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RECOGNITION OF THE LINEAR GROUPS OVER THE BINARY FIELD BY THE SET OF THEIR ELEMENT ORDERS

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ABSTRACT. In this paper we will prove that the simple group $PSL_{14}(2)$ is recognizable by the set of its element orders, that is to say if G is a finite group with the same set of element orders as $PSL_{14}(2)$, then $G \cong PSL_{14}(2)$.

1. INTRODUCTION

Let G be a finite group. The set of orders of elements of G is called the spectrum of G and is denoted by $\omega(G)$. Obviously the set $\omega(G)$ is closed and partially ordered by the divisibility relation, hence $\omega(G)$ is uniquely determined by the set $\mu(G)$ of its maximal elements. For a subset Ω of the set of natural numbers we define $h(\Omega)$ to be the number of isomorphism classes of finite groups G such that $\omega(G) = \Omega$. We set $h(G) = h(\omega(G))$ and call it the $h-$function on G. The group G is called recognizable if $h(G) = 1$. Examples of recognizable groups are given in [2], [3], [16] and [17]. The simple groups $PSL_n(2)$, $3 \leqslant n \leqslant 7$, are proved to be recognizable by the set of their element orders. In [4] it is also proved that the group $PSL_8(2)$ is recognizable. Based on the above facts it is natural to make the following assertion.

Conjecture. *The simple groups $PSL_n(2)$ for all $n \geqslant 3$ are recognizable.*

To support the above conjecture we proved in [5] and [6] that the groups $PSL_{11}(2)$ and $PSL_{13}(2)$ are recognizable. In this paper we intend to prove the same result for the group $PSL_{14}(2)$. Our main result is the following

Theorem. *Let G be a finite group. Then $\omega(G) = \omega(PSL_{14}(2))$ if and only if $G \cong PSL_{14}(2)$.*

Our notation throughout the paper is standard. In particular $\mathbb{Z}_n$ denotes the cyclic group of order n and $A : B$ denotes the semi-direct product of a group A by a group B, and if B is a cyclic group of order b then the above semi-direct product is denoted by $A : b$. In the case that both A and B are cyclic groups of order a and b respectively, then the above semi-direct product is denoted by $a : b$.

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2. Preliminaries

For a natural number n the set of prime divisors of n is denoted by $\pi(n)$. If G is a finite group, then we set $\pi(G) = \pi(|G|)$. The Gruenberg-Kegel graph $GK(G)$ of a group G, or the prime graph of G, is a graph with vertex set $\pi(G)$ such that two distinct vertices p and q are joined by an edge if and only if $pq \in \omega(G)$. The number of connected components of the graph $GK(G)$ is denoted by $s(G)$ and its components are denoted by $\pi_i(G), 1 \leqslant i \leqslant s(G)$. Hence $\pi_i(G)$ is a connected subgraph of $GK(G)$ and if no confusion arises we usually denote its vertex set by $\pi_i(G)$ again. If G is a group of even order, then we assume that $2 \in \pi_1(G)$, and we let $\pi_1 = \pi_1(G)$. The classification of finite groups with non-connected prime graphs was reduced to the same question about simple groups by K. Gruenberg and O. Kegel in 1975(unpublished). But later in [19] the prime graph of all the finite simple groups except the simple groups of Lie type in even characteristic were classified. In [11] the same classification was obtained for the simple groups of Lie type in even characteristic. Using the above classification many results on recognizability of finite groups were obtained. These results depend on the following theorem of Gruenberg and Kegel.

Gruenberg-Kegel's Theorem (see [19]). *If G is a finite group with non-connected prime graph, then G has one of the following structures:*
(a) *Frobenius or 2-Frobenius,*
(b) *simple,*
(c) *an extension of a π_1-group by a simple group,*
(d) *an extension of a simple group by a π_1-group,*
(e) *an extension of a π_1-group by simple by π_1-group.*

In the following we will explain some terminologies used in the above theorem. If $N \trianglelefteq G$ and $K = \frac{G}{N}$, then G is called an extension of the group N by the group K and it is denoted by $G = N.K$. If N is a π_1-group (a group, whose prime divisors are in π_1) and K is a simple group then $G = N.K$ is called an extension of a π_1-group by a simple group. Similarly a simple by π_1-group or a π_1 by simple by π_1-group is defined.

We remark that a group G is called 2-Frobenius if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ of G such that K is a Frobenius group with kernel H and $\frac{G}{H}$ is Frobenius group with kernel $\frac{K}{H}$. Note that 2-Frobenius groups are always solvable. This is because by Thompson's theorem [18] the Frobenius kernel is a nilpotent group. Hence H and $\frac{K}{H}$ are nilpotent and therefore solvable groups, implying that K is a solvable group. If $\left|\frac{G}{K}\right|$ is odd, then $\frac{G}{K}$ is solvable, hence G is solvable. Therefore we may assume that $|\frac{G}{K}|$ is even.

Now we assume that $\left|\frac{G}{K}\right|$ is even. In this case since the order of the Frobenius complement in the Frobenius group $\frac{G}{H}$ is even, by [14] $\frac{K}{H}$ must be abelian. But then $\frac{K}{H}$ is isomorphic to the direct product of its Sylow subgroups and, being a Frobenius complement, by [14] $\frac{K}{H}$ is isomorphic to the direct product of cyclic groups A_i where the A_i's are the Sylow subgroups of G. In particular $\frac{G}{H}$ is cyclic.

Now we put $A = \frac{K}{H}$ and $F = \frac{G}{H}$, where A is a cyclic group. Since A is an abelian Frobenius kernel we obtain $C_F(A) = A$ and therefore $\frac{F}{A}$ is isomorphic to a subgroup of $Aut(A)$. But since $Aut(A)$ is abelian we conclude that $\frac{F}{A}$ is abelian. Therefore $\frac{F}{A} \cong \frac{G}{K}$ is abelian and hence solvable. Now solvability of K implies that G is solvable and we are done.

