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IDEAL AMENABILITY OF VARIOUS CLASSES OF BANACH ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a Banach algebra. If $n \in \mathbb{N}$ and $\mathcal{I}$ is a closed two sided ideal in $\mathcal{A}$, then $\mathcal{A}$ is $n-\mathcal{I}-$weakly amenable if the first cohomology group of $\mathcal{A}$ with coefficients in the $n-$th dual space $\mathcal{I}^{(n)}$ is zero, i.e., $H^1(\mathcal{A}, \mathcal{I}^{(n)}) = \{0\}$. Further, $\mathcal{A}$ is $n-$ideally amenable (ideally amenable) if $\mathcal{A}$ is $n - \mathcal{I}-$weakly amenable ($1 - \mathcal{I}-$weakly amenable) for every closed two sided ideal $\mathcal{I}$ in $\mathcal{A}$. In this paper we investigate $(2m + 1) - \mathcal{I}-$weakly amenability of Banach algebras for $m \geq 1$, and ideal amenability of Segal algebras and triangular Banach algebras $T = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ (where $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras and $\mathcal{M}$ is a $\mathcal{A}, \mathcal{B}-$module).

1. Introduction

Let $\mathcal{A}$ be a Banach algebra, and suppose that X is a Banach $\mathcal{A}$-bimodule such that

$$\|a.x\| \leq \|a\|\|x\| \text{ and } \|x.a\| \leq \|a\|\|x\|$$

for all $a \in \mathcal{A}$ and $x \in X$.

We can define the right and left actions of $\mathcal{A}$ on the dual space X^* of X by

$$\langle x, \lambda.a\rangle = \langle a.x, \lambda\rangle,$$
$$\langle x, a.\lambda\rangle = \langle x.a, \lambda\rangle,$$

for all $a \in A$, $x \in X$ and $\lambda \in X^*$.

Similarly, the second dual X^{**} of X becomes a Banach $\mathcal{A}$-bimodule under the actions

$$\langle \lambda, a.\Lambda\rangle = \langle \lambda.a, \Lambda\rangle,$$
$$\langle \lambda, \Lambda.a\rangle = \langle b.\lambda, \Lambda\rangle,$$

for all $a \in \mathcal{A}$, $x \in X$, $\lambda \in X^*$, and $\Lambda \in X^{**}$.

Suppose that X is a Banach $\mathcal{A}$-bimodule. A derivation $D : \mathcal{A} \to X$ is a linear map that satisfies $D(ab) = a.D(b) + D(a).b$ for all $a, b \in \mathcal{A}$. A derivation δ is said to be inner if there exists $x \in X$ such that $\delta(a) = \delta_x(a) = a.x - x.a$ for all $a \in \mathcal{A}$. Denoting the linear space of bounded derivations from $\mathcal{A}$ into X by $Z^1(\mathcal{A}, X)$ and the linear subspace of inner derivations by $N^1(\mathcal{A}, X)$, we consider

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the quotient space $H^1(\mathcal{A},X)=Z^1(\mathcal{A},X)/N^1(\mathcal{A},X)$, called the first Hochschild cohomology group of $\mathcal{A}$ with coefficients in X. A Banach algebra $\mathcal{A}$ is said to be amenable if $H^1(\mathcal{A},X^*)=\{0\}$ for all Banach $\mathcal{A}$-bimodules X, and $\mathcal{A}$ is called weakly amenable if $H^1(\mathcal{A},\mathcal{A}^*)=\{0\}$ (see [12] and [13]). Let $n\in\mathbb{N}$. A Banach algebra $\mathcal{A}$ is called n-weakly amenable if $H^1(\mathcal{A},\mathcal{A}^{(n)})=\{0\}$ (see [2]). A Banach algebra $\mathcal{A}$ is called ideally amenable if $H^1(\mathcal{A},\mathcal{I}^*)=\{0\}$ for every closed two sided ideal $\mathcal{I}$ of $\mathcal{A}$ (see [6,8,9,11]). A Banach algebra $\mathcal{A}$ is $n-\mathcal{I}-$weakly amenable if $H^1(\mathcal{A},\mathcal{I}^{(n)})=\{0\}$, where $\mathcal{I}$ is a closed two sided ideal in $\mathcal{A}$, and $\mathcal{A}$ is $n-$ideally amenable if $\mathcal{A}$ is $n-\mathcal{I}-$weakly amenable for every closed two sided ideal $\mathcal{I}$ in $\mathcal{A}$ (see [5,7,10,11]). In this paper, for abbreviation we write *b.a.i.* instead of bounded approximate identity.

