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SOME PROPERTIES OF GROWTH FOR COMPOSITE ENTIRE FUNCTIONS WITH DEFICIENT FUNCTION

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ABSTRACT. In this paper, we solve a problem of C. T. Chuang and C. C. Yang concerning the characteristic function of the composite function.

1. INTRODUCTION

Chuang Chitai and Yang Chungchun [1] proposed the following problem:

Let f, g_1 and g_2 be entire functions. What conditions can assure that if $T(r, g_1) = \circ(T(r, g_2))$ $(r \to \infty)$, then

$$T(r, f(g_1)) = \circ(T(r, f(g_2)))\ (r \to \infty)? \tag{1.1}$$

In this work, we give some conditions such that (1.1) holds.

Theorem 1.1. *Let f_1, f_2 and g_1, g_2 be non-constant entire functions with $T(r, f_1) = O^*((\log r)^{\beta} e^{(\log r)^{\alpha}})(0 < \alpha < 1, \beta > 0)$ (i.e., there exist two positive constants K_1 and K_2 such that $K_1 \leqslant \frac{T(r,f_1)}{(\log r)^{\beta} e^{(\log r)^{\alpha}}} \leqslant K_2$) and the order of g_2 be $\rho_{g_2}(< \infty)$, let $a_i(z)(i = 1, 2, \cdots, n, n \leqslant \infty)$ be entire functions which satisfy $T(r, a_i(z)) = \circ(T(r, g_2))$ with $\sum_{i=1}^{n} \delta(a_i(z), g_2) = 1$ and $\delta(a_i(z), g_2) > 0, (a_i(z) \not\equiv \infty)$.*

If $T(r, f_1) \sim T(r, f_2)$ and $T(r, g_1) = \circ(T(r, g_2))$ $(r \to \infty)$, then

$$T(r, f_1(g_1)) = \circ(T(r, f_2(g_2))) \qquad (r \to \infty, r \notin E),$$

where E is a set of finite logarithmic measure.

Remark 1.1. If we take $f_1(z) = f_2(z) = f(z)$ in Theorem 1.1, then we obtain

$$T(r, f(g_1)) = \circ(T(r, f(g_2))) \qquad (r \to \infty).$$

This solves the problem of C. T. Chuang and C. C. Yang.

2. LEMMAS

Lemma 2.1. *Let g be an entire function of order ρ_g and lower order $\lambda_g(\lambda_g < +\infty)$, let $a_i(z)(i = 1, 2, \cdots n,\ n \leqslant \infty)$ be entire functions, which satisfy $T(r, a_i(z)) = \circ(T(r, g))$. If $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$ with $\delta(a_i(z), g) > 0$, $a_i(z) \not\equiv \infty$, then*

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(1) [2]: *$g(z)$ is of regular growth and $\rho_g = \lambda_g$ is a positive integer*

(2) [3]:
$$\lim_{r\to\infty} T(r,g)/logM(r,g) = 1/\pi.$$

(3) *For arbitrarily small $\epsilon_1 > 0$, there exist $a_1(z), a_2(z), \cdots, a_k(z)$ such that*

$$\sum_{i=1}^{k} \delta(a_i(z), g) = 1 > 1 - \frac{\epsilon_1}{2}. \tag{2.1}$$

Let $a_1(z), a_2(z), \cdots, a_h(z) (h \leqslant k)$ be a maximal linearly independent subset of $a_1(z), a_2(z), \cdots, a_k(z)$.
Put

$$L(g) = \begin{vmatrix} g(z) & a_1(z) & a_2(z) & \cdots & a_h(z) \\ g^{(1)}(z) & a_1^{(1)}(z) & a_2^{(1)}(z) & \cdots & a_h^{(1)}(z) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g^{(h)}(z) & a_1^{(h)}(z) & a_2^{(h)}(z) & \cdots & a_h^{(h)}(z) \end{vmatrix}.$$

Then

(i) [4] the order of $L(g)$ is equal to the lower order of $L(g)$ and $L(g)$, $g(z)$ have the same order.

