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ON THE DIFFERENCE EQUATION $x_{n+1} = \dfrac{\alpha x_{n-l} + \beta x_{n-k}}{A x_{n-l} + B x_{n-k}}$

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Abstract. In this paper we investigate the global convergence result, boundedness, and periodicity of solutions of the recursive sequence

$$x_{n+1} = \frac{\alpha x_{n-l} + \beta x_{n-k}}{A x_{n-l} + B x_{n-k}}, \quad n = 0, 1, ...$$

where the parameters α, β, A and B are positive real numbers and the initial conditions $x_{-p}, x_{-p+1}, ..., x_{-1}$ and $x_0 \in (0, \infty)$ where $p = \max\{l, k\}$.

1. Introduction

Our goal in this paper is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = \frac{\alpha x_{n-l} + \beta x_{n-k}}{A x_{n-l} + B x_{n-k}}, \tag{1.1}$$

where α, β, A and $B \in (0, \infty)$ with the initial conditions $x_{-p}, x_{-p+1}, ..., x_{-1}$ and $x_0 \in (0, \infty)$ where $p = \max\{l, k\}$.

Recently there has been a lot of interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations, see for example [1] and [6-8].

We first recall some notations and results which will be useful for our investigation.

Definition 1.1. The difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ... \tag{1.2}$$

is said to be persistence if there exist numbers m and M with $0 < m \leqslant M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in (0, \infty)$ there exists a positive integer N, which depends on the initial conditions, such that

$$m \leqslant x_n \leqslant M \qquad \text{for all} \quad n \geqslant N.$$

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Definition 1.2. (Stability)
(i) The equilibrium point $\overline{x}$ of Eq. (1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geqslant -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq. (1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq. (1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq. (1.2) is a global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq. (1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq. (1.2).
(v) The equilibrium point $\overline{x}$ of Eq. (1.2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq. (1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Theorem 1.1 ([5]). *Assume that* $p, q \in R$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

Remark 1. Theorem 1.1 can be easily extended to a general linear equation of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \tag{1.4}$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, ...\}$. Eq. (1.4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

The theory of the Full Limiting Sequences was indicated in [3] and [4]. The following theorem was given in [2].

Theorem 1.2. *Let $F \in C[I^{k+1}, I]$ for some interval I of the real numbers and for some non-negative integer k, and consider the difference equation*

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}). \tag{1.5}$$

Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (1.5), and suppose that there exist constants $A \in I$ and $B \in I$ such that

$$A \leqslant x_n \leqslant B \quad \text{for all} \quad n \geqslant -k.$$

Let ℓ_0 be a limit point of the sequence $\{x_n\}_{n=-k}^{\infty}$. Then the following statements are true.

(i) *There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of Eq. (1.5), called a full limiting sequence of $\{x_n\}_{n=-k}^{\infty}$, such that $L_0 = \ell_0$, and such that for every $N \in \{..., -1, 0, 1, ...\}$ L_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$.*

(ii) *For every $i_0 \leqslant -k$, there exists a subsequence $\{x_{r_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that*

$$L_N = \lim_{i \to \infty} x_{r_i + N} \quad \text{for every} \quad N \geqslant i_0.$$

2. Local stability of the equilibrium point

In this section we study the local stability character of the solutions of Eq. (1.1). Eq. (1.1) has a unique positive equilibrium point which is given by

$$\overline{x} = \frac{\alpha + \beta}{A + B}.$$

Let $f : (0, \infty)^2 \longrightarrow (0, \infty)$ be a continuous function defined by

$$f(u, v) = \frac{\alpha u + \beta v}{Au + Bv}. \tag{2.1}$$

Therefore

$$\frac{\partial f(u,v)}{\partial u} = \frac{(\alpha B - \beta A)v}{(Au + Bv)^2},$$

$$\frac{\partial f(u,v)}{\partial v} = \frac{-(\alpha B - \beta A)u}{(Au + Bv)^2}.$$

