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SINGULARITY OF PROBABILITY MEASURE IN FRACTAL GEOMETRY

VU THI HONG THANH, NGUYEN NHUY AND LE XUAN SON

Abstract. Let μ be the probability measure induced by $S = \sum_{i=1}^{\infty} 3^{-i} X_i$, where $X_1, X_2, \dots$ are independent identically distributed random variables each taking integer values $0, 1, a$ with equal probability $1/3$, where a is a parameter. Let $\alpha(s, a)$ (resp. $\underline{\alpha}(s, a)$, $\overline{\alpha}(s, a)$) denote the local dimension (resp. lower, upper local dimension) of $s \in \text{supp } \mu$, and let

$$E(a) = \{\alpha : \alpha(s, a) = \alpha \text{ for some } s \in \text{supp } \mu\},$$

$$\overline{\alpha}(a) = \sup\{\overline{\alpha}(s, a) : s \in \text{supp } \mu\};\ \underline{\alpha}(a) = \inf\{\underline{\alpha}(s, a) :\ s \in \text{supp } \mu\}.$$

In this paper, we prove that for $a = 4$ we have

$$\overline{\alpha}(4) = 1,\ \underline{\alpha}(4) = 1 - \frac{\log(1 + \sqrt{5}) - \log 2}{\log 3} \quad \text{and} \quad E = [\underline{\alpha}(4),\ \overline{\alpha}(4)].$$

1. Introduction

Let $X_1, X_2, \dots$ be a sequence of independent identically distributed random variables each taking values $a_1, a_2, \dots, a_m$ with respective probabilities p_1, p_2, $\dots$, p_m. For $0 < \rho < 1$, let

$$S = \sum_{i=1}^{\infty} \rho^i X_i,$$

and let μ (depending on ρ) be the probability measure induced by S, i.e.,

$$\mu(A) = \text{Prob}\{\omega :\ S(\omega) \in A\}.$$

By Jessen and Wintner's "pure theorem" [9], the measure μ is either purely singular or absolutely continuous.

If μ is purely singular, the degree of singularities of μ can be analyzed on a pointwise basis by studying its local dimensions. In this case

$$\lim_{h \to 0^+} \frac{\mu(B_h(x))}{h} = 0$$

for almost all x in the support of μ, where $B_h(x)$ denotes the ball centered at x with radius h.

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On the other hand, this value may be infinite at other points. Therefore, it is natural to find $\alpha > 0$ so that

$$0 < \lim_{h\to 0^+} \inf \frac{\mu(B_h(x))}{h^\alpha} \leqslant \lim_{h\to 0^+} \sup \frac{\mu(B_h(x))}{h^\alpha} < \infty.$$

If such an α exists, then for small h, $\mu(B_h(x)) \approx Ch^\alpha$ for some constant $C > 0$, or $\alpha \approx \frac{\log \mu(B_h(x))}{\log h}$. This suggests the *lower local dimension* $\underline{\alpha}(s)$ of μ for $s \in$ supp μ

$$\underline{\alpha}(s) = \lim_{h\to 0^+} \inf \frac{\log \mu(B_h(s))}{\log h}. \tag{1.1}$$

The *upper local dimension*, denoted by $\overline{\alpha}(s)$, is defined similarly by using the upper limit. If the two limits equal, then the common value is called the *local dimension* of μ at s, denoted by $\alpha(s)$. Thus, the local dimension is a function defined on supp μ. Denote

$$\overline{\alpha} = \sup\{\overline{\alpha}(s) : s \in \text{supp } \mu\}; \ \underline{\alpha} = \inf\{\underline{\alpha}(s) : s \in \text{supp } \mu\};$$

$$\alpha^* = \sup \{\alpha(s) : s \in \text{supp } \mu\}; \ \alpha_* = \inf\{\alpha(s) : s \in \text{supp } \mu\};$$

$$E = \{\alpha \in \mathbb{R} : \alpha(s) = \alpha \text{ for some } s \in \text{supp } \mu\}.$$

In this note we are interested in identifying the domain E in the special case when $m = 3$, $\rho = p_1 = p_2 = p_3 = 1/3$, $a_1 = 0, a_2 = 1$ and $a_3 = a$ is a parameter. For $a = 3$ it is known [8] that $E = [\underline{\alpha}, \ \overline{\alpha}] = [2/3, \ 1]$. We prove

Theorem 1.1. (Main Theorem). *For $a = 4$ we have*

$$\overline{\alpha} = \alpha^* = 1; \underline{\alpha} = \alpha_* = 1 - \frac{\log(1+\sqrt{5}) - \log 2}{\log 3} \text{ and } E = [\underline{\alpha}, \overline{\alpha}].$$

As in [8], the proof of our Main Theorem uses a combinatoric approach which depends on some careful counting the multiple representations of n-partial sum s_n of $s = \sum_{i=1}^{\infty} 3^{-i}x_i \in$ supp μ and the associated probabilities.

The paper is organized as follows. In Section 2 we derive some lemmas and propositions, which will reduce the computation of $\alpha(s)$, $s \in$ supp μ, to the calculation of $\#\langle s_n\rangle$, $s_n = \sum_{i=1}^n 3^{-i}x_i$, where $\#\langle s_n\rangle$ denotes the cardinality of $\langle s_n\rangle$. In Section 3 we establish some decomposition results for zero elements and recurrence equation for computing $\#\langle s_n\rangle$. The proof of Main Theorem will be given in the last section.