(ii) [5]
$$K[L(g)] = \overline{\lim_{r\to\infty}} \frac{N(r, L(g)) + N(r, 1/L(g))}{T(r, L(g)} = 0.$$

Lemma 2.2 ([6])**.** *Let $f(z)$ be a meromorphic function of lower order λ and order ρ, let P be the integer defined by $P \geqslant 1, P - \frac{1}{2} \leqslant \lambda < P + \frac{1}{2}$, $\rho < P + 1$. If for $A_0 > 0, 0 < \epsilon \leqslant 1$, we have*

$$K(f) = \overline{\lim_{r\to\infty}} \frac{N(r,f) + N(r,1/f)}{T(r,f)} < \frac{\epsilon}{A_0(P+1)},$$

then for $1 < \sigma \leqslant 36$, when $r > r_0$, we have

$$T(\sigma r, f) = \sigma^P T(r,f)(1 + \eta(r,\sigma)), \quad |\eta(r,\sigma)| < \epsilon. \tag{2.2}$$

Lemma 2.3 ([7])**.** *Let $f(z)$ be a meromorphic function, $a_i(z)$ $(i = 1, 2, \cdots, k)$ be distinct meromorphic functions which satisfy $T(r, a_i(z)) = \circ(T(r,f))$. Let $\{a_i(z)_{i=1}^h\}$ be a maximal linearly independent subset of $\{a_i(z)_{i=1}^k\}(h \leqslant k)$.*
Put

$$A_0 = \begin{vmatrix} a_1(z) & a_2(z) & \cdots & a_h(z) \\ a_1^{(1)}(z) & a_2^{(1)}(z) & \cdots & a_h^{(1)}(z) \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{(h)}(z) & a_2^{(h)}(z) & \cdots & a_h^{(h)}(z) \end{vmatrix} = A(a_1(z), a_2(z), \cdots, a_h(z)),$$

thus

$$L(f)=\frac{(-1)^h}{A_0}A(f,a_1,a_2,\cdots,a_h)=f^{(h)}+\frac{A_1}{A_0}f^{(h-1)}+\cdots+\frac{A_h}{A_0}f. \tag{2.3}$$

Then

$$\sum_{i=1}^{h}m(r,\frac{1}{f-a_i(z)})\leqslant m(r,\frac{1}{L(f)})+\circ(T(r,f)), \tag{2.4}$$

outside a set E of a finite linear measure except in positive real number axis (mes $E<+\infty$).

Lemma 2.4. *Let g be an entire function of order ρ and lower order $\lambda(\lambda<+\infty)$, let $a_i(z)(i=1,2,\cdots,n,n\leqslant\infty)$ be entire functions, which satisfy $T(r,a_i(z))=\circ(T(r,g))$. If $\sum_{i=1}^{n}\delta(a_i(z),g)=1$ with $\delta(a_i(z),g)>0,a_i(z)\not\equiv\infty$, then*

$$T(\sigma r,g)\sim\sigma^{\rho}T(r,g),\qquad (r\to\infty,1<\sigma\leqslant 36).$$

Proof. Step 1: Since $a_i(z)\not\equiv\infty$ and $\delta(a_i(z),g)>0$, we may assume that $a_1(z)\not\equiv\infty$ and $\delta(a_1(z),g)>0$.

By (2.3) we have

$$\sum_{i=1}^{h}m(r,\frac{1}{g-a_i(z)})\leqslant T(r,L(g))+\circ(T(r,g)),\qquad (r\notin E).$$

So

$$\underline{\lim}_{r\to\infty}\frac{T(r,L(g))}{T(r,g)}\geqslant\sum_{i=1}^{h}\underline{\lim}_{r\to\infty}\frac{m(r,\frac{1}{g-a_i(z)})}{T(r,g)}\geqslant\delta(a_1(z),g)>0.$$

Thus, there exists $A\geqslant 1$, such that when $r>A$, we obtain

$$T(r,g)<\frac{1}{c_1}T(r,L(g)), \tag{2.5}$$

where $c_1=\frac{1}{2}\delta(a_1(z),g)$.
By (2.3) and (2.4) we get

$$\sum_{i=1}^{h}\underline{\lim}_{r\to\infty}\frac{m(r,\frac{1}{g-a_i(z)})}{T(r,L(g))}\leqslant\delta(0,L(g)).$$

Since

$$N(r,L(g))\leqslant\sum_{i=1}^{h}N(r,\frac{A_i}{A_0})+\sum_{i=1}^{h}N(r,g^{(i)})+\circ(T(r,g))=\circ(T(r,g)),$$

so

$$T(r,L(g)) = m(r,L(g))+N(r,L(g)) \leqslant m(r,g)+m(r,\frac{L(g)}{g})+\circ(T(r,g)) = T(r,g)+\circ(T(r,g)) = (1+\circ(1))T(r,g). \tag{2.6}$$

