ACTA MATHEMATICA VIETNAMICA
Volume 33, Number 1, 2008, pp. 17-26

EXISTENCE RESULTS FOR INITIAL VALUE PROBLEMS FOR NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

A. SGHIR

ABSTRACT. Existence results for initial value problems for neutral functional differential inclusions are given. The results are proved by direct applications of topological transversality methods based on the degree theory for χ-Lipschitz multimappings.

1. INTRODUCTION

The topological transversality method (see $[1-3]$), which requires the existence of a priori bounds for solutions, is used to establish existence results for boundary value problems. Recently this method was applied to initial value problems for functional differential inclusions. See for example $[11, 17, 20]$. The advantage of this method is that it yields simultaneously the existence of a solution and the maximal interval of existence.

Our aim in this paper is to apply this method to initial value problems for neutral functional differential inclusions in Banach spaces in order to generalize some results such that papers of Ntouyas and al [16] and O'Regan and Lee [18]...

Initial value problems for neutral functional differential equations have been studied by many authors, see for example $[8, 10, 12, 16, ..]$ and references therein.

The paper is organized as follows. In section 2, we present notations and definitions. In section 3, we give a result of existence for initial value problem for neutral functional differential inclusions, by assuming a priori bounds on solutions and using an argument presented in [16], we give immediately our existence theorem for initial value problem for neutral functional differential inclusions. Finally in section 4, we apply our results to a control system governed by an integro-differential equation.

2. PRELIMINARIES

Let $(E, |.|_E)$ be a Banach space. For a fixed $r > 0$, we define

Received June 13, 2006; in revised form February 7, 2007.
1991 *Mathematics Subject Classification.* Primary 34A60, 34K15, 34K30, 34K40.
Key words and phrases. Neutral functional differential inclusion, existence, χ-Lipschitz multimapping, degree theory.

$\mathcal{C} = C([-r, 0]; E)$ to be the Banach space of continuous E-valued functions on $J = [-r, 0]$ with the usual supremum norm $\|.\|$.

For any continuous function x defined on the interval $[-r, \omega]$ $(\omega > 0)$, and any $t \in I = [0, \omega]$ we denote by x_t the element of $\mathcal{C}$, defined by

$$x_t(\theta) = x(t + \theta),\ \theta \in J.$$

The symbol $CC(E)$ denotes the collection of nonempty compact convex subsets of E.

Let B be a bounded set in E, the Kuratowski measure of noncompactness of $B, \chi(B)$ (see for example, [5, 15]) is defined as

$$\chi(B) = \inf\{d > 0 : B \text{ has a finite covering of diameter} \leqslant d\}.$$

Now, let K be a convex closed subset of E, $\Omega \subset K$ an open bounded set for the relative topology, $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω relative to K respectively. Let $\Gamma : \overline{\Omega} \to CC(K)$ be a closed and χ-Lipschitz map i.e.

$$\exists k \in [0, 1[: \forall B \subset \overline{\Omega}\ \ \chi(\Gamma(B)) \leqslant k\chi(B).$$

Assume that the fixed point set, $Fix\Gamma = \{x \in \overline{\Omega} : x \in \Gamma(x)\}$, has no intersection with the boundary $\partial\Omega$. Then, following the works [2, 3], the topological degree $\deg(\Gamma, \Omega)$ can be defined. This topological degree has the following usual properties:

i)

$$\deg(\Gamma, \Omega) = \begin{cases} 1 \text{ if } \Gamma(\overline{\Omega}) \subset \Omega \\ 0 \text{ if } \Gamma(\overline{\Omega}) \subset K \backslash \Omega. \end{cases}$$

ii) If the closed χ-Lipschitz maps $\Gamma_0, \Gamma_1 : \overline{\Omega} \to CC(K)$ are homotopic, i.e. there exists a closed χ-Lipschitz family of maps, $H : \overline{\Omega} \times [0, 1] \to CC(K)$ such that

$$[\bigcup_{\lambda \in [0,1]} FixH(., \lambda)] \cap \partial\Omega = \emptyset \ \text{ and } H(., 0) = \Gamma_0, H(., 1) = \Gamma_1,$$

then $\deg(\Gamma_0, \Omega) = \deg(\Gamma_1, \Omega)$.

iii) If $\deg(\Gamma, \Omega) \neq 0$ then $\emptyset \neq Fix\Gamma \subset \Omega$.