2. $(2m+1)-\mathcal{I}-$Weak Amenability of Banach Algebras

Lemma 2.1. *Suppose that $\mathcal{I}$ is a closed two sided ideal in $\mathcal{A}^{**}$. Then, $\mathcal{I}$ is a closed two sided ideal in $\mathcal{A}^{(2m)}$ for $m\geq 1$.*

Proof. Let $\mathcal{I}$ be a left ideal of $\mathcal{A}^{(2m)}$ for $m\geq 1$. By using module direct sum decomposition we have

$$\mathcal{A}^{(2m)}=(\mathcal{A}^*)^{\perp}+(\mathcal{A}^{**})^{\wedge}\quad\text{and}\quad\mathcal{A}^{(2m+1)}=(\mathcal{A})^{\perp}+(\mathcal{A}^*)^{\wedge},$$

where $\widehat{\mathcal{A}}$ is the image of $\mathcal{A}$ in $\mathcal{A}^{(2m)}$ under the canonical embedding and $\mathcal{A}^{\perp}=\{F\in\mathcal{A}^{(2m+1)}:F|_{\widehat{\mathcal{A}}}=0\}$. For $F\in\mathcal{A}^{(2m+1)}$ let $F=f_1+\widehat{f_2}$ be such that $f_1\in\mathcal{A}^{\perp}$ and $f_2\in\mathcal{A}^*$. Since $\mathcal{I}$ is a left ideal in $\mathcal{A}^{(2m)}$, it then holds $af_1=0$ for each $a\in\mathcal{I}$. Thus, we have

$$aF=a\widehat{f_2}=(af_2)^{\wedge}.$$

For $\Psi\in\mathcal{A}^{(2m+2)}$ let $\Psi=\psi+\widehat{\varphi}$ be such that $\psi\in(\mathcal{A}^*)^{\perp}$ and $\varphi\in\mathcal{A}^{**}$. Since $\psi\in(\mathcal{A}^*)^{\perp}$, it then holds $\langle(af_2)^{\wedge},\psi\rangle=0$ for each $a\in\mathcal{I}$ and $f_2\in\mathcal{A}^*$. Thus, we have

$$\langle F,\Psi a\rangle=\langle(af_2)^{\wedge},\psi+\widehat{\varphi}\rangle=\langle(af_2)^{\wedge},\widehat{\varphi}\rangle=\langle F,(\varphi a)^{\wedge}\rangle.$$

Thus, $\Psi a=(\varphi a)^{\wedge}\in\widehat{\mathcal{I}}$ for $a\in\mathcal{I}$ and $\Psi\in\mathcal{A}^{(2m+2)}$. Therefore, $\mathcal{I}$ is a left ideal of $\mathcal{A}^{(2m+2)}$. For the other case, the proof is similar.

□

Lemma 2.2. *Let $\mathcal{A}$ be a Banach algebra with a left (right) b.a.i.. Suppose that $\mathcal{I}$ is a closed two sided ideal in $\mathcal{A}$ and $\mathcal{J}$ is a w^*-closed ideal of $\mathcal{I}^*$. If the left (right) module action on $\mathcal{J}$ is trivial, then $H^1(\mathcal{A},\mathcal{J})=\{0\}$.*

Proof. Suppose that $D:\mathcal{A}\to\mathcal{J}$ is a continuous derivation. Let (e_i) be a left *b.a.i.* of $\mathcal{A}$, and let $b\in\mathcal{J}$ be a w^*-cluster of $D(e_i)$. Since $\mathcal{A}\mathcal{J}=\{0\}$ we have

$$D(a)=\lim D(ae_i)=ba=ba-ab\qquad(a\in\mathcal{A}).$$

Hence D is inner and therefore $H^1(\mathcal{A},\mathcal{J})=\{0\}$.