Now by Gruenberg-Kegel's theorem we have the following further reduction.

Lemma 2.1. *Let G be a finite group with non-connected prime graph. If G is not a solvable group, then there is a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that N and $\frac{G}{G_1}$ are π_1-groups and $\overline{G} := \frac{G_1}{N}$ is a simple group.*

Now we return to the group $PSL_{14}(2)$. We have $|PSL_{14}(2)| = 2^{91} \cdot 3^9 \cdot 5^3 \cdot 7^4 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 31^2 \cdot 43 \cdot 73 \cdot 89 \cdot 127^2 \cdot 8191$ and therefore $\pi(PSL_{14}(2)) = \{2, 3, 5, 7, 11, 13, 17, 23, 31, 43, 73, 89, 127, 8191\}$. In [11] and [19], $s(G)$ for all finite simple groups G are computed. According to this result we have

$$s(PSL_n(2)) = \begin{cases} 2, & \text{if} \quad n = p \text{ or } p+1, \\ 1, & \quad\quad \text{otherwise.} \end{cases}$$

where p is a prime number. In the case that the Gruenberg-Kegel's graph of $PSL_n(2)$ is disconnected we have

$$\pi_1 = \pi\left(2\prod_{i=1}^{p-1}(2^i - 1)\right) \quad \text{or} \quad \pi_1 = \pi\left(2(2^{p+1} - 1)\prod_{i=1}^{p-1}(2^i - 1)\right),$$

in the respective cases $n = p$ or $p + 1$. In any case the other component is $\pi_2 = \pi(2^p - 1)$. Therefore for the group $PSL_{14}(2)$ we have $\pi_1 = \{2, 3, 5, 7, 11, 13, 17, 23, 31, 43, 73, 89, 127\}$ and $\pi_2 = \{8191\}$.

In the next step we proceed to find the spectrum of $PSL_{14}(2)$. To do this we will use the notation used in [9] for conjugacy classes of the general linear group $GL_n(q)$, where q is the power of a prime number. Let $A \in GL_n(q)$ have characteristic polynomial $\det(xI - A) = f_1^{k_1} f_2^{k_2} \cdots f_N^{k_N}$, where $f_i = f_i(x), 1 \leqslant i \leqslant N$, are distinct monic irreducible polynomials over $GF(q)$ and $k_i \geqslant 0$. Here, of course, we exclude the polynomial x for the reason of invertibility of A. In this case A is conjugate to a block diagonal matrix of the form

$$diag(U_{\nu_1}(f_1), U_{\nu_2}(f_2), \cdots, U_{\nu_N}(f_N))$$

where $\nu_1, \nu_2, \cdots, \nu_N$ are certain partitions of $k_1, k_2, \cdots, k_N$ respectively, and $U_{\nu_i}(f_i)$ is a certain matrix which will be explained later. This conjugacy class of A is denoted by the symbol $c = (f_1^{\nu_1} f_2^{\nu_2} \cdots f_N^{\nu_N})$.

The order of $c = (f_1^{\nu_1} f_2^{\nu_2} \cdots f_N^{\nu_N})$ is equal to the least common multiple, (l.c.m), of the orders of the matrices $U_{\nu_i}(f_i), 1 \leqslant i \leqslant N$. For each partition $(\lambda) \equiv l_1 + l_2 + \cdots + l_p$, $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_p > 0$, of a positive integer k and each polynomial $f = f(x) \in GF(q)[x]$ the matrix $U_\lambda(f)$ is defined to be

$$U_\lambda(f) = diag(U_{l_1}(f), U_{l_2}(f), \cdots, U_{l_p}(f)).$$

Each matrix $U_{l_i}(f)$ is defined as follows.

Let $f(x) = a_0 + a_1x + \cdots + a_{d-1}x^{d-1} + x^d$ be a monic polynomial of degree d over $GF(q)$ and let

$$U(f) = U_1(f) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{d-1} \end{bmatrix}$$

be its companion matrix. Then for any natural number m the matrix $U_m(f)$ is defined by

$$U_m(f) = \begin{bmatrix} U(f) & I_d & 0 & \\ 0 & U(f) & I_d & \cdots \\ \vdots & & \ddots & \\ \cdots & \cdots & \cdots & U(f) \end{bmatrix}$$

with m diagonal blocks $U(f)$ where I_d is the $d \times d$ identity matrix. The characteristic polynomial of $U_m(f)$ is $f(x)^m$. It is easy to verify that the characteristic polynomial of $U_\lambda(f)$ is $f(x)^k$ provided λ is a partition of k.

As mentioned earlier, the order of an element in the class $c = (f_1^{\nu_1}\ f_2^{\nu_2}\ \cdots\ f_N^{\nu_N})$ is equal to the l.c.m of the orders of the matrices $U_{\nu_i}(f_i), 1 \leqslant i \leqslant N$. But the order of each $U_{\nu_i}(f_i)$ is equal to the l.c.m of the orders of $U_{l_{i_1}}(f_i)$, $U_{l_{i_2}}(f_i)$, $\cdots$, $U_{l_{i_p}}(f_i)$, where $\{l_{i_1}, l_{i_2}, \cdots, l_{i_p}\}$ is the partition of k_i associated to ν_i. Therefore our task is to find the order of a matrix of the form $U_m(f)$. This is related to the concept of order for polynomials over finite fields which is developed in [12].

Definition. Let $f(x)$ be a polynomial over $GF(q)$ with $f(0) \neq 0$. The least positive integer e for which $f(x)$ is a divisor of $x^e - 1$ is called the order of $f(x)$ and is denoted by $ord(f(x))$.

In the following we will establish a relationship between $ord(f)$ and the order of its companion matrix. Note that $ord(f) = e$ has the property that if $f(x)\,\big|x^k - 1$, then $e|k$.