In the sequel we suppose that E is a separable Banach space.

3. Main results

In this section we examine initial value problems for neutral functional differential inclusions of the type

$$P(\varphi) \begin{cases} \frac{d}{dt}[x(t) - g(t, x_t)] \in F(t, x_t),\ t \in I \\ x_0 = \varphi,\ \varphi \in \mathcal{C} \end{cases}$$

where g and F satisfy the following assumptions:

(H_1) $g : I \times \mathcal{C} \to E$ is a continuous function and satisfies the following conditions:

$i)$ there exists $k_1 \in [0, 1[$ such that for every nonempty bounded set $B \subset \mathcal{C}$ and for all $t \in I$,

$$\chi(g(t, B)) \leqslant k_1 \chi_0(B)$$

where χ (resp. χ_0) is the Kuratowski measure of noncompactness in E (resp. $\mathcal{C}$);

$ii)$ for any bounded set B in $C_\omega := C([-r, \omega]; E)$, the set $\{t \longmapsto g(t, x_t) : x \in B\}$ is equicontinuous in $C(I; E)$.

(H_2) $F : I \times \mathcal{C} \to CC(E)$ satisfies the following conditions:

$F_1)$ for every $\psi \in \mathcal{C}$, the multimapping $F(., \psi)$ admits a measurable selection. $F_1)$ is satisfied if for example $F(., \psi)$ is measurable for all $\psi \in \mathcal{C}$ (see [4]);

$F_2)$ for almost all $t \in I$, the multimapping $F(t, .)$ is upper semi-continuous i.e. $\{\psi \in \mathcal{C} : F(t, \psi) \subset V\}$ is an open subset of $\mathcal{C}$ for every open set $V \subset E$;

$F_3)$ for every nonempty bounded set $B \subset \mathcal{C}$ there exists $m \in L^1_+(I)$ such that, for all $\psi \in B$ and almost all $t \in I$

$$\|F(t, \psi)\| := \sup\{|y|_E : y \in F(t, \psi)\} \leqslant m(t)h(\|\psi\|)$$

where h is a continuous nondecreasing function on $[0, +\infty[$ and is positive on $]0, +\infty[$;

$F_4)$ there exists $k_2 \geqslant 0$ such that, for every nonempty bounded set $B \subset \mathcal{C}$ and all $t \in I$

$$\chi(F(t, B)) \leqslant k_2 \chi_0(B)$$

where $k_1 + k_2\omega < 1$.

For $x \in C_\omega$, let $G_x : I \to CC(E)$ be the multimapping defined by $G_x(t) = F(t, x_t)$. From conditions $F_1) - F_3)$ it follows that for all $x \in C_\omega$

$$I^1_{G_x} := \{f \in L^1(I; E) : f(t) \in G_x(t) \text{ a.e.}\} \neq \emptyset \text{ (see, for example [21]).}$$

Definition 3.1. A function $x \in C_\omega$ is said to be a solution of $P(\varphi)$, if there exists a function $f \in I^1_{G_x}$ such that

$$x(t) = \begin{cases} \varphi(0) - g(0, \varphi) + g(t, x_t) + \int_0^t f(s)ds & \text{if } t \in I \\ \varphi(t) & \text{if } t \in J. \end{cases}$$

We put $C_* = \{y \in C_\omega : y_0 = 0\}$ and

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(0), \ t \in I \\ \varphi(t), \ t \in J. \end{cases}$$

In order to investigate the existence of solutions of $P(\varphi)$ we shall use the multivalued integral operator Γ defined on the space C_* by

$$\Gamma(y) = \left\{ z \in C_*\colon \ z(t) = -g(0, \varphi) + g(t, y_t + \widetilde{\varphi}_t) + \int_0^t f(s)ds, \ t \in I \text{ and } f \in I^1_{G_{y+\widetilde{\varphi}}} \right\}$$

It is easy to see that x is a solution of $P(\varphi)$ if and only if $x = y + \widetilde{\varphi}$ where $y \in \Gamma(y)$ and the set $\Gamma(y)$ is nonempty and convex for every $y \in C_*$.

Now, let us describe the main properties of Γ.

Lemma 3.1. *The operator* Γ *is closed.*

Proof. We give a sketch of proof; see [11, 14, 20].

