Volume 32, Number 2-3, 2007, pp. 303-318

SOME OPEN QUESTIONS ON POLYNOMIAL AUTOMORPHISMS AND RELATED TOPICS

Abstract. This is a collection* of open questions presented at the Open Session of the International School and Workshop "Polynomial Automorphisms and Related Topics", October 9-20, 2006, Hanoi, Vietnam.

On (K/k)-forms of the algebraic tori

TAC KAMBAYASHI
Department of Mathematical Sciences
Tokyo Denki University
Saitama 350-0394, Japan
E-mail: tac@r.dendai.ac.jp

The multiplicative group $\mathbb{C}^*$ of the complex numbers as a complex Lie group has two "real forms", namely the real Lie groups $\mathbb{R}^*_{>0}$ and SO(2). (Each of the latter groups becomes isomorphic to $\mathbb{C}^*$ through base-field extension from $\mathbb{R}$ to $\mathbb{C}$, hence the name.) On the other hand, the underlying variety of $\mathbb{C}^*$ may be identified as the hyperbola ($XY = 1$ in $\mathbb{C}^2$), and this has three real forms, i.e., the hyperbola, the real circle ($X^2+Y^2 = 1$) and the imaginary circle ($X^2+Y^2 = -1$), all considered as affine varieties defined over $\mathbb{R}$. All this is classical and elementary knowledge.

In a recent work [2] (a gist of which was offered in the Hanoi Conference 2006) we expanded this knowledge to higher dimensions and to general separably algebraic base-field extensions. Our interest covered the forms of algebraic tori as well as those of their underlying varieties $\mathrm{Spec}(K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}])$. However, in the present short note, in the interest of brevity we shall confine our attention to *the algebraic tori only* and shall explain the known facts and open problems on the (K/k)-forms of these group-schemes.

1. Setting of the main problems and approach to solution

For any field K let $\mathbb{T}^n_K$ (often denoted as $(\mathbb{G}_{m,K})^n$ in the literature) be the n-dimensional algebraic torus split over K. Let K/k be a finite Galois extension

Key words and phrases. Polynomial Automorphisms, Jacobian Conjecture.
* Edited by Stefan Maubach and Nguyen Van Chau.

with Galois group $\Gamma = \mathrm{Gal}(K/k)$. Then, one is interested in finding all (K/k)-*forms of* $\mathbb{T}_K^n$. Namely:

THE BASIC PROBLEM. *For each given $n > 0$ find all k-group-schemes G, up to k-isomorphisms, such that*

$$K \otimes_k G \cong_K \mathbb{T}_K^n,$$

where the K-isomorphism "$\cong_K$" is that of K-group-schemes.

Consider the K-automorphism group of $\mathbb{T}_K^n$, which one may identify with $\mathrm{GL}_n(\mathbb{Z})$, irrespective of the field K. As is well-known [4], the k-isomorphism classes of (K/k)-forms of $\mathbb{T}_K^n$ are parametrized in total by $\mathrm{H}^1(\Gamma, \mathrm{GL}_n(\mathbb{Z}))$. Since Γ acts trivially on $\mathrm{GL}_n(\mathbb{Z})$, we see that $\mathrm{H}^1(\Gamma, \mathrm{GL}_n(\mathbb{Z})) \cong \mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{Z}))/\approx$, where $\approx$ means conjugacy by an element of $\mathrm{GL}_n(\mathbb{Z})$.

It follows that the study of (K/k)-forms of $\mathbb{T}_K^n$ begins with a study of *integral representation of* Γ. In particular, one wishes to determine what finite subgroups exist in $\mathrm{GL}_n(\mathbb{Z})$ of a given $n > 0$. One then takes up individual faithful map $\Gamma \longrightarrow \mathrm{GL}_n(\mathbb{Z})$, twist the Γ-action on $\mathbb{T}_K^n$ by that map to $\mathrm{GL}_n(\mathbb{Z})$, and then takes the quotient by the twisted action. All (K/k)-forms of $\mathbb{T}_K^n$ are obtainable that way.