2. Some primary results

Denote $D = \{0, 1, 4\}$ and for $s = \sum_{i=1}^{\infty} 3^{-i}x_i \in$ supp $\mu, x_i \in D$, let $s_n = \sum_{i=1}^{n} 3^{-i}x_i$ be its n-partial sum. Let

$$\langle s_n\rangle = \{(x_1, \ldots, x_n) \in D^n : \sum_{i=1}^{n} 3^{-i}x_i = s_n\}.$$

Because $X_1, X_2, \ldots$ are independent,

$$\mu_n(s_n) = 3^{-n}\#\langle s_n\rangle,$$

where μ_n is the probability measure induced by $S_n = \sum_{i=1}^{n} 3^{-i}X_i$.

Proposition 2.1. *For $s \in \text{supp } \mu$, we have*

$$\alpha(s) = \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n \log 3},$$

provided that the limit exists. Otherwise, we can replace $\alpha(s)$ by $\overline{\alpha}(s)$ ($\underline{\alpha}(s)$) and consider the upper (the lower) limits respectively.

We first prove

Lemma 2.1. *Let*

$$s_k = \sum_{i=1}^{k} 3^{-i}x_i > s'_k = \sum_{i=1}^{k} 3^{-i}x'_i > s''_k = \sum_{i=1}^{k} 3^{-i}x''_i,$$

where $x_i,\ x'_i,\ x''_i \in D$ for $i = 1, \ldots, k$, be consecutive numbers in supp μ_k. *Then*

(i) *if $s_k - s'_k = 3^{-k}$, then $x'_k = 0$.*
(ii) *either $s_k - s'_k \neq 3^{-k}$ or $s'_k - s''_k \neq 3^{-k}$.*
(iii) *if $s_{k+1} = s_k$, then $\#\langle s_{k+1}\rangle = \#\langle s_k\rangle$.*

Proof. (i) Assume that $s_k = s_{k-1} + 3^{-k}x_k,\ s'_k = s'_{k-1} + 3^{-k}x'_k$, then

$$s_k - s'_k = 3^{-k} \Leftrightarrow s_{k-1} - s'_{k-1} = \frac{1 + x'_k - x_k}{3} 3^{-(k-1)}. \tag{2.1}$$

Observe that for $s_n, s'_n \in \text{supp } \mu_n$, we have $s_n - s'_n = t3^{-n}$ for some $t \in \mathbb{Z}$, and if $(1 + a - b) \equiv 0 \pmod 3$ and $a,\ b \in D$, then

$$a = 0,\ b \in \{1, 4\}. \tag{2.2}$$

Therefore, from (2.1) we have $x'_k = 0$. Thus, (i) is proved.

(ii) Write $s''_k = s''_{k-1} + 3^{-k}x''_k$. Assume on the contrary that $s_k - s'_k = s'_k - s''_k = 3^{-k}$. Then by (2.1) and (2.2), $x'_k \in \{1, 4\}$. But, since $s_k - s'_k = 3^{-k}$, by (i), $x'_k = 0$, a contradiction.

(iii) Let $s_{k+1} = s_k = s'_k + 3^{-(k+1)}x'_{k+1}$, where $x'_{k+1} \in \{1, 4\}$ and $s'_k \in \text{supp } \mu_k$. Then $s_k - s'_k = \frac{x'_{k+1}}{3}3^{-k} = t3^{-k} (t \in \mathbb{N})$. This implies $x'_{k+1} \equiv 0 \pmod 3$ which contradicts $x'_{k+1} \in \{1, 4\}$. Hence $\#\langle s_{k+1}\rangle = \#\langle s_k\rangle$. □

From Lemma 2.1 we get

Corollary 2.1. (i) *Any element $s_{k+1} \in \text{supp } \mu_{k+1}$ has at most two representations in* supp μ_k, *and if $s_{k+1} = s_k$ then the representation is unique.*
(ii) *If s_{k+1} has two representations: $s_{k+1} = s_k + 3^{-(k+1)}4 = s'_k + 3^{-(k+1)}$, then s_k, s'_k are two consecutive points in* supp μ_k.

Lemma 2.2. *For any two consecutive points* $s_n = \sum_{i=1}^{n} 3^{-i}x_i$ *and* $s'_n = \sum_{i=1}^{n} 3^{-i}x'_i$, *we have*

$$\frac{\mu_n(s_n)}{\mu_n(s'_n)} \leqslant n. \tag{2.3}$$

Proof. We prove the lemma by induction. Clearly the inequality holds true for $n = 1$. We consider the case $n = k + 1$, assuming that the inequality holds true for all $n \leqslant k$. Let $s_{k+1} > s'_{k+1}$ be two arbitrary consecutive points in supp μ_{k+1}. Writing $s_{k+1} = s_k + 3^{-(k+1)}x_{k+1}$, we consider the following cases.
Case 1. $x_{k+1} = 0$. Then $s_{k+1} = s_k$. By Lemma 2.1(iii), $\#\langle s_{k+1}\rangle = \#\langle s_k\rangle$. Now we calculate the cardinality of $\langle s'_{k+1}\rangle$. Obviously, if $s^*_{k+1} = s^*_k + 3^{-(k+1)}x^*_{k+1} \in$ supp μ_{k+1} and $s^*_{k+1} < s_{k+1}$, then $s^*_k < s_k$. Letting