Then we see that

$$\begin{aligned} \frac{\partial f(\overline{x}, \overline{x})}{\partial u} &= \frac{\alpha B - \beta A}{(A + B)(\alpha + \beta)} = -a_1, \\ \frac{\partial f(\overline{x}, \overline{x})}{\partial v} &= \frac{\beta A - \alpha B}{(A + B)(\alpha + \beta)} = -a_0. \end{aligned}$$

Thus the linearized equation of Eq. (1.1) about $\overline{x}$ is

$$y_{n+1} + a_1 y_{n-l} + a_0 y_{n-k} = 0, \tag{2.2}$$

whose characteristic equation is

$$\lambda^{k+1} + a_1 \lambda^{k-l} + a_0 = 0. \tag{2.3}$$

Theorem 2.1. *Assume that*

$$(\alpha + \beta)(A + B) > 2 \left| \beta A - \alpha B \right|.$$

Then the positive equilibrium point of Eq. (1.1) is locally asymptotically stable.

Proof. It follows by Theorem 1.1 that Eq. (2.2) is asymptotically stable if all roots of Eq. (2.3) lie in the open disc $|\lambda| < 1$, that is, if

$$|a_1| + |a_0| < 1,$$

$$\left| \frac{\alpha B - \beta A}{(A+B)(\alpha+\beta)} \right| + \left| \frac{\beta A - \alpha B}{(A+B)(\alpha+\beta)} \right| < 1,$$

and so

$$2 \left| \beta A - \alpha B \right| < (A+B)(\alpha+\beta).$$

The proof is complete. □

3. Boundedness of solutions

Here we study the permanence of Eq. (1.1).

Theorem 3.1. *Every solution of Eq.* (1.1) *is bounded and persists.*

Proof. Let $\{x_n\}_{n=-p}^{\infty}$ be a solution of Eq. (1.1). It follows from Eq. (1.1) that

$$x_{n+1} = \frac{\alpha x_{n-l} + \beta x_{n-k}}{A x_{n-l} + B x_{n-k}} = \frac{\alpha x_{n-l}}{A x_{n-l} + B x_{n-k}} + \frac{\beta x_{n-k}}{A x_{n-l} + B x_{n-k}}.$$

Hence

$$x_n \leqslant \frac{\alpha}{A} + \frac{\beta}{B} = M \quad \text{for all} \quad n \geqslant 1. \tag{3.1}$$

Now we wish to show that there exists $m > 0$ such that

$$x_n \geqslant m \quad \text{for all} \quad n \geqslant 1.$$

The transformation

$$x_n = \frac{1}{y_n},$$

will lead Eq. (1.1) to the equivalent form

$$\begin{aligned} y_{n+1} &= \frac{B y_{n-l} + A y_{n-k}}{\beta y_{n-l} + \alpha y_{n-k}} \\ &= \frac{B y_{n-l}}{\alpha y_{n-k} + \beta y_{n-l}} + \frac{A y_{n-k}}{\alpha y_{n-k} + \beta y_{n-l}}. \end{aligned}$$

It follows that

$$y_{n+1} \leqslant \frac{B}{\beta} + \frac{A}{\alpha} = \frac{\beta A + \alpha B}{\alpha \beta} = H \quad \text{for all} \quad n \geqslant 1.$$

Thus we obtain

$$x_n = \frac{1}{y_n} \geqslant \frac{1}{H} = \frac{\alpha\beta}{\beta A + \alpha B} = m \quad \text{for all} \quad n \geqslant 1. \tag{3.2}$$

From (3.1) and (3.2) we see that

$$m \leqslant x_n \leqslant M \qquad \text{for all} \quad n \geqslant 1.$$

Therefore every solution of Eq. (1.1) is bounded and persists. □

4. Periodicity of solutions

In this section we study the existence of prime period two solutions of Eq. (1.1).

Theorem 4.1. *Eq.* (1.1) *has positive prime period two solutions if and only if*

(i) $4B\alpha < (A - B)(\beta - \alpha)$ *and* k *is odd,* l *is even.*

(ii) $4A\beta < (\alpha - \beta)(B - A)$ *and* k *is even,* l *is odd.*

Proof. First suppose that there exists a prime period two solution

$$..., p, q, p, q, ...$$

of Eq. (1.1). We will prove that condition (i) holds. We prove this for the case k is odd, l is even, the case k is even, l is odd is similar and will be omitted.