2. The known and the unknown facts

The known facts about the forms of algebraic tori are as follows:

Dimension 1 *All* k-forms of $\mathbb{T}_K^1$, more often denoted as $\mathbb{G}_{m,K}$, split at a quadratic extension of k, and there are exactly two 1-dimensional k-group schemes up to k-isomorphisms that split at a given $Q = k[\sqrt{d}]$: the trivial one $\mathbb{G}_{m,k}$ and the affine k-group-scheme

$$\mathbb{U}_1 := \mathrm{Spec}(k[X,Y]/\langle X^2 - d^{-1}Y^2 - 1\rangle), \tag{0.1}$$

whose group operations are easily defined. (This is well-known, but see [2] for details.)

Dimension 2 The k-isomorphism classes of *nontrivial* (K/k)-forms of $\mathbb{T}_K^2 = (\mathbb{G}_{m,K})^2$ correspond to the conjugacy classes of nontrivial *finite* subgroups $\subset$ GL(2,$\mathbb{Z}$). Such subgroups have all been known, and their conjugacy classes are represented by cyclic groups C_n of order n and dihedral groups D_n of order $2n$ for $n = 2, 3, 4$ or 6. (This fact is deduced from a theorem due to F. E. Diederichsen and I. Reiner [1, XI-§74], [3]; another method is to use the modular group acting on the upper half plane [2].) Now, in an outstanding paper [6], Voskresenskii gave a complete list of all nontrivial (K/k)-forms of $\mathbb{T}_K^2$ each corresponding to a cyclic or dihedral group as above. His description is clear; but the list is too long to reproduce here.

Dimension 3 Tahara [5] takes pains to list up all nontrivial finite subgroups $\subset \mathrm{GL}_3(\mathbb{Z})$. However, (K/k)-forms of $\mathbb{T}_K^3$ are not treated there. It seems too hard to calculate actual torus forms from finite subgroups on Tahara's list, as one seems in need of a new idea to avoid all-too-cumbersome computations.

Remark. We take note of the fact that the (K/k)-forms of $\mathbb{T}^2_K$ in Voskresenskii's list [6] are all described in terms of $\mathcal{R}_{L'|L}(G)$ and $\mathcal{R}^{(1)}_{L'|L}(G)$ for suitable fields $L' \supset L$, where $\mathcal{R}_{L'|L}$ is the Weil descent functor, $\mathcal{R}^{(1)}_{L'|L}(G)$ is the kernel of the norm map $\mathcal{R}_{L'|L}(G) \longrightarrow G$, and G is a known L'-group-scheme, usually the same as $\mathbb{T}^1_{L'}$.

Examples. (a) The $\mathbb{U}_1$ above is identifiable as $\mathcal{R}^{(1)}_{Q|k}(\mathbb{T}^1_Q)$.

(b) Let $Q := k[\sqrt{d}]$ be a quadratic extension, with $\mathrm{char}(k) \neq 2$. We gave in [2] an example of a (Q/k)-form of $\mathbb{T}^2_Q$, named $\mathbb{U}_2$, as follows: As an affine k-scheme, $\mathbb{U}_2 := \mathrm{Spec}(B_2)$, where

$$B_2 := k[X, Y, Z, Z^{-1}]/\langle X^2 - dY^2 - Z\rangle = k[x, y, z, z^{-1}]. \tag{0.2}$$

One can easily check that $Q \otimes_k B_2 \cong k[T_1, T_1^{-1}, T_2, T_2^{-1}]$, so that the underlying scheme of $\mathbb{U}_2$ is a (Q/k)-form of that of $\mathbb{T}^2_Q$. We omit here the description of its group structure. It is not hard to show that this $\mathbb{U}_2 = \mathcal{R}_{Q|k}(\mathbb{T}^1_Q)$.

Here is our main question:

Question. *Can one describe all (K/k)-forms of $\mathbb{T}^3_K$ up to k-isomorphisms? Most desirably, can one do it in such a manner that there should be (a) some simple ingredients like $\mathbb{T}^1 = \mathbb{G}_m$, (b) combinations of $\mathcal{R}_{L'|L}$'s and $\mathcal{R}^{(1)}_{L'|L}$'s applied to the items in (a), and these exhaustibly represent all 3-dimensional (K/k)-forms?*

As evidence suggesting the latter part of the preceding question might hold valid, we wish to mention Voskresenskii's result [6] as explained already (see Remark and Example above) *and* our result about the real forms of algebraic tori as outlined in Section 3 just below.

