambda_2 F_C\Big[*\big(\varphi_1,\psi_1,(\varphi_2 \underset{2}{*} \psi_2)\big)\Big],$$

$$\frac{1}{\Delta} = 1 + \frac{\lambda_1\lambda_2 F_C\Big[*\big(\varphi_1,\psi_1,(\varphi_2 \underset{2}{*} \psi_2)\big)\Big]}{1-\lambda_1\lambda_2 F_C\Big[*\big(\varphi_1,\psi_1,(\varphi_2 \underset{2}{*} \psi_2)\big)\Big]}$$

and

$$\Delta_f = \begin{vmatrix} F_C h & \lambda_1(F_S\varphi_1)(F_S\psi_1) \\ F_C k & 1\end{vmatrix} = F_C h - \lambda_1 F_C[*(\varphi_1,\psi_1,k)],$$

$$\Delta_g = \begin{vmatrix} 1 & F_C h \\ \lambda_2(F_S\varphi_2)(F_S\psi_2) & F_C k\end{vmatrix} = F_C k - \lambda_2 F_C[*(\varphi_2,\psi_2,h)].$$

According to Wiener-Levi's theorem (see [1]), there exists a function $l\in L(\mathbb{R}_+)$ such that

$$F_C l = \frac{\lambda_1\lambda_2 F_C\Big[*\big(\varphi_1,\psi_1,(\varphi_2 \underset{2}{*} \psi_2)\big)\Big]}{1-\lambda_1\lambda_2 F_C\Big[*\big(\varphi_1,\psi_1,(\varphi_2 \underset{2}{*} \psi_2)\big)\Big]}.$$

Thus

$$\begin{aligned} F_C f &= (1+F_C l)\{F_C h - \lambda_1 F_C[*(\varphi_1,\psi_1,k)]\} \\ &= F_C h + F_C\Big(l \underset{F_C}{*} h\Big) - \lambda_1 F_C[*(\varphi_1,\psi_1,k)] - \lambda_1 F_C\Big[*(\varphi_1,\psi_1,k) \underset{F_C}{*} l\Big], \end{aligned}$$

we obtain

$$f = h + l \underset{F_C}{*} h - \lambda_1\Big[*(\varphi_1,\psi_1,k) + *(\varphi_1,\psi_1,k) \underset{F_C}{*} l\Big] \in L(\mathbb{R}_+).$$

Likewise, we have

$$\begin{aligned} F_C g &= (1 + F_C l)\{F_C k - \lambda_2 F_C[*(\varphi_2, \psi_2, h)]\} \\ &= F_C k + F_C\left(l \underset{F_C}{*} k\right) - \lambda_2 F_C[*(\varphi_2, \psi_2, h)] - \lambda_2 F_C\left[*(\varphi_2, \psi_2, h) \underset{F_C}{*} l\right], \end{aligned}$$

which implies that

$$g = k + l \underset{F_C}{*} k - \lambda_2\left[*(\varphi_2, \psi_2, h) + *(\varphi_2, \psi_2, h) \underset{F_C}{*} l\right] \in L(\mathbb{R}_+).$$

□

In the same manner we have the following result.

Theorem 3.5. *Consider the integral equations*

$$f(x) + \lambda \int_0^\infty \int_0^\infty \varphi(u)\psi(v)\theta_4(x, u, v)dudv = h(x), \;\; x > 0 \tag{3.5}$$

where φ, ψ *and* h *are functions of* $L(\mathbb{R}_+)$, λ *is a complex constant,* f *is the unknown function. With the condition*

$$1 + \lambda F_S\varphi(x) F_S\psi(x) \neq 0, \;\; \forall x > 0$$

there exists a solution in $\mathrm{L}(\mathbb{R}_+)$ *of* (3.5) *which is given by*

$$f = h - \left(h \underset{F_C}{*} l\right) \in L(\mathbb{R}_+),$$

here $l \in L(\mathbb{R}_+)$ *and*

$$F_C l = \frac{\lambda F_C\left(\varphi \underset{2}{*} \psi\right)}{1 + \lambda F_C\left(\varphi \underset{2}{*} \psi\right)}.$$

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