□

Theorem 2.3. *Suppose that $\mathcal{A}$ is $\mathcal{I}-$weakly amenable, has a left (right) b.a.i. and $\mathcal{I}$ is a left ideal in $\mathcal{A}^{**}$. Then, $\mathcal{A}$ is $(2m+1)-\mathcal{I}-$weakly amenable for $m\geq 1$.*

Proof. Suppose that $\mathcal{A}$ has a left *b.a.i.*. We have

$$H^1(\mathcal{A},\mathcal{I}^{(2m+1)}) = H^1(\mathcal{A},\mathcal{I}^*) + H^1(\mathcal{A},\mathcal{I}^\perp).$$

If $\mathcal{A}$ is $\mathcal{I}-$weakly amenable, then $H^1(\mathcal{A},\mathcal{I}^*) = \{0\}$. $\mathcal{I}^\perp$ is a w^*-closed submodule of $\mathcal{I}^{(2m+1)}$. Since $\mathcal{I}$ is a left ideal in $\mathcal{A}^{**}$ it is left ideal in $\mathcal{A}^{(2m)}$ (Lemma 2.1), thus the left module action on $\mathcal{I}^\perp$ is trivial. Then, $H^1(\mathcal{A},\mathcal{I}^\perp) = \{0\}$ and therefore $H^1(\mathcal{A},\mathcal{I}^{(2m+1)}) = \{0\}$ (Lemma 2.2). This prove that $\mathcal{A}$ is $(2m+1)-\mathcal{I}-$weakly amenable for $m \geq 1$. □

3. Results For Segal Algebras

Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Then, $(\mathfrak{B}, \|\cdot\|')$ is an abstract Segal algebra with respect to $(\mathcal{A}, \|\cdot\|)$ if

(1) $\mathfrak{B}$ is a dense left ideal in $\mathcal{A}$, and $\mathfrak{B}$ is a Banach algebra with respect to $\|\cdot\|'$;

(2) There exists $M > 0$ such that $\|b\| \leq M\|b\|'$ for each $b \in \mathfrak{B}$;

(3) There exists $C > 0$ such that $\|ab\|' \leq C\|a\|'\|b\|'$ for each $a, b \in \mathfrak{B}$.

Let G be a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$ is said to be a Segal algebra if it satisfies the following conditions:

(i) $S^1(G)$ is dense in $L^1(G)$;

(ii) If $f \in S^1(G)$ then $L_xf \in S^1(G)$, i.e. $S^1(G)$ is left translation invariant;

(iii) $S^1(G)$ is a Banach space under some norm $\|\cdot\|_S$, and $\|L_xf\|_s = \|f\|_s$ for all $f \in S^1(G)$ and $x \in G$;

(iv) The map $x \mapsto L_xf$ from G into $S^1(G)$ is continuous.

Theorem 3.1. *Let $\mathcal{A}$ be a commutative ideally amenable Banach algebra. Then, any abstract Segal subalgebra of A having an approximate identity is ideally amenable.*

Proof. Let $\mathfrak{B}$ be an abstract Segal subalgebra of $\mathcal{A}$ satisfying the hypothesis of the theorem, with (e_α) an approximate identity of $\mathfrak{B}$. Let D be a continuous derivation from $\mathfrak{B}$ into $\mathcal{I}^*$, where $\mathcal{I}^*$ is an arbitrary two sided closed ideal of $\mathfrak{B}$. We define the maps $D_\alpha : \mathcal{A} \to \mathcal{I}^*$ by $D_\alpha(a) = D(e_\alpha.a) - D(e_\alpha).a$ for $a \in \mathcal{A}$.

Now, for all $a, b \in \mathcal{A}$ we have

$$\begin{aligned}
D_\alpha(a.b) &= D(e_\alpha.a.b) - D(e_\alpha)(a.b)\\
&= norm - \lim_\beta(D(e_\alpha.a.e_\beta.b) - D(e_\alpha)(a.b))\\
&= norm - \lim_\beta(D(e_\alpha.a).e_\beta.b + e_\alpha.a.D(e_\beta.b) - D(e_\alpha)(a.b))\\
&= D(e_\alpha.a).b - D(e_\alpha)(a.b) + w^* - \lim_\beta(e_\alpha.a.D(e_\beta.b))\\
&= (D(e_\alpha.a) - D(e_\alpha).a).b + w^* - \lim_\beta(e_\alpha.a.D(e_\beta.b))\\
&= (D(e_\alpha.a) - D(e_\alpha).a).b + w^* - \lim_\beta(a.D(e_\alpha.e_\beta.b) - a.D(e_\alpha)(e_\beta.b)\\
&= (D(e_\alpha.a) - D(e_\alpha).a).b + a.D(e_\alpha.b) - a.D(e_\alpha).b\\
&= D_\alpha(a).b + a.D_\alpha(b).
\end{aligned}$$

Therefore, each D_α is a derivation. Since $\mathcal{I}^*$ is a symmetric $\mathcal{A}-$bimodule, it follows that $D_\alpha = 0$ (see [15]). Hence $D = 0$. □