Let $(y_n), (z_n) \subset C_*$ be two sequences with $z_n \in \Gamma(y_n)$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty}(y_n) = y$, $\lim\limits_{n\to+\infty}(z_n) = z$. We shall prove that $z \in \Gamma(y)$. For every $n \in \mathbb{N}$, we have

$$z_n(t) = -g(0,\varphi) + g(t, y_t^n + \widetilde{\varphi}_t) + \int_0^t f_n(s)ds, \ t \in I$$

where $f_n(s) \in F(s, y_s^n + \widetilde{\varphi}_s)$ a.e. From condition F_4) it follows that for almost every $t \in I$,

$$\chi(\{f_n(t) : n \in \mathbb{N}\}) \leqslant k_2\chi_0(\{y_t^n + \widetilde{\varphi}_t : n \in \mathbb{N}\}) = 0,$$

hence for almost every $t \in I$, $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact in E, but from condition F_3)

$$|f_n(t)|_E \leqslant m(t)h(\|y_t^n + \widetilde{\varphi}_t\|) < +\infty$$

for every $n \in \mathbb{N}$. Then from Diestel's theorem (see [6]) it follows that the sequence (f_n) is relatively weakly compact in $L^1(I; E)$. We can assume that (f_n) converges weakly to a function $f \in L^1(I; E)$ (if necessary we can use a subsequence of (f_n)). By Mazur's theorem [7], there exists a sequence $(\widetilde{f_m})$ such that $\lim\limits_{m\to+\infty}(\widetilde{f_m}) = f$ in $L^1(I; E)$ and from condition F_2), we obtain $f \in I^1_{G_{y+\widetilde{\varphi}}}$ and then $z \in \Gamma(y)$. ☐

Lemma 3.2. *For every bounded set* $\Omega \subset C_*$*, the set* $\Gamma(\Omega)$ *is bounded and equicontinuous.*

Proof. i) Let $y \in \Omega$ and $z \in \Gamma(y)$, then for some $f \in I^1_{G_{y+\widetilde{\varphi}}}$ we have for $t \in I$

$$\begin{aligned} |z(t)|_E &\leqslant |g(0,\varphi)|_E + |g(t, y_t + \widetilde{\varphi}_t)|_E + \int_0^t |f(s)|_E\, ds \\ &\leqslant |g(0,\varphi)|_E + |g(t, y_t + \widetilde{\varphi}_t)|_E + \int_0^t m(s)h(\|y_s + \widetilde{\varphi}_s\|)ds \\ &\leqslant |g(0,\varphi)|_E + M_1 + h(\rho + \|\varphi\|)\,\|m\|_1 \\ &\leqslant |g(0,\varphi)|_E + M(1 + \|m\|_1) \end{aligned}$$

where $\|m\|_1 = \int_0^\omega m(s)ds$ and $M = \max(M_1, h(\rho + \|\varphi\|))$ with

$$M_1 = \sup\{|g(t, y_t + \widetilde{\varphi}_t|_E : t \in I, \ y \in \Omega\}$$

and $\rho > 0$ such that $\forall y \in \Omega$, $\|y\|_\omega := \sup\limits_{t\in[-r,\omega]} |y(t)|_E < \rho$.

ii) For every $t, t_0 \in I$ and $z \in \Gamma(y)$ where $y \in \Omega$, one has

$$|z(t) - z(t_0)|_E \leqslant |g(t, y_t + \widetilde{\varphi}_t) - g(t_0, y_{t_0} + \widetilde{\varphi_{t_0}})|_E + \left| \int_{t_0}^{t} |f(s)|_E \, ds \right|$$

where $f \in I^1_{G_{y+\widetilde{\varphi}}}$ then, for $\varepsilon > 0$ and by condition ii) of assumption (H_1), there exists $\eta_1 > 0$ such that for $|t - t_0| < \eta_1$, $|g(t, y_t + \widetilde{\varphi}_t) - g(t_0, y_{t_0} + \widetilde{\varphi_{t_0}})|_E < \frac{\varepsilon}{2}$, and from the integral absolute continuity there exists $\eta_2 > 0$ such that for $|t - t_0| < \eta_2$, $\left| \int_{t_0}^{t} |f(s)|_E \, ds \right| < \frac{\varepsilon}{2}$, we conclude that

$$\forall \varepsilon > 0, \exists \eta > 0 : |t - t_0| < \eta, \; \forall z \in \Gamma(\Omega), \; |z(t) - z(t_0)|_E < \varepsilon.$$

□

Lemma 3.3. *For every bounded set $\Omega \subset C_*$,*

$$\chi_0^1(\Gamma(\Omega)) \leqslant (k_1 + k_2\omega)\chi_0^1(\Omega),$$

where χ_0^1 is the Kuratowski measure of noncompactness in C_*.