$$s'_k = \max\{s^*_k \in \text{supp } \mu_k : s^*_k < s_k\}$$

we get $s_k > s'_k$ are two consecutive points in supp μ_k. Observe that

If $s_k - s'_k = 3^{-k}$, then $s_{k+1} = s_k = s'_k + 3^{-k} > s'_k + 3^{-(k+1)}$. Since $s'_{k+1} < s_{k+1}$ are consecutive points, we have $s'_{k+1} = s'_k + 3^{-(k+1)}$. Observe that if $s'_{k+1} = s^*_k + 3^{-(k+1)}x^*_{k+1}$ is another representation, then by Corollary 2.1(i), $x^*_{k+1} = 4$. Hence, $s'_{k+1} = s'_k + 3^{-(k+1)} = s^*_k + 3^{-(k+1)}4$. This implies $s'_k - s^*_k = 3^{-k} = s_k - s'_k$, contradicting Lemma 2.1(ii). Therefore, $\#\langle s'_{k+1}\rangle = \#\langle s'_k\rangle$. Thus,

$$\frac{\mu_{k+1}(s_{k+1})}{\mu_{k+1}(s'_{k+1})} = \frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s'_k\rangle} \leqslant k < n.$$

If $s_k - s'_k \geqslant 2.3^{-k}$, then for any $x^*_{k+1} \in D$ and $s^*_k < s_k$ we have

$$s_{k+1} = s_k \geqslant s'_k + 2.3^{-k} > s'_k + 3^{-(k+1)}4 \geqslant s^*_k + 3^{-(k+1)}x^*_{k+1}.$$

Thus, $s'_k + 3^{-(k+1)}4$ is the largest value in supp μ_{k+1} that is smaller than s_{k+1}. Hence, $s'_{k+1} = s'_k + 3^{-(k+1)}4$. This implies $\#\langle s'_{k+1}\rangle = \#\langle s'_k\rangle$. Therefore,

$$\frac{\mu_{k+1}(s_{k+1})}{\mu_{k+1}(s'_{k+1})} = \frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s'_k\rangle} \leqslant k < n.$$

Case 2. $x_{k+1} = 1$ or $x_{k+1} = 4$. The proof of the two cases are the same, and so we demonstrate the case $x_{k+1} = 4$ only.

When $x_{k+1} = 4$, we have $s_{k+1} = s_k + 3^{-(k+1)}4$. By Corollary 2.1(ii), if $s^*_k - s_k = 3^{-k}$ for some $s^*_k \in$ supp μ_k, then s^*_k, s_k are two consecutive points and s_{k+1} has two representations $s_{k+1} = s_k + 3^{-(k+1)}4 = s^*_k + 3^{-(k+1)}$. Hence,

$$\#\langle s_{k+1}\rangle = \#\langle s_k\rangle + \#\langle s^*_k\rangle.$$

Since $s_{k+1} = s^*_k + 3^{-(k+1)}$ and $s_{k+1} > s'_{k+1}$ are consecutive, we have $s'_{k+1} = s^*_k$. By Lemma 2.1(iii), $\#\langle s'_{k+1}\rangle = \#\langle s^*_k\rangle$. By inductive assumption,

$$\frac{\mu_{k+1}(s_{k+1})}{\mu_{k+1}(s'_{k+1})} = \frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle + \#\langle s^*_k\rangle}{\#\langle s^*_k\rangle} \leqslant k + 1 = n.$$

The lemma is proved. □

Proof of Proposition 2.1. Observe that for $h > 0$ and $n \in \mathbb{N}$ with
$$3^{-(n+1)} < h \leqslant 3^{-n},$$
we have
$$\mu(B_{3^{-(n+1)}}(s)) \leqslant \mu(B_h(s)) \leqslant \mu(B_{3^{-n}}(s)).$$
Hence,
$$\alpha(s) = \lim_{n\to\infty} \frac{|\log \mu(B_{3^{-n}}(s))|}{n \log 3}. \tag{2.4}$$
Since
$$|S_n - S| \leqslant 3^{-n} 4 \sum_{i=1}^{\infty} 3^{-i} = 3^{-n} 4 \frac{1}{2} = 2.3^{-n}$$
we have (where $r = 2$)
$$\begin{aligned}\mu_n(B_{(1+r)3^{-n}}(s)) &= \text{Prob}(s - (1+r)3^{-n} \leqslant S_n \leqslant s + (1+r)3^{-n}) \\ &\leqslant \text{Prob}(s - (1+r)3^{-n} - r3^{-n} \leqslant S \leqslant s + (1+r)3^{-n} + r3^{-n}) \\ &= \mu(B_{(1+2r)3^{-n}}(s)).\end{aligned}$$
Similarly, $\mu(B_{3^{-n}}(s)) \leqslant \mu_n(B_{(1+r)3^{-n}}(s))$. Thus,
$$\mu(B_{3^{-n}}(s)) \leqslant \mu_n(B_{(1+r)3^{-n})}(s)) \leqslant \mu(B_{(1+2r)3^{-n}}(s)). \tag{2.5}$$
Therefore,
$$\frac{|\log \mu(B_{3^{-n}}(s))|}{n \log 3} \geqslant \frac{|\log \mu_n(B_{(1+r)3^{-n}}(s))|}{n \log 3}. \tag{2.6}$$