3. Real forms of algebraic tori

Let $Q := k[\sqrt{d}]$ be a quadratic extension of k, where $\mathrm{char}(k) \neq 2$ and $d \in k\backslash k^2$. We shall now seek the (Q/k)-forms of $\mathbb{T}^n_Q$ for general n. Since $\mathrm{Gal}(Q/k) = C_2 \cong \mathbb{Z}/2\mathbb{Z}$, let us first look at integral representations of C_2. So, let M be a finite $\mathbb{Z}[C_2]$-module, $\mathbb{Z}$-free of rank n. We write $\mathbb{Z}[C_2] = \mathbb{Z}[\epsilon]$, with $\epsilon^2 = 1$. Now, draw upon Diederichsen-Reiner Theorem [1, Thm. 74.3] to find that, as $\mathbb{Z}[\epsilon]$-modules,

$$M \cong A \oplus B \oplus C, \tag{0.3}$$

where A, B are finite free $\mathbb{Z}$-modules such that $\epsilon \cdot a = a$ ($\forall a \in A$), $\epsilon \cdot b = -b$ ($\forall b \in B$); and where C is a direct sum of $\mathbb{Z}$-modules of the type $\mathbb{Z} \oplus \mathbb{Z}y$ such that $\epsilon \cdot z = -z$ ($\forall z \in \mathbb{Z} =$ the first summand) and $\epsilon \cdot y = 1 + y$.

In June of 2006 Tadao Oda and the author had a discussion on the real forms of algebraic tori, both of us then being unaware of the theorem just above. A little later, based on his own analysis, he communicated to this author the following *conjecture*: every $\mathbb{Z}[\epsilon]$-module $\mathbb{Z}$-free of finite rank is a direct sum of copies of $\mathbb{Z}[\epsilon], \mathbb{Z}[\epsilon]/\mathbb{Z}(1 - \epsilon)$ and $\mathbb{Z}[\epsilon]/(1 + \epsilon)$.

Rather recently the present author became aware of Diederichsen-Reiner Theorem and noticed that a proof of Oda's Conjecture follows easily from that theorem. Therefore, we have a ready-made

Theorem. *Every (Q/k)-form of algebraic torus of any dimension is k-isomorphic to a direct product of copies of $\mathbb{T}_k^1$, $\mathbb{U}_1 = \mathcal{R}^{(1)}_{Q|k}(\mathbb{T}^1_Q)$ and $\mathbb{U}_2 = \mathcal{R}_{Q|k}(\mathbb{T}^1_Q)$.*

This result implies that from dimension 3 and up there should occur no essentially new (Q/k)-forms of algebraic tori. Is this just a reflection of the fact $[Q:k]{=}2$, or also of the fact that algebraic tori are after all a direct product of conic curves? We do not know.

References

[1] C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience Publ. John Wiley & Sons, New York, 1962 (New edition available in two volumes.)

[2] T. Kambayashi, *On forms of the Laurent polynomial rings and algebraic tori in dimensions 1 and 2*, preprint (under revision).

[3] I. Reiner, *Integral representation of cyclic groups of prime order*, Proc. Amer. Math. Soc. **8** (1957), 142–146

[4] J.-P. Serre, Cohomologie Galoisienne, Lect. Notes in Math. No. 5, 5^e éd, Springer-Verlag 1994.