Proof. Using Lemma 3.2, the set $\Gamma(\Omega)$ is bounded and equicontinuous, and by Ambrosetti's theorem [5, 15]

$$\chi_0^1(\Gamma(\Omega)) = \sup_{t \in [-r, \omega]} \chi[(\Gamma(\Omega))(t)] = \sup_{t \in I} \chi[(\Gamma(\Omega))(t)].$$

For each $t \in I$,

$$\begin{aligned}
& \chi[(\Gamma(\Omega))(t)] \\
\leqslant \; & \chi(\{g(t, y_t + \widetilde{\varphi}_t) \colon \, y \in \Omega\}) + \chi(\{\int_0^t f(s) ds \text{ where } f \in I^1_{G_{y+\widetilde{\varphi}}}, \; y \in \Omega\}) \\
\leqslant \; & k_1 \chi_0(\Omega_t + \widetilde{\varphi}_t) + k_2 \omega \chi_0(\Omega_t + \widetilde{\varphi}_t) \\
\leqslant \; & k_1 \chi_0(\Omega_t) + k_2 \omega \chi_0(\Omega_t) \\
\leqslant \; & k_1 \chi_0^1(\Omega) + k_2 \omega \chi_0^1(\Omega) \text{ (see, for example [19])}
\end{aligned}$$

where $\Omega_t + \widetilde{\varphi}_t = \{y_t + \widetilde{\varphi}_t : y \in \Omega\}$. □

The properties of the integral multimapping Γ described above allow to use the theory of the relative topological degree of χ-Lipschitz multimappings for searching solutions of problem $P(\varphi)$.

Theorem 3.1. *Assuming that (H_1), (H_2) hold and there exists a positive constant ρ such that $\|x\|_\omega < \rho$ for each solution x of the following problem*

$$P_\lambda(\varphi) \begin{cases} \frac{d}{dt}[x(t) - \lambda g(t, x_t)] \in \lambda F(t, x_t), & t \in I, \; \lambda \in [0, 1] \\ x_0 = \varphi, \end{cases}$$

then the initial value problem $P(\varphi)$ has at least one solution.

Proof. In order to apply the topological degree principle, we take

$$\Omega = \{y \in C_* : \|y\|_\omega < 2\rho\}, \; K = \overline{co}(\{0\} \cup \Gamma(\overline{\Omega})) \text{ and } \Omega_K = \Omega \cap K.$$

We know that the multimapping $\Gamma : \overline{\Omega}_K \to CC(K)$ is closed and χ_0^1-Lipschitz. Furthermore $\forall \lambda \in [0,1], \forall y \in Fix\lambda\Gamma$ then $x = y + \widetilde{\varphi}$ is a solution of $P_\lambda(\varphi)$, thus $\|y\|_\omega \leqslant \|y + \widetilde{\varphi}\|_\omega + \|\widetilde{\varphi}\|_\omega < 2\rho$, and hence $Fix\lambda\Gamma \cap \partial\Omega_K = \emptyset$. Take $\Gamma_0 = 0$ and $\Gamma_1 = \Gamma$, since there exists a closed χ-Lipschitz family of maps $H : \overline{\Omega}_K \times [0,1] \to CC(K)$ such that

$$[\underset{\lambda\in[0,1]}{\cup} FixH(.,\lambda)] \cap \partial\Omega_K = \emptyset \text{ and } H(.,0) = \Gamma_0, H(.,1) = \Gamma_1,$$

(it suffices to take $H(.,\lambda) = \lambda\Gamma$), then Γ_0 and Γ_1 are homotopic. So, $\deg(\Gamma_0, \Omega_K) = \deg(\Gamma_1, \Omega_K)$, but $\deg(\Gamma_0, \Omega_K) = 1$ as $0 \in \Omega_K$,
hence $\deg(\Gamma, \Omega_K) = \deg(\Gamma_1, \Omega_K) = 1$, therefore $\emptyset \neq Fix\Gamma \subset \Omega_K$. Thus $P(\varphi)$ has at least a solution. ☐