By Lemma 2.1(ii), $B_{(1+r)3^{-n}}(s)$ contains at most 5 consecutive points s_n in supp μ_n, so we have
$$\mu_n(B_{(1+r)3^{-n}}(s)) = \frac{\sum\{\#\langle s_n\rangle : s_n \in B_{(1+r)3^{-n}}(s) \cap \text{supp } \mu_n\}}{3^n} \leqslant 5n^4 \mu_n(s_n).$$
Thus, from (2.6) we get
$$\frac{|\log \mu(B_{3^{-n}}(s))|}{n \log 3} \geqslant \frac{|\log \mu_n(B_{(1+r)3^{-n}}(s))|}{n \log 3} \geqslant \frac{|\log \mu_n(s_n) + \log(5n^4)|}{n \log 3}.$$
Consequently, by (2.4) we have
$$\alpha(s) \geqslant \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n \log 3}. \tag{2.7}$$
Now for $h > 0$ let $n \in \mathbb{N}$ be such that
$$(2r+1)3^{-n} < h \leqslant (2r+1)3^{-n+1}.$$
Then
$$-\log h \geqslant -\log[(2r+1)3^{-n+1}] = n \log 3 - \log[(2r+1)3]$$
and by (2.5) we have
$$|\log \mu(B_h(s))| \leqslant |\log \mu(B_{(1+2r)3^{-n}}(s))| \leqslant |\log \mu_n(B_{(1+r)3^{-n}}(s))|.$$

Hence,

$$\frac{\log \mu(B_h(s))}{\log h} = \frac{|\log \mu(B_h(s))|}{-\log h} \leqslant \frac{|\log \mu_n(B_{(1+r)3^{-n}}(s))|}{n\log 3 - \log[(2r+1)3]},$$

which implies

$$\alpha(s) \leqslant \lim_{n\to\infty} \frac{|\log \mu_n(B_{(1+r)3^{-n}}(s))|}{n\log 3}. \tag{2.8}$$

Observe that

$$\frac{|\log \mu_n(B_{(1+r)3^{-n}}(s))|}{n\log 3} \leqslant \frac{|\log \mu_n(s_n)|}{n\log 3}.$$

Therefore, from (2.8) we get

$$\alpha(s) \leqslant \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n\log 3}.$$

Consequently, by (2.7)

$$\alpha(s) = \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n\log 3}.$$

The proposition is proved. □

3. Prime sequences and multiple sequences

By Proposition 2.1, the key for calculating the local dimension is to determine the rate of the growth of $\#\langle s_n\rangle$. Observe that if $(y_1,\dots,y_n)$ and $(z_1,\dots,z_n)$ are two elements in $\langle s_n\rangle$, then $\sum_{i=1}^{n} 3^{-i}(y_i - z_i) = 0$.

Let $\Gamma = D - D = \{0,\pm1,\pm3,\pm4\}$. We say that $(x_1,\dots,x_n)\in\Gamma^n$ is a *zero sequence* if $\sum_{i=1}^{n} 3^{-i}x_i = 0$. An easy calculation shows that

$$(0,\dots,0);\pm(-1,3);\pm(1,-4,3)$$

$$\pm(1,-4,\underbrace{4,-4},\dots,\underbrace{4,-4},3) \text{ or}$$

$$\pm(-1,\underbrace{4,-4},\dots,\underbrace{4,-4},3) \tag{3.1}$$

are zero sequences. We prove

Proposition 3.1. *$x = (x_1,\dots,x_n)\in\Gamma^n$ is a zero sequence if and only if it can be decomposed uniquely as a concatenation of sequences of (3.1).*

Proof. Since a concatenation of zero sequences is a zero sequence, we need to prove the "only if" part only.

Let $x = (x_1,\dots,x_n)$ be a zero sequence. Then

$$\sum_{i=1}^{n} 3^{-i}x_i = 0, \text{ where } x_i\in\Gamma \text{ for } i = 1,\dots,n, \tag{3.2}$$

which implies $x_n \equiv 0 \pmod 3$. Without loss of generality we may assume that $x_n = 3$. Multiplying (3.2) by 3^{n-1} we obtain

$$x_{n-1} + 1 \equiv 0 \pmod 3, \tag{3.3}$$

so $x_{n-1} = -1$ or $x_{n-1} = -4$.

If $x_{n-1} = -1$, then $(x_{n-1}, x_n) = (-1, 3)$, which belongs to (3.1). Thus, we can repeat the above argument for the remaining zero sequence $(x_1, \dots, x_{n-2})$.

If $x_{n-1} = -4$, then $(x_{n-1}, x_n) = (-4, 3)$, so from (3.2) it follows that $x_{n-2} - 1 \equiv 0 \pmod 3$. Hence, $x_{n-2} = 1$ or $x_{n-2} = 4$. Consider two cases

Case 1. $x_{n-2} = 1$. Then $(x_{n-2}, x_{n-1}, x_n) = (1, -4, 3)$. Hence, from (3.1) we get the assertion.