Corollary 3.2. *Every Segal algebra on an abelian locally compact group is ideally amenable.*

4. Results For Triangular Banach Algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and suppose that $\mathcal{M}$ is a Banach $\mathcal{A},\mathcal{B}$-module. We define a triangular Banach algebra

$$T = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix},$$

with sum and product being given by the usual 2×2 matrix operations and internal module actions. The norm on T is

$$\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \| = \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}}.$$

As a Banach space, T is isomorphic to the ℓ^1-direct sum of $\mathcal{A},\mathcal{B}$ and $\mathcal{M}$, so we have $T^{(2m-1)} \simeq \mathcal{A}^{(2m-1)} \oplus_1 \mathcal{M}^{(2m-1)} \oplus_1 \mathcal{B}^{(2m-1)}$ and $T^{(2m)} \simeq \mathcal{A}^{(2m)} \oplus_\infty \mathcal{M}^{(2m)} \oplus_\infty \mathcal{B}^{(2m)}$ for each $m \geq 1$. We identify T^{**} with $\begin{bmatrix} \mathcal{A}^{**} & \mathcal{M}^{**} \\ & \mathcal{B}^{**} \end{bmatrix}$, and the module action of T on T^{**} coincides with the restriction of the (first or second) Arens product on T^{**} to image of T in T^{**} under the canonical embedding. The module action of T on $T^{(3)}$ coincides with the restriction of the dual action of T^{**} on $T^{(3)}$ to the image of T in T^{**} under the canonical embedding and we define the product $\circ$ of action of T on $T^{(3)}$ by:

$$\begin{bmatrix} \alpha & \gamma \\ & \beta \end{bmatrix} \circ \begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} \alpha \circ a & \gamma \circ a \\ & \gamma \circ m + \beta \circ \beta \end{bmatrix}$$

and

$$\begin{bmatrix} a & m \\ & b \end{bmatrix} \circ \begin{bmatrix} \alpha & \gamma \\ & \beta \end{bmatrix} = \begin{bmatrix} a \circ \alpha + m \circ \gamma & b \circ \gamma \\ & b \circ \beta \end{bmatrix}$$

for each $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in T$ and $\begin{bmatrix} \alpha & \gamma \\ & \beta \end{bmatrix} \in T^{(3)}$.

It is clear that if $\mathfrak{I}$ is a closed two sided ideal of T, then there exist closed ideals $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{J}$ of $\mathcal{B}$ and a closed $\mathcal{A},\mathcal{B}-$submodule $\mathcal{M}'$ of $\mathcal{M}$ such that $\mathfrak{I} = \mathcal{I} \oplus \mathcal{M}' \oplus \mathcal{J}$ and $\mathcal{I}\mathcal{M} \cup \mathcal{M}\mathcal{J} \subseteq \mathcal{M}'$. In this paper we identify every closed two sided ideal $\mathfrak{I}$ of T with $\begin{bmatrix} \mathcal{I} & \mathcal{M}' \\ & \mathcal{J} \end{bmatrix}$, where $\mathcal{I}$ is a closed two sided ideal of $\mathcal{A}$ and $\mathcal{J}$ is a closed two sided ideal of $\mathcal{B}$. The following theorem is proved in [1].

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras, let $\mathcal{M}$ be a unital Banach $\mathcal{A},\mathcal{B}-$module, and let $\mathfrak{I}$, $\mathcal{I}$ and $\mathcal{J}$ be closed two sided ideals of T, $\mathcal{A}$ and $\mathcal{B}$, respectively. Then,*

$$H^1(T, \mathfrak{I}^*) \simeq H^1(\mathcal{A}, \mathcal{I}^*) \oplus H^1(\mathcal{B}, \mathcal{J}^*).$$

Corollary 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a unital Banach $\mathcal{A},\mathcal{B}-$module. Then T is ideally amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are ideally amenable.*

Corollary 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and let $\mathcal{M}=0$. Then, $\mathcal{A}\oplus\mathcal{B}$ is ideally amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are ideally amenable.*

By repeating the same calculations as for derivations D from T into $\mathfrak{I}^*$, we have

$$H^1(T,\mathfrak{I}^{(3)})\simeq H^1(\mathcal{A},\mathcal{I}^{(3)})\oplus H^1(\mathcal{B},\mathcal{J}^{(3)}).$$

So, T is $3-$ideally amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are $3-$ideally amenable, and we have the following theorem:

Theorem 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and let $\mathcal{M}$ be a unital Banach $\mathcal{A},\mathcal{B}-$module. Then, for each $n\geq 1$ it holds*

$$H^1(T,\mathfrak{I}^{(2n-1)})\simeq H^1(\mathcal{A},\mathcal{I}^{(2n-1)})\oplus H^1(\mathcal{B},\mathcal{J}^{(2n-1)}).$$

So, T is $(2n-1)-$ideally amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are $(2n-1)-$ideally amenable.