[5] K.Tahara, *On the finite subgroups of* $GL(3, \mathbb{Z})$, Nagoya Math. J. **41** (1971), 169–209

[6] V. E. Voskresenskii, *On two-dimensional algebraic tori*, Amer. Math. Soc. Transl. **73** (1968), 190–195

MARIUS KORAS
Institute of Mathematics University of Warsaw
UL.Banacha2 02-097 Warszawa, Poland
E-mail: koras@mimuw.edu.pl

Let V be the hypersurface in $\mathbb{C}^4$ given by $x + x^2y + z^2 + u^3 = 0$. V is diffeomorphic to $\mathbb{C}^3$ but not isomorphic to $\mathbb{C}^3$ (the Makar-Limanow invariant of V equals $\mathbb{C}[x]$). On V we have a $\mathbb{C}^1$-action given by $t(x, y, z, u) = (t^6x, t^{-6}, t^3z, t^2u)$. The fixed point set of the cyclic subgroup $\mathbb{Z}_6 \subset \mathbb{C}^*$ is disconnected.

Problem. *Is $V \times \mathbb{C}^1$ isomorphic to $\mathbb{C}^4$?*

If the answer is yes, then we obtain an example of nonlinearizable $\mathbb{C}^*$-action on C4 and also an example of nonlinearizable $\mathbb{C}^* \times \mathbb{C}^*$-action on $\mathbb{C}^4$. If the answer is no, then we are able (more or less) to prove that all $\mathbb{C}^* \times \mathbb{C}^*$-actions on $\mathbb{C}^4$ are linearizable.

The Łojasiewicz exponents of nondegenerate singularities and polynomials

TADEUSZ KRASIŃSKI
Faculty of Mathematics and Computer Science
University of Łódź
90-238 Łódź, Banacha 22, Poland
E-mail: krasinsk@uni.lodz.pl

1. Local case

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ with isolated critical point at 0 i.e. the mapping $\operatorname{grad} f = (\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ has an isolated zero at 0 (as usual we identify a function-germ with its representative). In this case f is called a *singularity*. One of interesting invariants of the singularity f is *the Łojasiewicz exponent* $\mathcal{L}_0(f)$ *of* f *at* 0 which is defined by

$$\mathcal{L}_0(f) = \inf\{\theta : |\operatorname{grad} f(z)| \geqslant C\,|z|^\theta \quad \text{in a ngh. of 0 for some constant } C > 0\}.$$

There are many known properties and effective formulas for $\mathcal{L}_0(F)$ (see [CK], [L-JT], [L], [P]). The basic property of $\mathcal{L}_0(F)$ is that it is a rational positive number.

The simplest singularities are *non-degenerate singularities*. They are defined via *Newton polyhedrons* – a combinatorial object connected with f. Let us recall it. For simplicity we consider the case $n = 2$ (in the general case definitions are similar). In 2-dimensional case Newton polyhedrons are called Newton diagrams.

Let

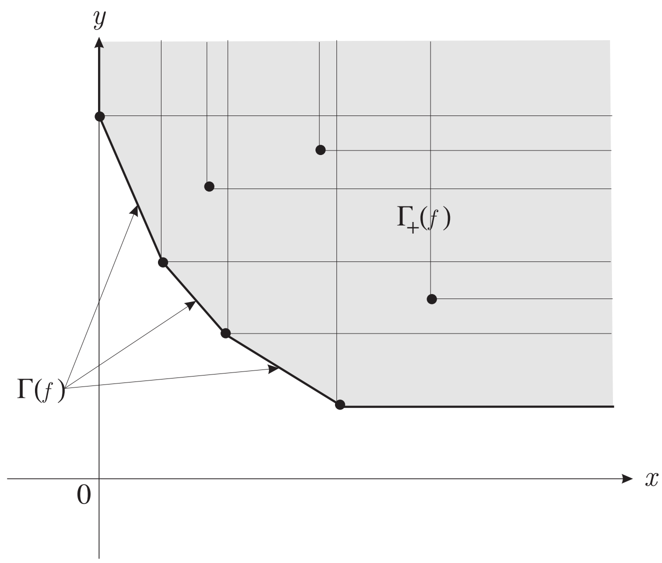
$$f(x, y) = \sum_{\alpha,\beta} a_{\alpha,\beta} x^\alpha y^\beta$$

be the expansion of f in a neighbourhood of $0 \in \mathbb{C}^2$ in a convergent Taylor series. We put

$$\operatorname{supp} f := \{(\alpha, \beta) \in \mathbb{N}_0^2 : \ a_{\alpha,\beta} \neq 0\}.$$

The *Newton diagram* $\Gamma_+(f)$ *of* f is the convex hull of $(\operatorname{supp} f) + \mathbb{R}^2_+$ (where $\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0\}$). The boundary of $\Gamma_+(f)$ is the union of two half-lines and a finite number of compact and pairwise non-parallel segments.