Step 2: By [8, (1.5.8)], we have

$$\sum_{i=1}^{h} m(r,\frac{1}{g-a_i(z)}) \leqslant m(r,\sum_{i=1}^{h}\frac{1}{g-a_i(z)})+\circ(T(r,g)) \leqslant T(r,L(g))-N(r,\frac{1}{L(g)})+\circ(T(r,g)).$$

Hence, by (3) in [5] we obtain

$$1-\frac{\epsilon_1}{2} < \sum_{i=1}^{n}\delta(a_i(z),g) \leqslant \underline{\lim}_{r\to\infty}\sum_{i=1}^{h}\frac{m(r,\frac{1}{g-a_i(z)})}{T(r,g)} \leqslant \underline{\lim}_{r\to\infty}(\frac{T(r,L(g))}{T(r,g)}-\frac{N(r,\frac{1}{L(g)})}{T(r,g)}).$$

Thus, for any $\epsilon>0(\epsilon>\frac{\epsilon_1}{2})$, we have

$$\underline{\lim}_{r\to\infty}\frac{T(r,L(g))}{T(r,g)} > 1-\frac{\epsilon_1}{2} > 1-\epsilon. \tag{2.7}$$

Step 3: Since $g(z)$ is an entire function which satisfies $\sum_{i=1}^{n}\delta(a_i(z),g)=1$, with $\delta(a_i(z),g)>0$, $(a_i(z)\not\equiv\infty)$, so, by Lemma 2.1(1), we see that $g(z)$ is of regular growth and $\rho=\lambda$ is a positive integer.

By Lemma 2.1(3): (i), we know that $L(g)$ is of regular growth and the order of $L(g)$ is equal to the lower order of $L(g)$ and $L(g)$, $g(z)$ have the same order. Put $P=\rho$ in Lemma 2.2, by Lemma 2.1(3):(ii), we see that $L(g)$ satisfies condition of Lemma 2.2, hence, for $0<\epsilon<1$, when $1<\sigma\leqslant 36$ and $r>r_0$, by (2.2), we get

$$T(\sigma r,L(g)) = \sigma^{\rho}T(r,L(g))(1+\eta(r,\sigma)), \quad |\eta(r,\sigma)|<\epsilon. \tag{2.8}$$

So, by (2.6), (2.7), (2.8) we obtain

$$\begin{aligned}
&\overline{\lim_{r\to\infty}}\frac{T(\sigma r,g)}{T(r,g)}\\
\leqslant\ & \overline{\lim_{r\to\infty}}\frac{T(\sigma r,g)}{T(\sigma r,L(g))}\cdot\overline{\lim_{r\to\infty}}\frac{T(\sigma r,L(g))}{T(r,L(g))}\cdot\overline{\lim_{r\to\infty}}\frac{T(r,L(g))}{T(r,g)}\\
\leqslant\ & (\underline{\lim}_{r\to\infty}\frac{T(\sigma r,g)}{T(\sigma r,L(g))})^{-1}\cdot\overline{\lim_{r\to\infty}}\frac{T(\sigma r,L(g))}{T(r,L(g))}\cdot\overline{\lim_{r\to\infty}}\frac{(1+\circ(1))T(r,g)}{T(r,g)}\\
\leqslant\ & \frac{1}{1-\epsilon}\cdot\sigma^{\rho}(1+\epsilon)=\frac{1+\epsilon}{1-\epsilon}\sigma^{\rho}.
\end{aligned}$$

Letting $\epsilon\to 0$, we have

$$\overline{\lim_{r\to\infty}}\frac{T(\sigma r,g)}{T(r,g)}\leqslant\sigma^{\rho}. \tag{2.9}$$

$$\begin{aligned}
\underline{\lim}_{r\to\infty}\frac{T(\sigma r,g)}{T(r,g)} \geqslant\ & \underline{\lim}_{r\to\infty}\frac{T(\sigma r,g)}{T(\sigma r,L(g))}\cdot\underline{\lim}_{r\to\infty}\frac{T(\sigma r,L(g))}{T(r,L(g))}\\
&\cdot\underline{\lim}_{r\to\infty}\frac{T(r,L(g))}{T(r,g)}\\
\geqslant\ & \underline{\lim}_{r\to\infty}\frac{\frac{1}{1+\circ(1)}T(\sigma r,L(g))}{T(\sigma r,L(g))}\cdot(\overline{\lim_{r\to\infty}}\frac{T(r,L(g))}{T(\sigma r,L(g))})^{-1}\cdot(1-\epsilon)\\
=\ & (\overline{\lim_{r\to\infty}}\frac{1}{\sigma^{\rho}(1+\eta(r,\sigma))})^{-1}\cdot(1-\epsilon)\\
\geqslant\ & (\overline{\lim_{r\to\infty}}\frac{1}{1-|\eta(r,\sigma)|})^{-1}\cdot(1-\epsilon)\sigma^{\rho}\\
\geqslant\ & (1-\epsilon)^2\sigma^{\rho}.
\end{aligned}$$