Lemma 2.2. *Let $A \in GL_n(q)$ have minimal polynomial $f(x)$. Then the order of A is equal to the order of $f(x)$.*

Proof. Assume $ord(f) = e$. Then by definition e is the smallest positive integer such that $f(x)|x^e - 1$. Therefore there is $g(x) \in GF(q)[x]$ with $x^e - 1 = f(x)g(x)$. Since $f(x)$ is the minimal polynomial of A, we have $A^e - I = 0$. Thus if $O(A) = k$, then $k|e$. Now $A^k = I$ implies that A satisfies the polynomial $x^k - 1$ over $GF(q)$. But since $f(x)$ is the minimal polynomial of A we obtain $f(x)|x^k - 1$, whence $e|k$. Therefore $e = k$ and the lemma is proved. $\square$

According to the above lemma the order of $U(f) = U_1(f)$ is equal to the order of $f(x)$. Now we will find a formula for the order of $U_m(f)$. Note that the characteristic polynomial of $U_m(f)$ is $f(x)^m$ where $f(x)$ is the characteristic polynomial of $U(f)$, hence $f(x)^m$ is the minimal polynomial of $U_m(f)$. Therefore, by Lemma 2.2 the order of $U_m(f)$ is equal to the order of $f(x)^m$. In the case where $f(x)$ is irreducible over $GF(q)$ the order of $f(x)^m$ can be obtained from the following lemma taken from [11, page 86].

Lemma 2.3. *Let $f(x) \in GF(q)[x]$ be irreducible with $f(0) \neq 0$. Let $q = p^r$ be the power of a prime p and $ord(f) = e$. Then $ord(f(x)^m) = ep^t$ where t is the smallest integer such that $p^t \geqslant m$.*

Therefore, to find the order of an element A of the conjugacy class $c = (f_1^{\nu_1} f_2^{\nu_2} \cdots f_N^{\nu_N})$ we assume the minimal polynomial of A is $f(x) = f_1(x)^{n_1} f_2(x)^{n_2} \cdots f_N(x)^{n_N}$ and then the order of $f(x)$ can be computed using the following lemma taken from [12].

Lemma 2.4. *Let $m(x) \in GF(q)[x], m(0) \neq 0$, have the factorization $m(x) = af_1(x)^{n_1} f_2(x)^{n_2} \cdots f_N(x)^{n_N}$ in terms of distinct monic irreducible polynomials $f_i(x)$ over $GF(q)$, where q is the power of prime number p and $a \in GF(q)$. Then $ord(m(x)) = ep^t$ where e is the l.c.m. of $ord(f_1), ord(f_2), \cdots, ord(f_N)$ and t is the smallest integer such that $p^t \geqslant \max\{n_1, n_2, \cdots, n_N\}$.*

Now suppose $A \in GL_n(q)$ has characteristic polynomial $m(x) = \det(xI - A) = f_1^{k_1} f_2^{k_2} \cdots f_N^{k_N}$ where $f_i = f_i(x), 1 \leqslant i \leqslant N$, are distinct monic irreducible polynomials over $GF(q)$ and $k_i > 0$. Conjugacy classes of $GL_n(q)$ whose elements have characteristic polynomial $m(x)$ are called conjugacy classes of type A. There is a conjugacy class of type A with minimal polynomial $m(x)$. This follows simply by taking $\nu_1 = k_1, \nu_2 = k_2, \cdots, \nu_N = k_N$ as partitions of the respective k_i's. It is obvious that if there is a conjugacy class of type A with minimal polynomial $t(x) = f_1^{n_1} f_2^{n_2} \cdots f_N^{n_N}$, then $t(x) | m(x)$ and therefore $ord(t(x)) | ord(m(x))$. Hence as far as the maximum order of elements in the conjugacy classes of type A is concerned we must calculate $ord(m(x))$ where $m(x)$ is given by $\sum_{i=1}^{N} k_i d_i = n$ where $d_i = \deg(f_i), 1 \leqslant i \leqslant N$.

By [11, page 84], if $f(x) \in GF(q)[x]$ is an irreducible polynomial of degree d, then $ord(f) \left| q^d - 1 \right.$. Furthermore, there is an irreducible polynomial over $GF(q)$ with order $q^d - 1$. Therefore, from Lemma 2.4 we obtain the following corollary.

Corollary 2.1. *Let $m(x) \in GF(q)[x]$, $m(0) \neq 0$, q be the power of a prime p, and let $m(x) = f_1(x)^{n_1} f_2(x)^{n_2} \cdots f_N(x)^{n_N}$, where $f_1(x), \cdots, f_N(x)$ are distinct monic irreducible polynomials of degree d_i over $GF(q), 1 \leqslant i \leqslant N$. Then $ord(m(x))$ divides $p^t \times l.c.m(q^{d_1} - 1, q^{d_2} - 1, \cdots, q^{d_N} - 1)$, where t is the smallest non-negative integer such that $p^t \geqslant \max\{n_1, n_2, \cdots, n_N\}$. Moreover if $\sum_{i=1}^{N} n_i d_i = n$, then $GL_n(q)$ has an element with the above order.*

Therefore, as far as $\mu(GL_n(q))$, the set of maximal elements of $\omega(GL_n(q))$, is concerned, we first find all the irreducible polynomials of degree up to n over $GF(q)$ and call them $f_1, f_2, \cdots, f_N$. Then we consider all the possible factorizations of the form $f_1^{n_1} f_2^{n_2} \cdots f_N^{n_N}$ where $n_i \geqslant 0$ and $\sum_{i=1}^{N} n_i d_i = n,\ d_i = \deg(f_i)$. Finally numbers of the form $p^t \times l.c.m(q^{d_1} - 1, q^{d_2} - 1, \cdots, q^{d_N} - 1)$ are elements of $\mu(GL_n(q))$ where t is explained above.

In the following we will find $\mu\,(PSL_{14}(2))$ which is needed for our result.