We suppose below that g and F satisfy the assumptions

(H_1') g satisfies (H_1) and there exist constants $0 \leqslant c_1 < 1$ and $c_2 \geqslant 0$ such that

$$|g(t,\psi)|_E \leqslant c_1 \|\psi\| + c_2 \text{ for every } t \in I \text{ and } \psi \in \mathcal{C}.$$

(H_2') F satisfies the conditions $(F_1), (F_2), (F_4)$ and (F_3'): there exists $m \in L^1_+(I)$ such that, for all $\psi \in \mathcal{C}$ and almost all $t \in I$

$$\|F(t,\psi)\| \leqslant m(t)h(\|\psi\|),$$

where h is a continuous nondecreasing function on $[0,+\infty[$ and is positive on $]0,+\infty[$.

Theorem 3.2. *Assuming that $(H_1'), (H_2')$ hold, then the initial value problem $P(\varphi)$ has a solution if*

$$\frac{1}{1-c_1}\int_0^\omega m(s)ds < \int_c^{+\infty} \frac{ds}{h(s)}$$

where $c = \frac{1}{1-c_1}[(1+c_1)\|\varphi\| + 2c_2]$.

Proof. To prove the existence of a solution of $P(\varphi)$ we apply Theorem 3.1. In order to apply this theorem we must establish a priori bounds for the initial value problem $P_\lambda(\varphi)$. Let x be a solution of $P_\lambda(\varphi)$. From

$$x(t) = \varphi(0) - \lambda g(0,\varphi) + \lambda g(t,x_t) + \lambda\int_0^t f(s)ds, \ t \in I \text{ and } f \in I^1_{G_x},$$

we have

$$|x(t)|_E \leqslant (1+c_1)\|\varphi\| + c_1\|x_t\| + 2c_2 + \int_0^t m(s)h(\|x_s\|)ds.$$

Using an argument presented in [16], we put

$$u(t) = \sup\{|x(s)|_E : -r \leqslant s \leqslant t\} \text{ for } t \in I$$

and let $t^* \in [-r,t]$ be such that $u(t) = |x(t^*)|_E$.

If $t^* \in [0,t]$, by the previous inequality we have

$$\begin{aligned} u(t) &= |x(t^*)|_E \leqslant (1+c_1)\|\varphi\| + c_1\|x_{t^*}\| + 2c_2 + \int_0^{t^*} m(s)h(\|x_s\|)ds \\ &\leqslant (1+c_1)\|\varphi\| + c_1 u(t) + 2c_2 + \int_0^t m(s)h(u(s))ds,\ t \in I \end{aligned}$$

or

$$u(t) \leqslant \frac{1}{1-c_1}[(1+c_1)\|\varphi\| + 2c_2 + \int_0^t m(s)h(u(s))ds],\ t \in I. \quad (*)$$

If $t^* \in J$, then $u(t) = \|\varphi\|$ and $(*)$ is verified.

Put

$$v(t) = \frac{1}{1-c_1}[(1+c_1)\|\varphi\| + 2c_2 + \int_0^t m(s)h(u(s))ds],\ t \in I,$$

we have $u(t) \leqslant v(t),\ t \in I, v(0) = c$ and

$$v'(t) = \frac{1}{1-c_1} m(t)h(u(t)) \leqslant \frac{1}{1-c_1} m(t)h(v(t)),\ t \in I.$$

Then

$$\int_{v(0)}^{v(t)} \frac{ds}{h(s)} \leqslant \frac{1}{1-c_1}\int_0^{\omega} m(s)ds < \int_c^{+\infty} \frac{ds}{h(s)},\ t \in I.$$

This inequality implies that there exists a constant $\rho > 0$ such that $v(t) < \rho$ and hence $\|x\|_{\omega} < \rho$, where ρ depends only on ω and the functions m and h. □

Remark 1. If $g = 0$, the initial value problem $P(\varphi)$ becomes

$$(P)\quad \begin{cases} x'(t) \in F(t, x_t),\ t \in I \\ x_0 = \varphi. \end{cases}$$

In this case Theorem 3.2 leads immediately to the following corollary.