Letting $\epsilon\to 0$, we get

$$\underline{\lim}_{r\to\infty}\frac{T(\sigma r,g)}{T(r,g)}\geqslant\sigma^{\rho}. \tag{2.10}$$

So, by (2.9), (2.10), we have

$$T(\sigma r,g)\sim\sigma^{\rho}T(r,g),\qquad (r\to\infty,1<\sigma\leqslant 36).$$ □

Lemma 2.5 ([8]). *Let f and g be entire functions and $g(0)=0$. Then for all $r>0$, we have*

$$T(r,f(g))\leqslant T(M(r,g),f).$$

Lemma 2.6 ([9]). *Let f be a transcendental entire function with*

$$T(r,f)=O^{*}((\log r)^{\beta}e^{(\log r)^{\alpha}})\quad(0<\alpha<1,\beta>0).$$

(i.e., there exist two positive constants K_1 and K_2 such that

$$K_1\leqslant\frac{T(r,f)}{(\log r)^{\beta}e^{(\log r)^{\alpha}}}\leqslant K_2).$$

Then

1. $T(r,f) \sim \log M(r,f) \qquad (r \to \infty, r \notin E)$;
2. $T(\sigma r,f) \sim T(r,f) \qquad (r \to \infty, \sigma \geqslant 2, r \notin E)$,

where E is a set of finite logarithmic measure .

Lemma 2.7 ([10]). *Let f and g be entire functions. Then*

$$M(r, f(g)) \geqslant M(M(r/6, g), f).$$

3. Proof of the Theorem 1.1

Step 1: Since $\sum_{i=1}^{n} \delta(a_i(z), g_2) = 1$ and $\delta(a_i(z), g_2) > 0, a_i(z) \not\equiv \infty$, by Lemma 2.1 and Lemma 2.4, we have

$$T(r, g_2) \sim (1/\pi) \log M(r, g_2) \qquad (r \to \infty),$$

and

$$T(\sigma r, g_2) \sim \sigma^{\rho_{g_2}} T(r, g_2) \qquad (r \to \infty). \tag{3.1}$$

So

$$\log M(r, g_2) = \pi T(r, g_2)(1 + \circ(1)) = \frac{\pi}{2^{\rho_{g_2}}} T(2r, g_2)(1 + \circ(1)). \tag{3.2}$$

Since

$$\log M(r, g_1) \leqslant 3T(2r, g_1) = 3 \cdot \circ(T(2r, g_2)) = \circ(T(2r, g_2)) \qquad (r \to \infty), \tag{3.3}$$

by (3.1) and (3.2) we get

$$\frac{\log M(r, g_1)}{\log M(r, g_2)} \leqslant \frac{\circ(T(2r, g_2))}{(\pi/2^{\rho_{g_2}}) \cdot T(2r, g_2)(1 + \circ(1))} \to 0 \qquad (r \to \infty),$$

and so

$$\log M(r, g_1) = \circ(\log M(r, g_2)) \qquad (r \to \infty). \tag{3.4}$$

Step 2: We may assume $g_1(0) = 0$. Otherwise, we only need to make a transformation: $G(z) = g_1(z) - g_1(0)$.

By Lemma 2.5, Lemma 2.1, (3.4) and (3.2), we have

$$\begin{aligned} & T(r, f_1(g_1)) \\ \leqslant\ & T(M(r, g_1), f) \\ =\ & O^*((\log M(r, g_1))^{\beta} e^{(\log M(r,g_1))^{\alpha}}) \\ \leqslant\ & K_2 (\log M(r, g_1))^{\beta} e^{(\log M(r,g_1))^{\alpha}} \\ =\ & K_2 (\circ(\log M(r, g_2))^{\beta} e^{(\circ(\log M(r,g_2)))^{\alpha}} \\ =\ & K_2 (\circ(\pi T(r, g_2)(1 + \circ(1))))^{\beta} e^{(\circ(\pi T(r,g_2)(1+\circ(1))))^{\alpha}} \\ =\ & K_2 (\circ(T(r, g_2)(1 + \circ(1))))^{\beta} e^{\circ((T(r,g_2)(1+\circ(1)))^{\alpha})} \qquad (r \to \infty). \end{aligned} \tag{3.5}$$