Lemma 2.5. *We have*

$$\begin{aligned}\mu\,(PSL_{14}(2)) =\{&240,\ 336,\ 496,\ 504,\ 840,\ 1260,\ 1736, 2040,\ 3048,\ 3556,\\ &3720,\ 4088,\ 4092,\ 5115,\ 5334,\ 5355,\ 6132,\ 6141,\ 6510,\\ &7140,\ 7161,\ 7620,\ 7665,\ 7812,\ 7874,\ 7905,\ 8001,\ 8188,\\ &8190,\ 8191,\ 11811,\ 13335,\ 14329,\ 15841,\ 16383\}.\end{aligned}$$

Proof. Obviously $PSL_{14}(2) = GL_{14}(2)$, so elements of $PSL_{14}(2)$ are non-singular 14×14 matrices and we use the notation used above for the conjugacy classes of $GL_{14}(2)$. Since we use polynomials of the form $f_1^{k_1} f_2^{k_2} \cdots\ f_N^{k_N}$, f_i is irreducible of degree d_i with the condition $\sum_{i=1}^{N} k_i d_i = 14$ and the shape of the polynomial is not important for us, we use the term $[d_1^{k_1}][d_2^{k_2}] \cdots [d_N^{k_N}]$ to represent the above polynomial.

Now in the following all the possible factorizations of polynomials of degree 14 with maximum orders are written. Note that the number of all possible factorizations is equal to the number of partitions of 14 that is 135.

Polynomial	Order	Polynomial	Order	Polynomial	Order
$[1^{14}]$	$2^4 = 16$	$[1^8][3^2]$	$2^3{\cdot}7 = 56$	$[1^6][2][3^2]$	$2^3{\cdot}3{\cdot}7=168$
$[1^{12}][2]$	$2^4{\cdot}3 = 48$	$[1^8][2^3]$	$2^3{\cdot}3 = 24$	$[1^6][2^4]$	$2^3{\cdot}3=24$
$[1^{11}][3]$	$2^4{\cdot}7 = 112$	$[1^7][7]$	$2^3{\cdot}127=1016$	$[1^6][2][6]$	$2^3{\cdot}63=504$
$[1^{10}][4]$	$2^4{\cdot}15 = 240$	$[1^7][2][5]$	$2^3{\cdot}3{\cdot}31=744$	$[1^6][2^2][4]$	$2^3{\cdot}15=120$
$[1^{10}][2^2]$	$2^4{\cdot}3 = 48$	$[1^7][3][4]$	$2^3{\cdot}7{\cdot}15=840$	$[1^5][9]$	$2^3{\cdot}511=4088$
$[1^9][5]$	$2^4{\cdot}31 = 496$	$[1^7][2^2][3]$	$2^3{\cdot}3{\cdot}7 = 168$	$[1^5][2][7]$	$2^3{\cdot}3{\cdot}127=3048$
$[1^9][2][3]$	$2^4{\cdot}3{\cdot}7 = 336$	$[1^6][8]$	$2^3{\cdot}255=2040$	$[1^5][3][6]$	$2^3{\cdot}63=504$
$[1^8][6]$	$2^3{\cdot}63 = 504$	$[1^6][3][5]$	$2^3{\cdot}7{\cdot}31=1736$	$[1^5][4][5]$	$2^3{\cdot}15{\cdot}31=3720$
$[1^8][4][2]$	$2^3{\cdot}15 = 120$	$[1^6][4^2]$	$2^3{\cdot}15=120$	$[1^5][2^2][5]$	$2^3{\cdot}3{\cdot}31=744$

Polynomial	Order	Polynomial	Order	Polynomial	Order
$[1^5][2][3][4]$	$2^3{\cdot}7{\cdot}15=840$	$[1^4][2][3][5]$	$2^2{\cdot}3{\cdot}7{\cdot}31=2604$	$[1^3][2][3][6]$	$2^2{\cdot}63=252$
$[1^5][2^3][3]$	$2^3{\cdot}3{\cdot}7=168$	$[1^4][2][4^2]$	$2^2{\cdot}15=60$	$[1^3][2][4][5]$	$2^2{\cdot}15{\cdot}31=1860$
$[1^5][3^3]$	$2^3{\cdot}7=56$	$[1^4][2^3][4]$	$2^2{\cdot}15=60$	$[1^3][2^3][5]$	$2^2{\cdot}3{\cdot}31=372$
$[1^4][10]$	$2^2{\cdot}1023=4092$	$[1^4][2^2][3^2]$	$2^2{\cdot}3{\cdot}7=84$	$[1^3][2^2][3][4]$	$2^2{\cdot}7{\cdot}15=420$
$[1^4][2][8]$	$2^2{\cdot}255=1020$	$[1^4][2^5]$	$2^2{\cdot}3=12$	$[1^3][2][3^3]$	$2^2{\cdot}3{\cdot}7=84$
$[1^4][3][7]$	$2^2{\cdot}7{\cdot}127=3556$	$[1^4][3^2][4]$	$2^2{\cdot}7{\cdot}15=420$	$[1^3][2^4][3]$	$2^2{\cdot}3{\cdot}7=84$
$[1^4][4][6]$	$2^2{\cdot}15{\cdot}21=1260$	$[1^3][11]$	$2^2{\cdot}2047=8188$	$[1^3][3][8]$	$2^2{\cdot}7{\cdot}255=7140$
$[1^4][5^2]$	$2^2{\cdot}31=124$	$[1^3][2][9]$	$2^2{\cdot}3{\cdot}511=6132$	$[1^3][3^2][5]$	$2^2{\cdot}7{\cdot}31=868$
$[1^4][2^2][6]$	$2^2{\cdot}63=252$	$[1^3][2^2][7]$	$2^2{\cdot}3{\cdot}127=1524$	$[1^3][3][4^2]$	$2^2{\cdot}7{\cdot}15=420$