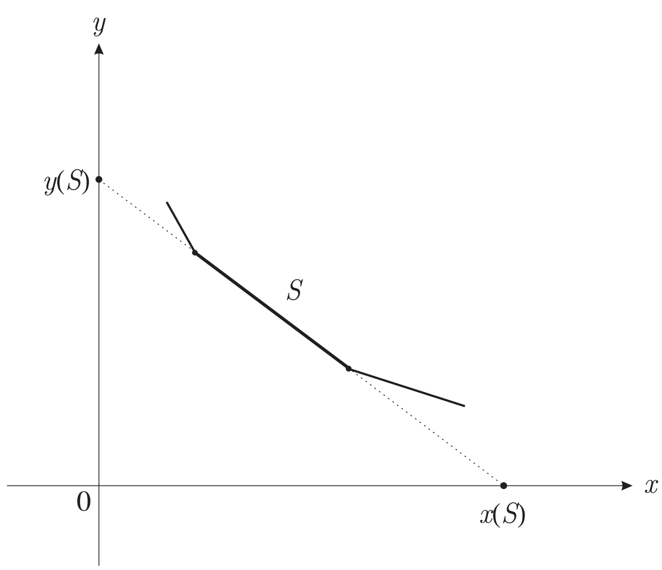
The Newton polygon $\Gamma(f)$ is the set of these compact segments.



For each segment $S \in \Gamma(f)$ we define:

1. $x(S)$ – the abscissa of the point, when the line determined by S intersects the horizontal axis,

2. $y(S)$ – the ordinate of the point, when the line determined by S intersects the vertical axis,

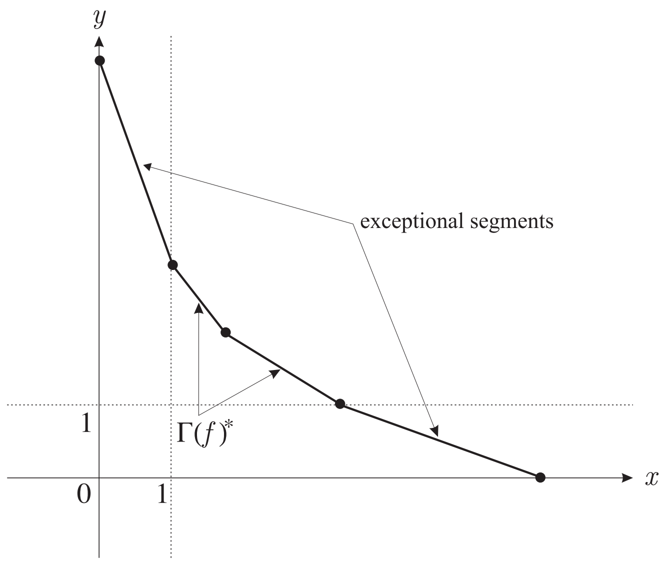


3. $\mathrm{in}(S) = \sum_{(\alpha,\beta)\in S} a_{\alpha,\beta} x^\alpha y^\beta$.

The *reduced Newton polygon* $\Gamma(f)^*$ is obtained from $\Gamma(f)$ by omitting the *exceptional segments* according to the rule:

(a) we omit the first segment if abscissas of their ends lie on the lines $x = 0$ and $x = 1$, respectively,

(b) we omit the last segment if ordinates of their ends lie on the lines $y = 0$ and $y = 1$, respectively.



In other words exceptional segments are segments in $\Gamma(f)$ which lie in the wall of thickness 1 around the axes.