Corollary 4.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}=0$. Then, $\mathcal{A}\oplus\mathcal{B}$ is $(2n-1)-$ideally amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are $(2n-1)-$ideally amenable.*

Derivations from T into $\mathfrak{I}^{(2n)}$ are different from those into $\mathfrak{I}^{(2n-1)}$ for $n\geq 1$. Therefore, we have different results in this case.

Lemma 4.6. *Let $\delta: T\to\mathfrak{I}^{(2n)}$ be a continuous derivation. Then, there exist $\gamma_\delta\in\mathcal{M}'^{(2n)}$ continuous derivations $\delta_1:\mathcal{A}\to\mathcal{I}^{(2n)}$, $\delta_4:\mathcal{B}\to\mathcal{J}^{(2n)}$, and a continuous map $\rho:\mathcal{M}\to\mathcal{M}'^{(2n)}$ such that*

(i) $\delta\begin{bmatrix} a & 0 \\ & 0\end{bmatrix}=\begin{bmatrix}\delta_1(a) & a.\gamma_\delta \\ & 0\end{bmatrix}$ *for all* $a\in\mathcal{A}$,

(ii) $\delta\begin{bmatrix} 0 & 0 \\ & b\end{bmatrix}=\begin{bmatrix}0 & -\gamma_\delta.b \\ & \delta_4(b)\end{bmatrix}$ *for all* $a\in\mathcal{B}$,

(iii) $\delta\begin{bmatrix} 0 & m \\ & 0\end{bmatrix}=\begin{bmatrix}0 & \rho(m) \\ & 0\end{bmatrix}$ *for all* $m\in\mathcal{M}$.

(iv) $\rho(a.m)=\delta_1(a).m+a.\rho(m)$,

(v) $\rho(m.b)=\rho(m).b+m.\delta_4(b)$,

(vi) *if $\delta_{\mathcal{A}}:\mathcal{A}\to\mathcal{I}^{(2n)}$ and $\delta_{\mathcal{B}}:\mathcal{B}\to\mathcal{J}^{(2n)}$ are continuous derivations and $\rho_{\mathcal{M}}:\mathcal{M}\to\mathcal{M}'^{(2n)}$ is a continuous map that satisfies (iv) and (v), then $D:T\to\mathfrak{I}^{(2n)}$ defined by $\begin{bmatrix} a & m \\ & b\end{bmatrix}\mapsto\begin{bmatrix}\delta_{\mathcal{A}}(a) & \rho_{\mathcal{M}}(m) \\ & \delta_{\mathcal{B}}(b)\end{bmatrix}$ is a continuous derivation.*

In [4] the following sets are defined. For every $n \geq 1$ we denote the centralizer of $\mathcal{A}$ in $\mathcal{I}^{(2n)}$ by $Z_{\mathcal{A}}(\mathcal{I}^{(2n)}) = \{x \in \mathcal{I}^{(2n)} : x.a = a.x \text{ for all } a \in \mathcal{A}\}$, and similarly $Z_{\mathcal{B}}(\mathcal{J}^{(2n)}) = \{z \in \mathcal{J}^{(2n)} : z.b = b.z \text{ for all } b \in \mathcal{B}\}$. The set

$$ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)}) = \{\rho_{x,z} : \mathcal{M} \to \mathcal{M}^{'(2n)} : x \in Z_{\mathcal{A}}(\mathcal{I}^{(2n)}), z \in Z_{\mathcal{B}}(\mathcal{J}^{(2n)})\}$$

is called the set of central Rosenblum operators on $\mathcal{M}$ with coefficients in $\mathcal{M}^{'(2n)}$ and $\rho_{x,z}(m) = x.m - m.z$ ($\mathcal{M}$ is unital). We also have

$$\begin{aligned} Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)}) \quad = \quad & \{\varphi : \mathcal{M} \to \mathcal{M}^{'(2n)} : \varphi(a.m) = a.\varphi(m), \\ & \varphi(m.b) = \varphi(m).b, \text{ for all } a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}\}. \end{aligned}$$

Theorem 4.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and let $\mathcal{M}$ be a unital Banach $\mathcal{A},\mathcal{B}-$module. Then, we have*

(i) $ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)}) \subseteq Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)})$.