Case 2. $x_{n-2} = 4$. Then $(x_{n-2}, x_{n-1}, x_n) = (4, -4, 3)$ and from (3.2), it follows that $x_{n-3} + 1 \equiv 0 \pmod 3$. Thus, the assertion follows from (3.3). Repeating the above argument we get

$$(x_i, \dots, x_n) = (-1, \underbrace{4, -4}, \dots, \underbrace{4, -4}, 3) \text{ or}$$

$$(x_i, \dots, x_n) = (1, -4, \underbrace{4, -4}, \dots, \underbrace{4, -4}, 3)$$

for some $i \geqslant 1$. □

By Proposition 2.1, the maximum (minimum) value of the local dimension will occur at a point $x = (x_1, x_2, \dots) \in D^\infty$ such that $\#\langle s_n \rangle$ ($s_n = \sum_{i=1}^{n} 3^{-i} x_i$) attains a minimum (maximum) value for all sufficient large n. So we will introduce some notions which will be used to calculate the extreme local dimension.

Two sequences $x = (x_1, \dots, x_n) \in D^n$ and $y = (y_1, \dots, y_n) \in D^n$ are said to be *equivalent*, denoted by $x \approx y$, if $x - y$ is a zero sequence. It is easy to see that "$\approx$" is an equivalence relation. Let $\langle x \rangle$ denote the equivalence class of x. Note that if $s_n = \sum_{i=1}^{n} 3^{-i} x_i$, then $\langle x \rangle = \langle s_n \rangle$, so $\#\langle x \rangle = \#\langle s_n \rangle$.

We call $x = (x_1, \dots, x_n) \in D^n$ a *prime sequence* if $\#\langle x \rangle = 1$, and $x = (x_1, x_2, \dots) \in D^\infty$ a *prime sequence* if every finite segment of x is a prime sequence, and by a *segment* of a sequence we mean a consecutive subsequence of the form $(x_i, x_{i+1}, \dots, x_{i+n})$. A sequence (finite or infinite) is called a *multiple sequence* if it is not a prime sequence.

Proposition 3.2. *$x = (x_1, \dots, x_n) \in D^n$ is a prime sequence if and only if it contains no segment of the form (0,4) or (1,1).*

Proof. Since (0,4) $\approx$ (1,1), if x contains (0,4) or (1,1) then $\#\langle x \rangle \geqslant 2$. Hence, x is a multiple sequence.

Conversely, if $\#\langle x \rangle \geqslant 2$, then there is an $y = (y_1, \dots, y_n) \in D^n$ with $y \neq x$ such that $x - y$ is a zero sequence. Hence, by Proposition 3.1, $x - y$ contains a segment of (3.1). Without loss of generality assume that

$$x - y = (-1, \underbrace{4, -4}, \dots, \underbrace{4, -4}, 3) \quad \text{or} \tag{3.4}$$

$$x - y = (1, -4, \underbrace{4, -4}, \dots, \underbrace{4, -4}, 3). \tag{3.5}$$

If $x - y$ belongs to (3.4), then

$$x = (0, 4, \dots, 0, 4) \text{ and } y = (1, 0, 4, \dots, 0, 4, 1),$$

and if $x - y$ belongs to (3.5), then

$$x = (1, 0, 4, \dots, 0, 4) \quad \text{and} \quad y = (0, 4, 0, 4, \dots, 0, 4, 1).$$

Thus, x always contains (0, 4). The proposition is proved. □

For $n \in \mathbb{N}^*$, let $Z_n = \{(x_1, \dots, x_n) \in D^n\}$ which $(x_1, \dots, x_n) = (1, \dots, 1)$, or $(x_1, \dots, x_n) = (0, 4, \dots, 0, 4)$, or $(x_1, \dots, x_n)$ is concatenated by form segments of $(1, \dots, 1)$ or $(0, 4)$.

Note that x contains a segment $(0, 4)$ or a segment $(1, 1)$ for every $x \in Z_n$. By Proposition 3.2, it is a multiple sequence.

The members of Z_n are called *basic multiple sequences* of length n. Clearly, Z_n is an equivalence class in D^n. Moreover, if $x \in Z_n$, then $\langle x \rangle = Z_n$. Hence, $\#\langle x \rangle = \#Z_n$.

Observe that by placing a digit 1 at the beginning or at the end of a basic multiple finite sequence, then we get another basic multiple sequence of larger length. So we call an infinite sequence $x = (x_1, x_2, \dots) \in D^\infty$ a *basic multiple sequence* if $x_i = 1$ for every $i \in \mathbb{N}$ or, if $x_i \neq 1$ for some $i \in \mathbb{N}$ then $(x_i, x_{i+1}) = (0, 4)$, where i is the smallest such that $x_i \neq 1$.

A multiple segment of a sequence $x = (x_1, x_2, \dots)$ is *maximal* if it contains no other proper subsegments.

Proposition 3.3. *Any sequence $x = (x_1, x_2, \dots) \in D^\infty$ is a unique concatenation of maximal basic multiple segments and prime segments.*

Proof. By Proposition 3.2, if x does not contain $(0, 4)$ or $(1, 1)$ then x is a prime sequence. Otherwise, we check from x_1 to x_2 and so on until we get $(0, 4)$ or $(1, 1)$. Then we can write

$$\begin{aligned} x &= (x_1, \dots, x_k, 0, 4, x_{k+3}, \dots) \text{ or} \\ x &= (x_1, \dots, x_k, 1, 1, x_{k+3}, \dots). \end{aligned}$$

Thus, $(x_1, \dots, x_k)$ is a prime segment by Proposition 3.2. Now we continue to check from x_{k+3}. There are two cases.