We see from Eq. (1.1) that

$$p = \frac{\alpha q + \beta p}{Aq + Bp},$$

and

$$q = \frac{\alpha p + \beta q}{Ap + Bq}.$$

Hence

$$Apq + Bp^2 = \alpha q + \beta p, \tag{4.1}$$

and

$$Apq + Bq^2 = \alpha p + \beta q. \tag{4.2}$$

Subtracting (4.1) from (4.2) gives

$$B(p^2 - q^2) = (\beta - \alpha)(p - q).$$

Since $p \neq q$, it follows that

$$p + q = \frac{(\beta - \alpha)}{B}. \tag{4.3}$$

Also, since p and q are positive, $(\beta - \alpha)$ should be positive.

Again, adding (4.1) and (4.2) yields

$$2Apq + B(p^2 + q^2) = (p + q)(\alpha + \beta). \tag{4.4}$$

It follows by (4.3), (4.4) that

$$2(A-B)pq = \frac{2\alpha(\beta-\alpha)}{B}.$$

Again, since p and q are positive and $\beta > \alpha$, we see that $A > B$. Thus

$$pq = \frac{\alpha(\beta-\alpha)}{B(A-B)}. \tag{4.5}$$

Now it is clear from Eq. (4.3) and Eq. (4.5) that p and q are the two positive distinct roots of the quadratic equation

$$t^2 - \frac{(\beta-\alpha)}{B}t + \frac{\alpha(\beta-\alpha)}{B(A-B)} = 0, \tag{4.6}$$

and so

$$\left[\frac{\beta-\alpha}{B}\right]^2 - \frac{4\alpha(\beta-\alpha)}{B(A-B)} > 0.$$

Since $A-B$ and $\beta-\alpha$ have the same sign,

$$\frac{\beta-\alpha}{B} > \frac{4\alpha}{(A-B)},$$

which is equivalent to

$$4B\alpha < (A-B)(\beta-\alpha).$$

Therefore condition (i) holds.

Conversely suppose that condition (i) is true. We will show that Eq. (1.1) has a prime period two solution.

Assume that

$$p = \frac{\dfrac{\beta-\alpha}{B} - \sqrt{\left[\dfrac{\beta-\alpha}{B}\right]^2 - \dfrac{4\alpha(\beta-\alpha)}{B(A-B)}}}{2},$$

and

$$q = \frac{\dfrac{\beta-\alpha}{B} + \sqrt{\left[\dfrac{\beta-\alpha}{B}\right]^2 - \dfrac{4\alpha(\beta-\alpha)}{B(A-B)}}}{2}.$$

We see from condition (i) that

$$(A-B)(\beta-\alpha) > 4B\alpha,$$

or

$$[\beta-\alpha]^2 > \frac{4B\alpha(\beta-\alpha)}{(A-B)},$$

which is equivalent to

$$\left[\frac{\beta-\alpha}{B}\right]^2 > \frac{4\alpha(\beta-\alpha)}{B(A-B)}.$$

Therefore p and q are distinct positive real numbers. Set

$$x_{-k} = q, \;\; x_{-1} = p, ..., \qquad \text{and} \quad x_0 = p.$$

We wish to show that

$$x_1 = x_{-1} = q \quad \text{and} \qquad x_2 = x_0 = p.$$

It follows from Eq. (1.1) that

$$x_1 = \frac{\alpha x_0 + \beta x_{-k}}{A x_0 + B x_{-k}} = \frac{\alpha p + \beta q}{Ap + Bq}$$