Remark 1. Since we assume that f has an isolated critical point at $0 \in \mathbb{C}^2$, then it is easy to show that $\Gamma(f)^* = \emptyset$ if and only if in an appropriate linear system of coordinates in $\mathbb{C}^2$

$$f(x, y) = xy + \sum_{\substack{\alpha,\beta \\ \alpha+\beta\geqslant 3}} h_{\alpha,\beta} x^\alpha y^\beta.$$

For such f we have $\mathcal{L}_0(f) = 1$. Then, in the sequel, the assumption that $\Gamma(f)^* \neq \emptyset$ is not very restrictive.

We say f is *nondegenerate (in the Kouchnirenko's sense)* [K] if for every $S \in \Gamma(f)$ the system of equations

$$(*) \qquad \frac{\partial}{\partial x}\,\mathrm{in}(S) = 0, \qquad \frac{\partial}{\partial y}\,\mathrm{in}(S) = 0,$$

has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$.

Remark 2. It is easy to prove that for exceptional segments $S \in \Gamma(f)$ f is nondegenerate on S. Hence it suffices to consider the system

Theorem 1. ([L], Thm 2.1) *If f is a nondegenerate singularity at $0 \in \mathbb{C}^2$ and $\Gamma(f)^* \neq \emptyset$, then*

$$(**) \qquad \mathcal{L}_0(f) = \max_{S \in \Gamma(f)^*} (x(S), y(S)) - 1.$$

By this theorem and Remark 2 if f is nondegenerate then $\mathcal{L}_0(f)$ can be read of its Newton diagram.

In n-dimensional case one can analogously define the above notions for a singularity $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Namely, we define

1. $\mathcal{L}_0(f)$,
2. $\operatorname{supp} f$,
3. $\Gamma_+(f)$, $\Gamma(f)$,
4. $z_1(S), \ldots, z_n(S)$ for each face $S \in \Gamma(f)$, where $(z_1, \ldots, z_n)$ are coordinates in $\mathbb{C}^n$.
5. nondegenerateness of f.

Now, we may pose the problem

Problem 0.1. *Define appropriately exceptional faces of $\Gamma(f)$ such that*

$$\mathcal{L}_0(f) = \max_{S \in \Gamma(f)^*} (z_1(S), \ldots, z_n(S)) - 1.$$

2. Global case

If $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial one can similarly define *the Łojasiewicz exponent $\mathcal{L}_\infty(f)$ of f at infinity* by

$$\mathcal{L}_\infty(f) = \sup\{\theta : |\operatorname{grad} f(z)| \geqslant C\,|z|^\theta \quad \text{for } z \gg 0 \text{ for some constant } C > 0\}$$

and similar notions connected with the Newton polyhedron of f at infinity (see [L2], [BA], [O], [S]).

Problem 0.2. *Find effective formulas for $\mathcal{L}_\infty(f)$ in nondegerate case in n- dimensional case.*

Remark 3. As in local case the 2-dimensional case at infinity is completely solved [L2].

References

[BA] C. Bivià-Ausina, *Łojasiewicz exponents at infinity and injectivity of real polynomial maps,* preprint, 2006.

[CK] J. Chadzyński and T. Krasiński, *The Łojasiewicz exponent of an analytic mapping of two complex variables at an isolated zero,* in: Singularities, Banach Center Publ. 20, PWN, Warszawa 1988, 139–146

[K] A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor,* Invent. Math. **32** (1976), 1–31.

[L-JT] M. Lejeune-Jalabert and B. Teissier, *Cloture integrale des idéaux et equisingularité,* École Polytechnique 1974.

[L1] A. Lenarcik, *On the Łojasiewicz exponent of the gradient of a holomorphic function,* in: Singularities Symposium–Łojasiewicz 70. Banach Center Publ. 44, PWN, Warszawa 1998, 149–166.

[L2] A. Lenarcik, *On the Łojasiewicz exponent of the gradient of a polynomial function,* Ann. Polon. Math. **71** (1999), 211–239.

[O] M. Oka, *Note on boundary obstruction to jacobian conjecture of two variables,* Acta Math. Vietnam. (in this volume).

[P] A. Płoski, *Multiplicity and the Łojasiewicz exponent,* in: Singularities, Banach Center Publ. 20, PWN, Warszawa 1988, 353–364.