Case 1. $(x_{k+3}, x_{k+4}, \dots)$ is a basic multiple infinite sequence. Then x is concatenated by two parts: the first one is a prime segment $(x_1, \dots, x_k)$ and the second is a basic multiple infinite sequence $(x_{k+1}, x_{k+2}, \dots)$.

Case 2. $(x_{k+1}, x_{k+2}, \dots)$ is not a basic multiple infinite sequence. Let x_{k+t} $(t \in \mathbb{N}, t \geqslant 3)$ be the first co-ordinate with $x_{k+t} \neq 1$ or $(x_{k+t}, x_{k+t+1}) \neq (0, 4)$. Then

$$(x_{k+1}, \dots, x_{k+t-1}) = (0, 4, 1, \dots, 1) \quad \text{and} \quad (x_{k+1}, \dots, x_{k+t-1}) = (1, \dots, 1)$$

are maximal basic multiple segments. Thus, x is concatenated by three parts: the prime sequence $(x_1, \dots, x_k)$, the maximal basic multiple segment $(x_{k+1}, \dots, x_{k+t-1})$ and the infinite subsequence $(x_{k+t}, x_{k+t+1}, \dots)$. Using the above argument, we continue to decompose the infinite part $(x_{k+t}, x_{k+t+1}, \dots)$ to obtain the assertion. ☐

Proposition 3.4. *For any basic multiple sequence $x \in Z_n$, let $F_n = \#Z_n = \#\langle x \rangle$. Then*

$$F_1 = 1, F_2 = 2 \quad \textit{and} \quad F_n = F_{n-1} + F_{n-2} \quad \textit{for} \quad n \geqslant 3. \tag{3.6}$$

Proof. We prove the proposition by induction. It is easy to check that $F_1 = 1, F_2 = 2$ and $F_3 = F_1 + F_2$. Suppose that (3.6) holds true for all $n \leqslant k$. We will show that $F_{k+1} = F_k + F_{k-1}$. Let $x = (x_1, \dots, x_{k+1}) \in Z_{k+1}$ and $s_{k+1} = \sum_{i=1}^{k+1} 3^{-i} x_i$. Then

$$\#\langle s_{k+1} \rangle = \#Z_{k+1} = F_{k+1}.$$

Without loss of generality we may assume that $x = (x_1, \dots, x_{k+1}) = (1, 1, \dots, 1)$. Then we have

$$s_{k+1} = \sum_{i=1}^{k+1} 3^{-i} x_i = s_k + 3^{-(k+1)},$$

where $\langle s_k \rangle = \langle (1, 1, \dots, 1) \rangle = Z_k$.

Let $s'_k = s_k - 3^{-k}$. Then

$$\langle s'_k \rangle = \langle (1, 1, \dots, 1, 0) \rangle, \ s_{k+1} = s'_k + 3^{-(k+1)} 4 \ \text{ and } s'_k = s'_{k-1},$$

where $\langle s'_{k-1} \rangle = \langle (1, 1, \dots, 1) \rangle = Z_{k-1}$. By Lemma 2.1(iii), we have

$$\#\langle s'_k \rangle = \#\langle s'_{k-1} \rangle = \#Z_{k-1} = F_{k-1}.$$

Consequently,

$$F_{k+1} = \#\langle s_{k+1} \rangle = \#\langle s_k \rangle + \#\langle s'_k \rangle = \#\langle s_k \rangle + \#\langle s'_{k-1} \rangle = F_{k-1} + F_k.$$

The proposition is proved. ☐

From Proposition 3.4 it follows that if x is a basic multiple sequence of length n, then by Fibonacci formula, we have

$$F_n = \#Z_n = \#\langle x \rangle = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n+1} \right]. \tag{3.7}$$

Proposition 3.5. *For any $n \in \mathbb{N}, n \neq 0$, we have*

$$F_n \geqslant \#\langle t_n \rangle \ \textit{for all} \ \ t_n \in \operatorname{supp} \mu_n.$$

Proof. We prove the proposition by induction. Clearly the inequality is true for $n = 1$. We consider the case $n = k + 1$, assuming that the inequality is true for all $n \leqslant k$. Let

$$y = (y_1, \dots, y_{k+1}) \notin Z_{k+1}, \; t_{k+1} = \sum_{i=1}^{k+1} 3^{-i} y_i.$$

Write $t_{k+1} = t_k + 3^{-(k+1)} y_{k+1}$, where $t_k \in$ supp μ_k, $y_{k+1} \in D$. We consider the following cases.