$$= \frac{\alpha \left[\frac{\beta-\alpha}{B} - \sqrt{\left[\frac{\beta-\alpha}{B}\right]^2 - \frac{4\alpha(\beta-\alpha)}{B(A-B)}} \right] + \beta \left[\frac{\beta-\alpha}{B} + \sqrt{\left[\frac{\beta-\alpha}{B}\right]^2 - \frac{4\alpha(\beta-\alpha)}{B(A-B)}} \right]}{A \left[\frac{\beta-\alpha}{B} - \sqrt{\left[\frac{\beta-\alpha}{B}\right]^2 - \frac{4\alpha(\beta-\alpha)}{B(A-B)}} \right] + B \left[\frac{\beta-\alpha}{B} + \sqrt{\left[\frac{\beta-\alpha}{B}\right]^2 - \frac{4\alpha(\beta-\alpha)}{B(A-B)}} \right]}.$$

Dividing the denominator and numerator by $\frac{\beta-\alpha}{B}$ gives

$$x_1 = \frac{(\alpha+\beta) + (\beta-\alpha) \left[\sqrt{1 - \frac{4B\alpha}{(\beta-\alpha)(A-B)}}\right]}{(A+B) + (B-A) \left[\sqrt{1 - \frac{4B\alpha}{(\beta-\alpha)(A-B)}}\right]}.$$

Multiplying the denominator and numerator by

$$(A+B) - (B-A) \left[\sqrt{1 - \frac{4B\alpha}{(\beta-\alpha)(A-B)}}\right]$$

gives

$$\begin{aligned} x_1 \;\; = \;\; & \frac{(\alpha+\beta)(A+B) - [\beta-\alpha][B-A] - 4B\alpha}{(A+B)^2 - [B-A]^2 \left[1 - \frac{4B\alpha}{(\beta-\alpha)(A-B)}\right]} \\ & + \frac{[\beta(A+B-B+A) + \alpha(-A-B-B+A)]\sqrt{1 - \frac{4B\alpha}{(\beta-\alpha)(A-B)}}}{(A+B)^2 - [B-A]^2 \left[1 - \frac{4B\alpha}{(\beta-\alpha)(A-B)}\right]} \end{aligned}$$

$$= \frac{[2\beta A - 2B\alpha] + [2\beta A - 2B\alpha]\sqrt{1 - \dfrac{4B\alpha}{(\beta-\alpha)(A-B)}}}{4BA - \dfrac{4B\alpha[B-A]}{(\beta-\alpha)}}$$

$$= \frac{\dfrac{\beta-\alpha}{B} + \sqrt{\left[\dfrac{\beta-\alpha}{B}\right]^2 - \dfrac{4\alpha(\beta-\alpha)}{B(A-B)}}}{2} = q.$$

Similarly as before one can easily show that

$$x_2 = p.$$

Then it follows by induction that

$$x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all} \quad n \geqslant -1.$$

Thus Eq. (1.1) has the positive prime period two solution

$$\ldots,\ p,\ q,\ p,\ q,\ldots$$

where p and q are the distinct roots of the quadratic equation (4.6) and the proof is complete. □

Lemma 4.1. *If k is even and l is even (or k is odd and l is odd) then Eq.* (1.1) *has no periodic solution of prime period two.*

Proof. Assume the contrary that there exist distinctive positive real numbers p and q such that

$$\ldots,\ p,\ q,\ p,\ q,\ldots$$

be a period two solution of Eq. (1.1). Then for k is even and l is even (or k is odd and l is odd) we see from Eq. (1.1) that

$$p = \frac{\alpha q + \beta q}{Aq + Bq} = \frac{\alpha+\beta}{A+B}, \quad \textit{or} \quad p = \frac{\alpha p + \beta p}{Ap + Bp} = \frac{\alpha+\beta}{A+B},$$

and

$$q = \frac{\alpha p + \beta p}{Ap + Bp} = \frac{\alpha+\beta}{A+B}, \quad \textit{or} \quad q = \frac{\alpha q + \beta q}{Aq + Bq} = \frac{\alpha+\beta}{A+B},$$

which implies

$$p = q,$$

a contradiction. □

5. Global stability of Eq. (1.1)

In this section we investigate the global asymptotic stability of Eq. (1.1).