Corollary 3.1. *Assuming that (H_2') hold, then the initial value problem (P) has a solution if*

$$k_2\omega < 1 \text{ and } \int_0^{\omega} m(s)ds < \int_{\|\varphi\|}^{+\infty} \frac{ds}{h(s)}.$$

4. Application

In this section we shall consider some applications of the obtained results to the optimal control of the systems described by the integral-differential equations having the following form

$$(N)\begin{cases} \frac{\partial}{\partial t}[v(t,s)-\int_{-r}^{0}k(t+\theta)\ v(t+\theta,s)d\theta]=f(t,s,v(t,s),\int_0^\omega q(t,s,\nu)v(t,\nu)d\nu) \\ +\sum\limits_{i=1}^{m}u_i(t)e_i(t,s,v(t,s)) \\ \text{where } t,s\in I \text{ and } (u_1(t),...,u_m(t))\in\mathcal{U}(v(t-r,.))\ (r>0) \\ v(t,0)=v(t,\omega),\ t\in I \\ v(\theta,s)=\varphi(\theta)(s),\ (\theta,s)\in J\times I. \end{cases}$$

Put $E=\{y\in C(I;\mathbb{R}):y(0)=y(\omega)\}$ and suppose that if $q(t,s,\nu)$ is the kernel, then $q(t,0,\nu)=q(t,\omega,\nu)$ and the integral operator $Q(t)$ defined by

$$(Q(t)y)(s)=\int_0^\omega q(t,s,\nu)y(\nu)d\nu$$

is compact in the space E and continuous with respect to t.

The function $f:I^2\times\mathbb{R}^2\to\mathbb{R}$ is such that

$$f(t,0,\nu,\tau)=f(t,\omega,\nu,\tau),$$

$$|f(t,s,\nu,\tau)|\leqslant m_1(t)h_1(|\nu|)$$

where $m_1\in L^1_+(I)$, h_1 is a continuous nondecreasing function on $[0,+\infty[$,

$$|f(t,s,\nu,\tau)-f(t,s,\nu_1,\tau)|\leqslant\gamma(s)\left|\nu-\nu_1\right|,\quad(\alpha)$$

where $\gamma\in C(I)$ with $\|\gamma\|\leqslant\alpha$ and generates the continuous operator $\mathbf{f}\colon I\times E^2\to E$ defined by

$$\mathbf{f}(t,y,z)(s)=f(t,s,y(s),z(s)).$$

The functions $e_i:I^2\times\mathbb{R}\to\mathbb{R}$ are such that

$$e_i(t,0,\nu)=e_i(t,\omega,\nu),\ \text{and}\ |e_i(t,s,\nu)|\leqslant M_i,\ i=1,...,m,$$

and generate the operators $\mathbf{e}_i:I\times E\to E$ defined by

$$\mathbf{e}_i(t,y)(s)=e_i(t,s,y(s)),\ i=1,...,m$$

and $t\longmapsto\mathbf{e}_i(t,.)$ are continuous mappings and

$$|\mathbf{e}_i(t,y)-\mathbf{e}_i(t,z)|_E\leqslant\beta_i\left|y-z\right|_E,\ t\in I\text{ and }y,z\in E.\ (\beta)$$

The controls u_i are measurable functions satisfying the following delay feedback condition

$$(u_1(t),...,u_m(t))\in\mathcal{U}(v(t-r,.))$$

where $\mathcal{U}:E\to CC(U)$ is an upper semicontinuous multimapping, $U\subset\mathbb{R}^m$ is a bounded closed convex set.

Let $\beta = \sup_{(u_1(t),...,u_m(t))\in U} \sum_{i=1}^{m} |u_i(t)|\, \beta_i$. Define the multimapping $F : I \times \mathcal{C} \to P(E)$ by

$$F(t,\psi)$$
$$=\mathbf{f}(t,\psi(0),Q(t)\psi(0))+\{\sum_{i=1}^{m} u_i(t)\mathbf{e}_i(t,\psi(0)) \colon\ (u_1(t),...,u_m(t)) \in \mathcal{U}(\psi(-r))\}.$$

For this multimapping the assumption (H_2') is fulfilled (see for example [11]). Indeed, (F_1) follows from the continuity of **f**, $\mathbf{e}_i$ and measurability of u_i $(i = 1, ..., m)$, (F_2) follows from the upper semicontinuity of $\mathcal{U}$, (F_3') follows from the boundedness of U with

$$m(t) = \max[m_1(t), m_2(t) := \sup_{(u_1(t),...,u_m(t))\in U} \sum_{i=1}^{m} |u_i(t)|\, M_i]$$

$$h(t) = h_1(t) + 1,$$

and (F_4) follows from (α) and (β) where $k_2 = \alpha + \beta$.