If $y_{k+1} = 0$, then $t_{k+1} = t_k$. By Lemma 2.1(iii) and inductive assumption, we have

$$\#\langle t_{k+1} \rangle = \#\langle t_k \rangle \leqslant F_k < F_{k+1}.$$

If $y_{k+1} \neq 0$, let $t_{k+1} = t_k + 3^{-(k+1)} = t'_k + 3^{-(k+1)}4$ be two representations of t_{k+1} in supp μ_k. Then $t_k = t'_k + 3^{-k}$. Assume that $t'_k = \sum_{i=1}^{k} 3^{-i} y'_i$, then by Lemma 2.1(i), $y'_k = 0$. Hence, $t'_k = t'_{k-1}$, which implies $\#\langle t'_k \rangle = \#\langle t'_{k-1} \rangle$. By the induction assumption and by Proposition 3.4, we have

$$\#\langle t_{k+1} \rangle \leqslant \#\langle t_k \rangle + \#\langle t'_k \rangle = \#\langle t_k \rangle + \#\langle t'_{k-1} \rangle \leqslant F_k + F_{k-1} = F_{k+1}.$$

The proposition is proved. □

Proposition 3.6. *If $x = (x_1, \dots, x_n) \in D^n$ is concatenated by prime segments and m maximal basic multiple sequences with lengths $l_1, \dots, l_m$ respectively, $l_1 + \dots + l_m \leqslant n$, then*

$$\#\langle x \rangle = \prod_{i=1}^{m} F_{l_i} \leqslant F_n.$$

Proof. By the multiplication principle it is easy to see that $\#\langle x \rangle = \prod_{i=1}^{m} F_{l_i}$. To prove the inequality we first show that, for any $n \in \mathbb{N}, n \geqslant 2$ and for any $n_1, n_2 \in \mathbb{N}$ with $n_1 + n_2 = n$, one has

$$F_{n_1} F_{n_2} \leqslant F_n. \tag{3.8}$$

The inequality (3.8) can be proved by induction. The inequality holds trivially for all $n \leqslant 5$. Suppose that it holds for all $n \leqslant k$, $k \geqslant 5$, we prove it also holds for $n = k + 1$. Let $k_1 \leqslant k_2$ be such that $k_1 + k_2 = k + 1$. By Proposition 3.4 and by the induction assumption, we get

$$\begin{aligned} F_{k_1} F_{k_2} &= F_{k_1}(F_{k_2-1} + F_{k_2-2}) \\ &= F_{k_1} F_{k_2-1} + F_{k_1} F_{k_2-2} \\ &\leqslant F_{k_1+k_2-1} + F_{k_1+k_2-2} \\ &= F_{k_1+k_2} = F_{k+1}. \end{aligned}$$

From (3.8) we have

$$\prod_{i=1}^{m} F_{l_i} \leqslant F_{l_1+l_2} \prod_{i=3}^{m} F_{l_i} \leqslant \ldots \leqslant F_{l_1+\ldots+l_m} \leqslant F_n.$$

The proposition is proved. □

4. Proof of the Main Theorem

The following proposition establishes the values of $\overline{\alpha}$, α^*, $\underline{\alpha}$, α_* of Main Theorem.

Proposition 4.1. *The following equalities hold true:*

$$\overline{\alpha} = \alpha^* = 1, \textit{ and } \underline{\alpha} = \alpha_* = 1 - \frac{\log(1+\sqrt{5}) - \log 2}{\log 3}.$$

Proof. Observe that if $x = (x_1, x_2, \ldots) \in D^\infty$ is a prime sequence, then $\#\langle s_n\rangle = 1$ for every n, where $s_n = \sum_{i=1}^{n} 3^{-i}x_i$. Hence, $\mu_n(s_n) = 3^{-n}\#\langle s_n\rangle = 3^{-n}$ for every n. By Proposition 2.1, for $s = \sum_{i=1}^{\infty} 3^{-i}x_i \in \operatorname{supp}\mu$ we have

$$\overline{\alpha} = \alpha^* = \alpha(s) = 1.$$

We prove the second equality. Let $s = \sum_{i=1}^{\infty} 3^{-i}x_i \in \operatorname{supp}\mu$ and $s_n = \sum_{i=1}^{n} 3^{-i}x_i \in \operatorname{supp}\mu_n$. By Propositions 2.1, 3.5 and (3.7), we have

$$\begin{aligned}
\alpha(s) &= \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n\log 3} = \lim_{n\to\infty} \frac{|\log 3^{-n}\#\langle s_n\rangle|}{n\log 3} \\
&\geqslant 1 - \lim_{n\to\infty} \frac{|\log \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}]|}{n\log 3} \\
&= 1 - \frac{\log(1+\sqrt{5}) - \log 2}{\log 3}.
\end{aligned}$$

Therefore,

$$\underline{\alpha}, \alpha_* \geqslant 1 - \frac{\log(1+\sqrt{5}) - \log 2}{\log 3}.$$

On the other hand, let $x = (x_1, x_2, \ldots) = (1, 1, \ldots) \in D^\infty$. Then $\#\langle s_n\rangle = F_n$ for all $n \in \mathbb{N}$, where $s_n = \sum_{i=1}^{n} 3^{-i}x_i$. Hence, for $s = \sum_{i=1}^{\infty} 3^{-i}x_i$, we have

$$\alpha(s) = 1 - \frac{\log(1+\sqrt{5}) - \log 2}{\log 3}.$$

This shows that

$$\underline{\alpha}, \alpha_* \leqslant 1 - \frac{\log(1+\sqrt{5}) - \log 2}{\log 3}.$$

Thus,

$$\underline{\alpha} = \alpha_* = 1 - \frac{\log(1+\sqrt{5}) - \log 2}{\log 3}.$$

The second equality is proved. □

Let $\alpha \in (\underline{\alpha}, \overline{\alpha})$. Write

$$\alpha = r(1 - \frac{\log a}{\log 3}) + (1-r)1 = 1 - r\frac{\log a}{\log 3}$$

for some $r \in (0,1)$, where $a = \frac{1+\sqrt{5}}{2}$. Let

$$l_i = \begin{cases} 2i & \text{if } i \text{ is even} \\ [\frac{2i(1-r)}{r}] & \text{if } i \text{ is odd,} \end{cases}$$

where $[t]$ is the largest integer not exceeding than t. Let

$$E_j = \{i : i \text{ is even}, i \leqslant j\}; O_j = \{i : i \text{ is odd}, i \leqslant j\};$$

$$e_j = \sum_{i \in E_j} l_i;\ o_j = \sum_{i \in O_j} l_i, \text{ and } n_j = \sum_{i=1}^{j} l_i.$$

Then $n_j = e_j + o_j$.