(ii) *If $\varphi \in Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)})$ then*

$$\Delta_\varphi \begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} 0 & \varphi(m) \\ & 0 \end{bmatrix} \in Z^1(T, \mathfrak{I}^{(2n)}),$$

and Δ_φ is inner if and only if φ is a central Rosenblum operator on $\mathcal{M}$ with coefficients in $\mathcal{M}^{'(2n)}$.

(iii) *If $\mathcal{A}$ and $\mathcal{B}$ are $(2n)-$ideally amenable, then*

$$H^1(T, \mathfrak{I}^{(2n)}) \simeq \frac{Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)})}{ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)})}.$$

Proof. The proof of statements (i) and (ii) is clear. For (iii), let the linear map $\mathfrak{f} : Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)}) \to H^1(T, \mathfrak{I}^{(2n)})$ be defined by $\varphi \mapsto \overline{\Delta}_\varphi$, where $\overline{\Delta}_\varphi$ denotes the equivalence class of Δ_φ in $H^1(T, \mathfrak{I}^{(2n)})$. Then $\mathfrak{f}$ is surjective, so we have

$$H^1(T, \mathfrak{I}^{(2n)}) \simeq \frac{Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)})}{\ker \mathfrak{f}}.$$

If $\varphi \in \ker \mathfrak{f}$ then, by statement (ii), $\mathfrak{f}(\varphi)$ is inner and, again by (ii), $\varphi \in ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)})$. Hence,

$$H^1(T, \mathfrak{I}^{(2n)}) \simeq \frac{Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)})}{ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{'(2n)})}.$$

□

Theorem 4.8. *Let T be a triangular Banach algebra and let $\mathfrak{I}$ be a two sided closed ideal of T. If $D : T \to \mathfrak{I}$ is a continuous derivation, then $D^{**} : T^{**} \to \mathfrak{I}^{**}$ is a continuous derivation.*

Proof. It is clear that D^{**} is a continuous linear operator. Let $\alpha_1, \alpha_2 \in \mathcal{A}^{**}$, $\beta_1, \beta_2 \in \mathcal{B}^{**}$ and $\gamma_1, \gamma_2 \in M^{**}$. Then, there are nets $(a_i), (a_j)$ in $\mathcal{A}$, $(b_i), (b_j)$ in $\mathcal{B}$ and $(m_i), (m_j)$ in $\mathcal{M}$ such that $\alpha_1 \Box \alpha_2 = w^* - \lim_i \lim_j a_i a_j$, $\beta_1 \Box \beta_2 = w^* - \lim_i \lim_j b_i b_j$, $\gamma_1 \Box \gamma_2 = w^* - \lim_i \lim_j m_i m_j$, $\alpha_1 \Box \gamma_2 = w^* - \lim_i \lim_j a_i m_j$ and

$\gamma_1\Box\beta_2 = w^* - \lim_i \lim_j a_i b_j$. Let $\Phi = \begin{bmatrix} \alpha_1 & \gamma_1 \\ & \beta_1 \end{bmatrix}$ and $\Psi = \begin{bmatrix} \alpha_2 & \gamma_2 \\ & \beta_2 \end{bmatrix} \in T^{**}$. Then, we have

$$\begin{aligned} D^{**}(\Phi\Box\Psi) &= D^{**}\left(\begin{bmatrix} \alpha_1 & \gamma_1 \\ & \beta_1 \end{bmatrix}\Box\begin{bmatrix} \alpha_2 & \gamma_2 \\ & \beta_2 \end{bmatrix}\right) \\ &= D^{**}\left(\begin{bmatrix} \alpha_1\Box\alpha_2 & \alpha_1\Box\gamma_2 + \gamma_1\Box\beta_2 \\ & \beta_1\Box\beta_2 \end{bmatrix}\right) \\ &= D^{**}\left(w^* - \lim_i \lim_j \begin{bmatrix} a_i a_j & a_i m_j + m_i b_j \\ & b_i b_j \end{bmatrix}\right) \\ &= w^* - \lim_i \lim_j D\left(\begin{bmatrix} a_i a_j & a_i m_j + m_i b_j \\ & b_i b_j \end{bmatrix}\right) \\ &= \Phi.D^{**}(\Psi) + D^{**}(\Psi).\Psi. \end{aligned}$$

□

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