Polynomial	Order	Polynomial	Order	Polynomial	Order
$[1^3][4][7]$	$2^2{\cdot}15{\cdot}127=7620$	$[1^2][4^3]$	$2^2{\cdot}15=60$	$[1^2][5][7]$	$2{\cdot}31{\cdot}127=7874$
$[1^3][5][6]$	$2^2{\cdot}31{\cdot}63=7812$	$[1^2][2^2][3][5]$	$2{\cdot}3{\cdot}7{\cdot}31=1302$	$[1^2][2^2][8]$	$2{\cdot}255=510$
$[1^2][12]$	$2{\cdot}4095=8190$	$[1^2][3][4][5]$	$2{\cdot}7{\cdot}15{\cdot}31=6510$	$[1^2][4][8]$	$2{\cdot}255=510$
$[1^2][2^6]$	$2^3{\cdot}3=24$	$[1^2][2][5^2]$	$2{\cdot}3{\cdot}31=186$	$[1^2][3][9]$	$2{\cdot}511=1022$
$[1^2][2^3][3^2]$	$2^2{\cdot}3{\cdot}7=84$	$[1^2][2^3][6]$	$2^2{\cdot}63=252$	$[1^2][2][10]$	$2{\cdot}1023=2046$
$[1^2][3^4]$	$2^2{\cdot}7=28$	$[1^2][3^2][6]$	$2{\cdot}63=126$	$[1][13]$	$8191=8191$
$[1^2][2^4][4]$	$2^2{\cdot}15=60$	$[1^2][2][4][6]$	$2{\cdot}5{\cdot}63=630$	$[1][2^5][3]$	$2^3{\cdot}3{\cdot}7=168$
$[1^2][2][3^2][4]$	$2{\cdot}7{\cdot}15=210$	$[1^2][6^2]$	$2{\cdot}63=126$	$[1][2^2][3^3]$	$2^2{\cdot}3{\cdot}7=84$
$[1^2][2^2][4^2]$	$2{\cdot}15=30$	$[1^2][2][3][7]$	$2{\cdot}3{\cdot}7{\cdot}127=5334$	$[1][2^3][3][4]$	$2^2{\cdot}7{\cdot}15=420$

Polynomial	Order	Polynomial	Order	Polynomial	Order
$[1][3^3][4]$	$2^2 \cdot 7 \cdot 15 = 420$	$[1][2][5][6]$	$31 \cdot 63 = 1953$	$[1][3][10]$	$1023 \cdot 7 = 7161$
$[1][2][3][4^2]$	$2 \cdot 7 \cdot 15 = 210$	$[1][2^3][7]$	$3 \cdot 127 = 381$	$[1][2][11]$	$3 \cdot 2047 = 6141$
$[1][2^4][5]$	$2^2 \cdot 15 \cdot 31 = 1860$	$[1][3^2][7]$	$7 \cdot 127 = 889$	$[14]$	$127 \cdot 3 \cdot 43 = 16\,383$
$[1][2][3^2][5]$	$2 \cdot 3 \cdot 7 \cdot 31 = 1302$	$[1][2][4][7]$	$15 \cdot 127 = 1905$	$[2^7]$	$2^3 \cdot 3 = 24$
$[1][2^2][4][5]$	$2 \cdot 15 \cdot 31 = 930$	$[1][6][7]$	$63 \cdot 127 = 8001$	$[2^4][3^2]$	$2^2 \cdot 3 \cdot 7 = 84$
$[1][4^2][5]$	$2 \cdot 15 \cdot 31 = 930$	$[1][2][3][8]$	$7 \cdot 255 = 1785$	$[2][3^4]$	$2^2 \cdot 3 \cdot 7 = 84$
$[1][3][5^2]$	$2 \cdot 7 \cdot 31 = 434$	$[1][5][8]$	$31 \cdot 255 = 7905$	$[2^5][4]$	$2^3 \cdot 15 = 120$
$[1][2^2][3][6]$	$2 \cdot 63 = 126$	$[1][2^2][9]$	$2 \cdot 3 \cdot 511 = 3066$	$[2^2][3^2][4]$	$2^2 \cdot 7 \cdot 15 = 420$
$[1][3][4][6]$	$5 \cdot 63 = 315$	$[1][4][9]$	$15 \cdot 511 = 7665$	$[2^3][4^2]$	$2^2 \cdot 15 = 60$

Polynomial	Order	Polynomial	Order	Polynomial	Order
$[3^2][4^2]$	$2 \cdot 15 = 30$	$[2^2][4][6]$	$2 \cdot 5 \cdot 63 = 630$	$[3^2][8]$	$2 \cdot 7 \cdot 255 = 3570$
$[2][4^3]$	$2^2 \cdot 15 = 60$	$[4^2][6]$	$2 \cdot 5 \cdot 63 = 630$	$[2][4][8]$	$255 = 255$
$[2^3][3][5]$	$2^2 \cdot 3 \cdot 7 \cdot 31 = 2604$	$[3][5][6]$	$31 \cdot 63 = 1953$	$[6][8]$	$21 \cdot 255 = 5355$
$[3^3][5]$	$2^2 \cdot 7 \cdot 31 = 868$	$[2][6^2]$	$2 \cdot 63 = 126$	$[2][3][9]$	$3 \cdot 511 = 1533$
$[2][3][4][5]$	$7 \cdot 15 \cdot 31 = 3255$	$[2^2][3][7]$	$2 \cdot 3 \cdot 7 \cdot 127 = 5334$	$[5][9]$	$31 \cdot 511 = 15\,841$
$[2^2][5^2]$	$2 \cdot 3 \cdot 31 = 186$	$[3][4][7]$	$7 \cdot 15 \cdot 127 = 13\,335$	$[2^2][10]$	$2 \cdot 1023 = 2046$
$[4][5^2]$	$2 \cdot 15 \cdot 31 = 930$	$[2][5][7]$	$3 \cdot 31 \cdot 127 = 11\,811$	$[4][10]$	$5 \cdot 1023 = 5115$
$[2^4][6]$	$2^2 \cdot 63 = 252$	$[7^2]$	$2 \cdot 127 = 254$	$[3][11]$	$7 \cdot 2047 = 14\,329$
$[2][3^2][6]$	$2 \cdot 63 = 126$	$[2^3][8]$	$2^2 \cdot 255 = 1020$	$[2][12]$	$65 \cdot 63 = 4095$