To complete the proof of Main Theorem, it remains to prove that for any $\alpha \in (\underline{\alpha}, \overline{\alpha})$, there exists s in supp μ for which $\alpha(s) = \alpha$. We prove

Claim 4.1. *For*

$$x = (x_1, x_2, \dots) = (\underbrace{4, \dots, 4}_{l_1}, \underbrace{1,1,1,1}_{l_2=4}, \underbrace{4, \dots, 4}_{l_3}, \underbrace{1,1,1,1,1,1,1,1}_{l_4=8}, \dots) \in D^\infty,$$

we have $\alpha(s) = \alpha$, *where* $s = \sum_{i=1}^{\infty} 3^{-i} x_i \in \text{supp } \mu$.

Proof. Observe that any segment $(x_1, \dots, x_{n_j})$ contains $[\frac{j}{2}]$ maximal basic multiple sequences with lengths $l_2,\ l_4,\ \dots,\ l_{2[\frac{j}{2}]}$ respectively, where $l_2 + l_4 + \dots + l_{2[\frac{j}{2}]} = e_j$ and contains $(j - [\frac{j}{2}])$ prime segments. Let $s_{n_j} = \sum_{i=1}^{n_j} 3^{-i} x_i$. By Proposition 3.6, we have

$$\#\langle s_{n_j} \rangle = \prod_{i \in E_j} F_{l_i} \leqslant F_{e_j} = \frac{1}{\sqrt{5}}(a^{e_j+1} + \frac{(-1)^{e_j}}{a^{e_j+1}}) < \frac{1}{\sqrt{5}} a^{e_j+2}. \tag{4.1}$$

Observe that

$$F_{l_i} = \frac{1}{\sqrt{5}}(a^{l_i+1} + \frac{(-1)^{l_i}}{a^{l_i+1}}) = \frac{1}{\sqrt{5}}(a^{l_i+1} + \frac{1}{a^{l_i+1}}) > \frac{1}{\sqrt{5}} a^{l_i+1}$$

for any $i \in \mathbb{N}$ and i is even. Hence,

$$\#\langle s_{n_j} \rangle = \prod_{i \in E_j} F_{l_i} > (\frac{1}{\sqrt{5}})^{[\frac{j}{2}]} a^{e_j + [\frac{j}{2}]}. \tag{4.2}$$

For any $n \in \mathbb{N}, n \neq 0$ let $j \in \mathbb{N}$ with $n_{j-1} \leqslant n < n_j$. Since $\#\langle s_n \rangle$ is an increasing function with respect to n, by (4.1) and (4.2), we have

$$(\frac{1}{\sqrt{5}})^{j/2} a^{e_{j-1}+j/2-1} \leqslant \#\langle s_{n_{j-1}} \rangle \leqslant \#\langle s_n \rangle \leqslant \#\langle s_{n_j} \rangle \leqslant \frac{1}{\sqrt{5}} a^{e_j+2}.$$

Hence,

$$\frac{|\log 3^{-n_j} (\frac{1}{\sqrt{5}})^{j/2} a^{e_{j-1}+j/2-1}|}{n_{j-1} \log 3} \geqslant \frac{|\log \mu_n(s_n)|}{n \log 3} \geqslant \frac{|\log 3^{-n_{j-1}} \frac{1}{\sqrt{5}} a^{e_j+2}|}{n_j \log 3}. \tag{4.3}$$

Observe that

$$\lim_{j \to \infty} \frac{j}{n_j} = 0, \quad \lim_{j \to \infty} \frac{n_{j-1}}{n_j} = 1.$$

Let

$$u_i = \begin{cases} \frac{1}{2} l_i & \text{if } i \text{ is even} \\ \frac{1}{2} l_{i-1} & \text{if } i \text{ is odd,} \end{cases}$$

and $v_i = \frac{1}{2}(l_i + l_{i-1})$. An easy computation (see [8]) yields

$$\lim_{j \to \infty} \frac{e_j}{n_j} = \lim_{j \to \infty} \frac{u_j}{v_j} = r. \tag{4.4}$$

From (4.3), (4.4) and Proposition 2.1, we obtain

$$\alpha(s) = 1 - \frac{r \log a}{log 3} = \alpha.$$

Thus $\alpha(s) = \alpha$, which proves Claim 4.1 and consequently Main Theorem is proved. □

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Department of Mathematics
Pedagogical College of Nghe An
Nghe An, Vietnam

E-mail address: Vu_Hong_Thanh@yahoo.com

Journal of Science
Vietnam National University, Hanoi

E-mail address: nnhuyna@yahoo.com

Department of Mathematics
Vinh University, Nghe An

E-mail address: Lexuanson@gmail.com