On the other hand, since $T(r, f_1) \sim T(r, f_2)$ $(r \to \infty)$, one gets

$$T(r, f_2) = T(r, f_1)(1 + \circ(1)) = O^*((\log r)^{\beta} e^{(\log r)^{\alpha}})(1 + \circ(1)).$$

and so, by Lemma 2.6, we obtain

$$\log M(r,f_2)=T(r,f_2)(1+\circ(1))=T(r,f_1)(1+\circ(1))^2$$

$$=O^*((\log r)^\beta e^{(\log r)^\alpha})(1+\circ(1))^2. \tag{3.6}$$

Thus, by Lemma 2.7, Lemma 2.1 and (3.1), (3.6), we obtain

$$\begin{aligned}
&T(r,f_2(g_2)) \qquad (3.7)\\
\geqslant&\frac{1}{3}\log M(\frac{r}{2},f_2(g_2))\geqslant\frac{1}{3}\log M(M(\frac{r}{12},g_2),f_2)\\
=&\frac{1}{3}O^*((\log M(\frac{r}{12},g_2))^\beta e^{\log M(\frac{r}{12},g_2))^\alpha})(1+\circ(1))^2\\
\geqslant&\frac{1}{3}K_1(\log M(\frac{r}{12},g_2))^\beta e^{(\log M(\frac{r}{12},g_2))^\alpha}(1+\circ(1))^2\\
=&\frac{1}{3}K_1(\pi T(\frac{r}{12},g_2)(1+\circ(1)))^\beta e^{(\pi T(\frac{r}{12},g_2)(1+\circ(1)))^\alpha}\\
=&\frac{1}{3}K_1(\pi(\frac{1}{12})^{\rho_{g_2}}T(r,g_2)(1+\circ(1)))^\beta e^{(\pi(\frac{1}{12})^{\rho_{g_2}}T(r,g_2)(1+\circ(1))^\alpha}(1+\circ(1)))^2\\
&\qquad\qquad\qquad\qquad(r\to\infty).
\end{aligned}$$

Thus, by (3.5) and (3.7), we have

$$\begin{aligned}
&\frac{T(r,f_1(g_1))}{T(r,f_2(g_2))}\\
\leqslant&\frac{K_2(\circ(T(r,g_2)(1+\circ(1))))^\beta e^{(\circ(T(r,g_2)(1+\circ(1))))^\alpha}}{\frac{1}{3}K_1(\pi(\frac{1}{12})^{\rho_{g_2}}T(r,g_2)(1+\circ(1)))^\beta e^{(\pi(\frac{1}{12})^{\rho_{g_2}}T(r,g_2)(1+\circ(1)))^\alpha}(1+\circ(1))^2}\\
=&\frac{3K_2}{K_1}(T(r,g_2)(1+\circ(1)))^\beta[\frac{(\circ(T(r,g_2)(1+\circ(1))))^\beta}{(T(r,g_2)(1+\circ(1)))^\beta}\cdot\frac{1}{(\pi(\frac{1}{12})^{\rho_{g_2}})^\beta(1+\circ(1))^2}]\\
&\cdot e^{(T(r,g_2)(1+\circ(1)))^\alpha[\frac{(\circ(T(r,g_2)(1+\circ(1))))^\alpha}{(T(r,g_2)(1+\circ(1)))^\alpha}-(\pi(\frac{1}{12})^{\rho_{g_2}})^\alpha]}\\
=&\frac{3K_2}{K_1}[\frac{(\circ(T(r,g_2)(1+\circ(1))))^\beta}{(T(r,g_2)(1+\circ(1)))^\beta}\cdot\frac{1}{(\pi(\frac{1}{12})^{\rho_{g_2}})^\beta(1+\circ(1))^2}]\\
&\cdot e^{\beta\log(T(r,g_2)(1+\circ(1)))+(T(r,g_2)(1+\circ(1)))^\alpha[\frac{(\circ(T(r,g_2)(1+\circ(1))))^\alpha}{(T(r,g_2)(1+\circ(1)))^\alpha}-(\pi(\frac{1}{12})^{\rho_{g_2}})^\alpha]}\\
&\to 0\qquad(r\to\infty,\rho_{g_2}>0).
\end{aligned}$$

So

$$T(r,f_1(g_1))=\circ(T(r,f_2(g_2)))\qquad(r\to\infty).$$

This completes the proof of Theorem 1.1. □

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