From this table we can find $\mu(PSL_{14}(2))$ as indicated in the lemma. $\square$

Now it is easy to deduce $\omega(PSL_{14}(2))$ from $\mu(PSL_{14}(2))$. This is obtained as follows:

$\omega(PSL_{14}(2)) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20,$
$21, 22, 23, 24, 26, 28, 30, 31, 33, 34, 35, 36, 39, 40, 42, 43, 44, 45, 46, 48, 51,$
$55, 56, 60, 62, 63, 65, 66, 68, 69, 70, 72, 73, 77, 78, 80, 84, 85, 89, 90, 91, 92,$
$93, 102, 105, 112, 117, 119, 120, 124, 126, 127, 129, 130, 132, 136, 140,$
$146, 153, 155, 161, 165, 168, 170, 178, 180, 182, 186, 195, 204, 210, 217,$
$219, 231, 234, 238, 240, 248, 252, 254, 255, 267, 273, 279, 280, 292, 310,$
$315, 336, 340, 341, 356, 357, 365, 372, 381, 390, 408, 420, 434, 438, 455,$
$465, 476, 496, 504, 508, 510, 511, 527, 546, 558, 584, 585, 595, 620, 623,$
$630, 635, 651, 680, 682, 714, 744, 762, 765, 819, 840, 868, 876, 889, 910,$
$930, 1016, 1020, 1022, 1023, 1071, 1085, 1095, 1116, 1143, 1170, 1190,$
$1240, 1260, 1270, 1302, 1364, 1365, 1428, 1524, 1533, 1581, 1638, 1705,$
$1736, 1778, 1785, 1860, 1905, 1953, 2040, 2044, 2046, 2047, 2170, 2263,$
$2380, 2387, 2540, 2555, 2604, 2635, 2667, 2730, 3048, 3066, 3255, 3556,$
$3570, 3720, 3810, 3906, 3937, 4088, 4092, 4094, 4095, 4445, 5115, 5334,$
$5355, 5461, 6132, 6141, 6510, 7140, 7161, 7620, 7665, 7812, 7874, 7905,$
$8001, 8188, 8190, 8191, 11811, 13335, 14329, 15841, 16383\}$

3. Properties of groups G with $\omega(G) = \omega(PSL_{14}(2))$

In this section we will prove some results which are needed to prove our main theorem.

Proposition 3.1. *If G is a simple group with*

$$\{2,8191\} \subseteq \pi(G) \subseteq \{2,3,5,7,11,13,17,23,31,43,73,89,127,8191\}$$

then $G \cong PSL_{13}(2),\ PSL_{14}(2)$ *or* $PSL_2(8191)$.

Proof. Obviously G is a non-abelian simple group. By the classification of finite simple groups, G is one of the alternating, sporadic or a finite simple group of Lie type. By checking the orders of the sporadic simple groups in [1] we find out that G cannot be a sporadic simple group. If G is isomorphic to an alternating group $\mathbb{A}_n$, then as $8191 \,||G|$ we must have $n \geqslant 8191$ which contradicts the list of prime divisors of $|G|$. Therefore it remains to consider the possibilities of the isomorphism of G with a simple group of Lie type.

Suppose G is isomorphic to a simple group of Lie type in characteristic p. As $p|\,|G|$ we must have $p \in \pi(G)$. The orders of the simple groups of Lie type are given in [1] and these are products of numbers of the form $p^e \pm 1$ or powers of p. We distinguish two cases.

Case(1). $p \in \{7,11,13,17,23,31,43,73,89,127,8191\}$.
Let $p = 8191$. Since $8191^3 \pm 1$ contain primes outside $\pi(G)$ we conclude that only $p^i \pm 1$, $i = 1,2$, can divide $|G|$. But $p^2 - 1 = 8190 \cdot 2^{13}$ and $G = PSL_2(8191)$ is a possibility.

Suppose then that $p \neq 8191$. Let $p = 7$. Since $8191|\,|G|$ we must have $8191|7^e \pm 1$ for some $e \in \mathbb{N}$. But by inspection we see that $7^6 \pm 1$ contains a prime divisor outside $\pi(G)$ and yet $8191 \nmid 7^i \pm 1$ for $1 \leqslant i \leqslant 6$. This contradiction shows that $p = 7$ is impossible. For the rest of possibilities of p similar investigation yields a contradiction.

Case(2). $p \in \{2,3,5\}$.
First we assume $p = 2$. An easy calculation shows that there is no $k \in \mathbb{N}$ such that $8191|2^k + 1$, and if $8191|2^e - 1$, $e \in \mathbb{N}$, then $13|e$. But $2^{13} - 1 = 8191$, $2^{14} - 1 = 3 \cdot 43 \cdot 127$, $2^{15} - 1 = 7 \cdot 31 \cdot 151$, hence $G \cong PSL_{13}(2)$ and $PSL_{14}(2)$ are the only possibilities in this case.

For $p = 3$ we observe that $3^7 - 1 = 2 \cdot 1093$ and $3^4 + 1 = 2 \cdot 41$ contain primes outside $\pi(G)$ yet $8191 \nmid 3^e \pm 1$ for $e \leqslant 7$. Therefore in this case we do not get a possibility. Similar argument rules out $p = 5$ and the proposition is proved. □

Lemma 3.1. *Let G be a finite group with the same spectrum as $PSL_{14}(2)$. Then G is neither a Frobenius nor a 2-Frobenius group.*

Proof. First we will prove that G is a non-solvable group. Suppose G is a solvable group. Let H be a $\{11,13,17\}$ –Hall subgroup of G. By Lemma 2.5, H does not contain elements of order $11 \cdot 13$, $11 \cdot 17$ or $13 \cdot 17$, hence by Theorem 4.1 in [10] we must have $\pi(H) \leqslant 2$, a contradiction. Therefore G is a non-solvable group. We already mentioned that 2-Frobenius groups are always solvable, hence G can not be a 2-Frobenius group.