We define the mapping $g : I \times \mathcal{C} \to E$ by

$$g(t,\psi)(s) = \int_{-r}^{0} k(t+\theta)\psi(\theta)(s)d\theta$$

where $k : [-r,\omega] \to \mathbb{R}$ is a continuous function. It is trivial to see that the assumption (H_1') is verified with $k_1 = r \sup_{t\in[-r,\omega]} |k(t)| = c_1$ and $c_2 = 0$.

If we suppose that $k_1 + \omega k_2 < 1$, and

$$\frac{1}{1-c_1}\int_0^{\omega} m(s)ds < \int_c^{+\infty} \frac{ds}{h(s)}$$

where $c = \frac{1+c_1}{1-c_1}\|\varphi\|$, then the problem (N) has at least one solution.

References

[1] R. Akhmerov, M. Kamenskii, A. Potapov, A. Rodkina and B. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Nauka, Novosibirsk (1986) (in Russian).

[2] Y. G. Borisovich, B. D. Gel'Man, A. D. Myshkis and V. V. Obukhovskij, *Topological methods in the fixed-point theory of multivalued maps*, Russ. Math. Surveys **35** (1980), 65-143.

[3] Y. G. Borisovich, B. D. Gel'Man, A. D. Myshkis and V. V. Obukhovskij, *Multivalued mappings, Mathematical Analysis*, J. Sov. Math. **24** (1984), 719-791.

[4] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, **580**, Springer-Verlag, 1977.

[5] K. Deimling, *Nonlinear Analysis*, Springer-Verlag, 1985.

[6] J. Diestel, *Remarks on weak compactness in* $L^1(\mu;X)$, Glasgow Math. J. **18** (1977), 87-91.

[7] I. Ekeland and R. Temam,*Convex Analysis and Variational Problems*, North-Holland Amsterdam, 1976.

[8] L. Erbe, W. Krawcewicz and J. Wu, *A composite coincidence degree with applications to boundary value problems of neutral equations*, Trans. Amer. Math. Soc. **335** (1993) 459-478.

[9] B. Gel'Man and V. Obukhovskii, *Some fixed point theorems for multivalued mappings of a condensing type, in algebraic problems of analysis and topology*, Voronezh University Press (1990) 110-115.
[10] J. Hale, *Functional Differential Equations*, Springer-Verlag, 1977.
[11] M. Kamenskii and V. Obukhovskii, *Condensing multioperators and periodic solutions of parabolic functional differential inclusions in Banach spaces*, Nonlinear Anal. **20** (7) (1993), 781-792.
[12] V. Kolmanovskij and A. Myshkis, *Applied theory of functional-differential equations*, Mathematics and its applications (Soviet series).
[13] M. Krasnoselskii, P. Zabreiko, E. Pustyl'nik and P. Sobolevskii, *Integral Operators in Spaces of Summable Functions*, Noordhoff, Groningen, 1975.
[14] V. Lakshmikantham and R. Bernfeld Stephen, *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, 1974.
[15] H. Robert and J. R. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, John-Wiley, 1976.
[16] S. Ntouyas, Y. Sficas and P. Tsamatos, *Existence results for initial value problems for neutral functional differential equations*, J. Diff. Equations **114** (1994), 523-537.
[17] V. V. Obukhovskij, *On semilinear functional-differential inclusions in a Banach space and control systems of a parabolic type*, Avtomatika **3** (1991), 73-81.
[18] D. O'Regan and J. Lee, *Topological transversality: Applications to initial value problems*, Ann. Pol. Math. **48** (1988) 247-253.
[19] M. Roseau, *Equations differentielles*, Masson, Paris, 1976.
[20] A. Sghir, *Existence results for functional differential inclusions in Banach spaces*, to appear in Journal Math-Recherche et Applications, Kénitra-Maroc.
[21] A. Tolstongov, *Differential Inclusions in Banach Spaces*, Nauka, Novosibirsk, 1986.

Université Cadi Ayyad
Faculté des Sciences Semlalia
Département de Mathématiques
B. P. 2390. Marrakech, Morocco