Lemma 5.1. *For any values of the quotient $\frac{\alpha}{A}$ and $\frac{\beta}{B}$, the function $f(u,v)$ defined by Eq.* (2.1) *has the monotonic behavior in its two arguments.*

Proof. The proof follows by some computations and it will be omitted. □

Theorem 5.1. *The equilibrium point $\overline{x}$ is a global attractor of Eq.* (1.1) *if one of the following statements holds*

$$(1)\ \alpha B \geqslant \beta A \ \text{ and } \ \beta \geqslant \alpha. \tag{5.1}$$

$$(2)\ \alpha B \leqslant \beta A \ \text{ and } \ \alpha \geqslant \beta. \tag{5.2}$$

Proof. Let $\{x_n\}_{n=-p}^{\infty}$ be a solution of Eq. (1.1) and again let f be a function defined by Eq. (2.1)

We will prove the theorem for Case (1), the proof for Case (2) is similar and is left to the reader.

Assume that (5.1) is true, then it is easy to see that the function $f(u, v)$ is non-decreasing in u and non-increasing in v. Thus from Eq. (1.1), we see that

$$x_{n+1} = \frac{\alpha x_{n-l} + \beta x_{n-k}}{A x_{n-l} + B x_{n-k}} \leqslant \frac{\alpha x_{n-l} + \beta(0)}{A x_{n-l} + B(0)} = \frac{\alpha}{A}.$$

Then

$$x_n \leqslant \frac{\alpha}{A} = H \quad \text{for all} \quad n \geqslant 1. \tag{5.3}$$

$$\begin{aligned} x_{n+1} \quad &= \quad \frac{\alpha x_{n-l} + \beta x_{n-k}}{A x_{n-l} + B x_{n-k}} \geqslant \frac{\alpha(0) + \beta x_{n-k}}{A(0) + B x_{n-k}} \\ &\geqslant \quad \frac{\beta x_{n-k}}{B x_{n-k}} \geqslant \frac{\beta}{B} = h \quad \text{ for all } \quad n \geqslant 1. \end{aligned} \tag{5.4}$$

From Eqs. (5.3) and (5.4), we see that

$$h = \frac{\beta}{B} \leqslant x_n \leqslant \frac{\alpha}{A} = H \quad \text{for all} \ \ n \geqslant 1.$$

It follows by the Method of Full Limiting Sequences that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq. (1.1) with

$$I = I_0 = \lim_{n\to\infty} \inf x_n \leqslant \lim_{n\to\infty} \sup x_n = S_0 = S,$$

where

$$I_n, S_n \in [I, S]\,, \quad n = 0, -1, ...$$

It suffices to show that $I = S$.

Now it follows from Eq. (1.1) that

$$I = \frac{\alpha I_{-l-1} + \beta I_{-k-1}}{A I_{-l-1} + B I_{-k-1}} \geqslant \frac{\alpha I + \beta S}{A I + B S},$$

and so

$$\alpha I + \beta S - A I^2 \leqslant B S I. \tag{5.5}$$

Similarly, we see from Eq. (1.1) that

$$S = \frac{\alpha S_{-l-1} + \beta S_{-k-1}}{A S_{-l-1} + B S_{-k-1}} \leqslant \frac{\alpha S + \beta I}{A S + B I},,$$

and so

$$\alpha S + \beta I - AS^2 \geqslant BSI. \tag{5.6}$$

Therefore it follows from Eqs. (5.5) and (5.6) that

$$\alpha I + \beta S - AI^2 \leqslant \alpha S + \beta I - AS^2,$$

that is

$$\alpha(S - I) + \beta(I - S) + A(I + S)(I - S) \geqslant 0,$$

or equivalently

$$(I - S)[A(I + S) + \beta - \alpha] \geqslant 0.$$

Hence

$$I \geqslant S \quad \text{if} \quad A(I + S) + \beta - \alpha \geqslant 0.$$

Now, we know by (5.1) that $\beta \geqslant \alpha$, and so it follows that $I \geqslant S$. Therefore $I = S$. This completes the proof. □

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