Suppose G is a Frobenius group with kernel A and complement B. If B is a solvable group, then G will be solvable which is not the case. Therefore B is non-solvable and by the structure of non-solvable Frobenius complements given in [14] B has a normal subgroup B_0 such that $[B : B_0] \leqslant 2$ and $B_0 \cong SL_2(5) \times Z$, where every Sylow subgroup of Z is cyclic and $\pi(Z) \cap \{2, 3, 5\} = \varnothing$. By Lemma 2.5 G does not contain elements of order $5 \cdot 23$ and $5 \cdot 43$. Hence we must have $\{23, 43\} \subseteq \pi(A)$, and since A is nilpotent we conclude that it must contain an element of order $23 \cdot 43$ which is impossible by Lemma 2.5. This contradiction proves that G is not a Frobenius group and the lemma is proved. □

4. Proof of the main theorem

Our main theorem states that if G is a finite group with the same spectrum as $PSL_{14}(2)$, then G is isomorphic to $PSL_{14}(2)$. Since $PSL_{14}(2)$ has a disconnected Gruenberg-Kegel graph, we can deduce the structure of G from the Gruenberg-Kegel's theorem stated earlier. Since by Lemma 3.1 G is not a solvable group, we have by Lemma 2.1 that there is a normal series of G

$$1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$$

such that N and $\frac{G}{G_1}$ are π_1- groups and $\overline{G}_1 = \frac{G_1}{N}$ is a non-abelian simple group. Our aim is to prove that $N = 1$ and $G \cong PSL_{14}(2)$. Keeping the above notation fixed we continue our proof with a series of results.

Proposition 4.1. *$Aut\,(PSL_{14}(2))$ contains an element of order 86.*

Proof. The mapping $\theta : PSL_{14}(2) \to PSL_{14}(2)$ defined by $\theta(A) = \left(A^{-1}\right)^t$, $\forall A \in PSL_{14}(2)$, is an outer automorphism of the group $PSL_{14}(2)$ and therefore we can write $G^+ = Aut\,(PSL_{14}(2)) = PSL_{14}(2) :< \theta >$. By Theorem 4.1 in [7] the group $Aut\,(PSL_{14}(2))$ has two conjugacy classes of elements of order 2 with representatives θI and θJ. By the proof of Theorem 4.1 in [7] we have $|C_{G^+}(\theta J)| = 2\,|SP_{14}(2)|$, hence

$$|C_{G^+}(\theta J)| = 2^{50} \cdot 3^9 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 43 \cdot 127$$

and from the above it is evident that G^+ has an element of order $2 \cdot 43 = 86$ and the proposition is proved. □

By Lemma 2.5 the group $PSL_{14}(2)$ does not contain elements of order 86, hence by Proposition 4.1 we obtain $\omega\,(Aut\,(PSL_{14}(2))) \neq \omega\,(PSL_{14}(2))$.

Before proceeding to the next result, Lemma 2.1 of [13] is useful and it will be stated below.

Theorem 4.1. *Let N be a normal subgroup of the finite group H. Assume that $\frac{H}{N}$ is a Frobenius group with Frobenius kernel A and cyclic Frobenius complement B. If $(|A|\,,\ |N|) = 1$ and A is not contained in $\frac{NC_H(N)}{N}$ then $p\,|B| \in \omega(H)$, for some prime divisor p of $|N|$.*

Since our proof depends on the existence of suitable Frobenius groups we mention a few of them.

Lemma 4.1. *Let F be a field and C be a multiplicative subgroup of F^*. Then*

$$G = \{\lambda_{a,b} : F \to F|\ \lambda_{a,b}(x) = ax + b,\ a \in C,\ b \in F,\ x \in F\}$$

is a Frobenius group with kernel isomorphic to F and complement C. If F is a finite field, then G has $|C|$ linear characters and $\frac{|F|-1}{|C|}$ irreducible complex characters of degree $|C|$. Moreover if p is a prime number and $p \nmid |F|\,|C|$, then the non-linear characters of G may be realized over $\mathbb{Z}_p$.

Proof. Clearly G is a group acting transitively on F. The stabilizer of $0 \in F$ is $G_0 = \{\lambda_{a,0} \mid a \in C\} \cong C$ and for any $0 \neq t \in F$ we have $G_{0,t} = 1$. Therefore G is a Frobenius group with complement C. The kernel is $\{\lambda_{1,b}|\ b \in F\} \cong F$. The rest of proof follows from [7, page 68, Theorem 4.13.8]. $\square$

If F is a finite field with q elements, the Frobenius group G constructed above is denoted by $q\colon \frac{q-1}{|C|}$ where $|C|$ *is* any proper divisor of $q-1$.

Lemma 4.2. $\overline{G}_1 \cong PSL_{14}(2)$.

Proof. Since G and $PSL_{14}(2)$ have the same spectrum, the Gruenberg-Kegel graph of G has two components and they are $\pi_1 = \{2, 3, 5, 7,\ 11, 13,\ 17,\ 23,\ 31,$ $43,\ 73,\ 89,\ 127\}$ and $\pi_2 = \{8191\}$. By the conditions stated before Proposition 4.1 we deduce that π_2 is a component of the Gruenberg-Kegel graph of $\overline{G}_1$, hence $\{2, 8191\} \subseteq \pi\left(\overline{G}_1\right) \subseteq \pi(G)$. Now by Proposition 3.1 we have one of the possibilities $\overline{G}_1 \cong PSL_2(8191)$, $PSL_{13}(2)$ or $PSL_{14}(2)$. We will rule out the first two possibilities as follows.

Case(1). $\overline{G}_1 \cong PSL_2(8191)$.

We have $\overline{G}_1 = \frac{G_1}{N} \cong PSL_2(8191)$. If $N = 1$, then the normal series mentioned at the beginning of section 4 reduces to $1 \trianglelefteq G_1 \trianglelefteq G$. Since the Gruenberg-Kegel graph of G has two components, we obtain $C_G(G_1) = 1$, hence G is isomorphic to a subgroup of $Aut(PSL_2(8191))$. But $|Aut(PSL_2(8191))| = 2\,|PSL_2(8191)| =$ $8192.8191.8190$ and we see that $Aut(PSL_2(8191))$ does not contains an element of order 43, contradicting Lemma 2.5. Therefore we assume $N \neq 1$. By Lemma 4.1, the group $\overline{G}_1 = \frac{G_1}{N} \cong PSL_2(8191)$ has a Frobenius subgroup $\frac{H}{N} = 8191 : \frac{8191-1}{2} = A : B$ where A and B are cyclic groups of orders 8191 and 4095 respectively. Since $8191 \nmid |N|$ we have $(|A|\,,\ |N|) = 1$. Moreover if A is contained in $\frac{NC_H(N)}{N} \cong \frac{C_H(N)}{N \cap C_H(N)}$, then 8191 divides $|C_H(N)|$. But this will imply that $\overline{G}_1$ contain an element of order $8191p$, where p is a prime divisor of $|N|$, contradicting Lemma 2.5. Now all the conditions of Theorem 4.1 are fulfilled, hence $43\,|B| = 43 \cdot 4095 \in \omega(H)$ again contradicting Lemma 2.5. Therefore Case(1) leads to a contradiction.

Case(2). $\overline{G}_1 \cong PSL_{13}(2)$.

As we mentioned in case(1), we have $43 \mid |N|$. Since $\pi(N) \subseteq \pi_1$, hence $8191 \nmid |N|$.

By Lemma 4.1 the group $PSL_{13}(2)$ has a Frobenius subgroup of shape $8191 : \frac{8191-1}{630} = 8191 : 13$. We denote this group by $\frac{H}{N} = A : B$ where $A \cong \mathbb{Z}_{8191}$ and $B \cong \mathbb{Z}_{13}$. With the same reasoning as in case(1) we see that all the conditions of Theorem 4.1 are fulfilled, hence $43 \cdot 8191 \in \omega(H)$ contradicting Lemma 2.5.

This final contradiction leaves the remaining case, i.e. $\overline{G}_1 \cong PSL_{14}(2)$, as the only possibility and the proof is complete. □

Lemma 4.3. $N = 1$.

Proof. We have the normal series

$$1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$$

such that N and $\frac{G}{G_1}$ are π_1-groups and by Lemma 4.2 $\overline{G}_1 = \frac{G_1}{N} \cong PSL_{14}(2)$. Suppose $N \neq 1$. Without loss of generality we may assume that $N = O_r(G)$ for some prime $r \in \pi_1$ with N an elementary abelian group and $C_{G_1}(N) = N$.

The group $PSL_{14}(2) = GL_{14}(2)$ has the following subgroup. Let A be an element of order $2^{13} - 1$ in $PSL_{13}(2)$. The set

$$L = \left\{ \begin{bmatrix} 1 & a_1 & a_2 \cdots a_{13} \\ 0 & & X \end{bmatrix} |a_i \in \mathbb{Z}_2,\ X \in < A >,\ 1 \leqslant i \leqslant 13 \right\}$$

is a subgroup of $PSL_{14}(2)$. It is easy to verify that L is a Frobenius group of the shape $L = 2^{13} : (2^{13} - 1)$. Now if $r \neq 2$, then by Theorem 4.1 the group G would contain an element of order $(2^{13} - 1)r$, contradicting Lemma 2.5.

Thus we assume $r = 2$. Because of our assumption N is an elementary abelian 2-group of order 2^k and $\overline{G}_1 = \frac{G_1}{N}$ is isomorphic to $PSL_{14}(2) = GL_{14}(2)$. The group $GL_{14}(2)$ has a subgroup isomorphic to $GL_3(2) \times GL_{11}(2)$ and if σ is an element of order 7 in $GL_{14}(2)$, then it is easy to see that $C_{GL_{14}(2)}(\sigma) \cong < \sigma > \times GL_{11}(2)$. Modules N for the group $GL_3(2) = L_3(2)$ can be constructed using the Group Algorithm Programming [15]. Using these modules we deduce that $C_N(\sigma) = K \neq 1$ for any module N of $GL_3(2)$. But considering K as a module for $GL_{11}(2)$ we form the subgroup $H = KGL_{11}(2)$ of G_1 and since $\frac{H}{K} \cong GL_{11}(2)$, by [5] it has a Frobenius subgroup of the shape 23:11, hence by Theorem 4.1 we deduce that $KGL_{11}(2)$ has an element of order 22. But since $KGL_{11}(2)$ is centralized by σ this implies that $G_1 = NGL_{14}(2)$ has an element of order $7 \times 22 = 154$, contradicting lemma 2.5. This final contradiction shows that $N = 1$ and the lemma is proved. □

Proposition 4.2. $G \cong PSL_{14}(2)$.

Proof. By Lemma 4.3 we have $N = 1$. Hence the normal series reduces to $1 \trianglelefteq G_1 \trianglelefteq G$. Since the Gruenberg-Kegel graph of G has two components we obtain $C_G(G_1) = 1$. Hence G is isomorphic to a subgroup of $Aut(G_1) = Aut\,(PSL_{14}(2)) = PSL_{14}(2) : 2$. Therefore $PSL_{14}(2) \leqslant G \leqslant PSL_{14}(2) : 2$. By Proposition 4.1 we have $\omega\,(PSL_{14}(2) : 2) \neq \omega(G)$, thus $G = PSL_{14}(2)$ as claimed. Therefore the proposition and our main result are proved. □

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