[S] P.T. So'n, *On the effective computation of Lojasiewicz exponents via Newton polyhedra,* Math. Hung. Per. (to appear).

Hilbert's Fourteenth Problem and the invariant fields of $\mathbb{Z}/2\mathbb{Z}$

SHIGERU KURODA
Department of Mathematics and Information Sciences
Tokyo Metropolitan University
1-1 Minami-Ohsawa, Hachioji
Tokyo 192-0397, Japan
E-mail: kuroda@tmu.ac.jp

Let $k[X]$ be the polynomial ring in n variables over a field k for some $n \in \mathbf{N}$, and $k(X)$ the field of fractions of $k[X]$. Then, *Hilbert's Fourteenth Problem* asks whether the k-algebra $L \cap k[X]$ is finitely generated whenever L is a subfield of $k(X)$ containing k. In 1958, Nagata solved this problem by giving a counterexample, where $k(X)$ is transcendental over L. In the case where $k(X)$ is algebraic over L, we gave a counterexample of extension degree $[k(X) : L] = d$ for each $d \geqslant 3$ when $n \geqslant 3$, and for each $d \geqslant 2$ when $n \geqslant 4$. On the other hand, $L \cap k[X]$ is always finitely generated if $n \leqslant 2$ due to Zariski. Clearly, $L \cap k[X] = k[X]$ if $[k(X) : L] = 1$, i.e., $L = k(X)$. However, the following problem remains open.

Problem. *Assume that $n = 3$, and L is a subfield of $k(X)$ containing k. Is the k-algebra $L \cap k[X]$ finitely generated if $[k(X) : L] = 2$?*

Note that $[k(X) : L] = 2$ if and only if L is the invariant subfield of $k(X)$ for some action of $\mathbb{Z}/2\mathbb{Z}$ over k.

Are locally finite polynomial automorphisms linked to locally finite derivations?

STEFAN MAUBACH
Department of Mathematics Radboud University
6525 ED Nijmegen, The Netherlands
E-mail: s.maubach@science.ru.nl

We define a polynomial map $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ to be *locally finite* if $deg(F^n)$ is bounded, i.e. $max(deg(F^n))$ is finite. This definition is equivalent to "for each $g \in \mathbb{C}^{[n]}$ the vector space generated by $g, F(g), F^2(g), \ldots$ is finite dimensional", and also to "there exist $n \in \mathbb{N}, a_i \in \mathbb{C}$ such that $\sum_{i=0}^{n} a_i F^i = 0$".

Now, as we know, if D is a locally finite derivation, then $F_D := exp(D)$ exists and is an automorphism of $\mathbb{C}^{[n]}$. It is also a locally finite polynomial automorphism: given $g \in \mathbb{C}^{[n]}$, we know that $g, D(g), D^2(g), \ldots$ is finite dimensional, which implies that $g, F_D(g), F_D^2(g), \ldots$ is finite dimensional. Now the obvious conjecture is: does the converse hold?

Conjecture. *Is a locally finite polynomial automorphism an exponent of a locally finite derivation?*

Note that it is conjectured that the exponents of locally finite derivations generate the automorphism group (this is equivalent to stating that the exponents of locally nilpotent derivations, plus the affine maps, generate the automorphism group).

The conjecture is proven in case the linear part of F has different nonzero eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1^{e_1}\lambda_2^{e_2}\cdots\lambda_n^{e_n} = 1$ where $e_i \in \mathbb{N}$, implies $e_1 = e_2 = \ldots = e_n = 0$.

References

[1] JJ-Ph. Furter, S. Maubach, *Locally finite polynomial endomorphisms*, to appear in J. Pure Appl. Algebra.

Do there exist odd polynomial automorphisms over $\mathbb{F}_4, \mathbb{F}_8, \ldots$?

STEFAN MAUBACH
Department of Mathematics Radboud University
6525 ED Nijmegen, The Netherlands
E-mail: s.maubach@science.ru.nl

Write $\mathbb{F}_q$ for the field with $q = p^m$ elements. Given a polynomial automorphism F of $\mathbb{F}_q^{[n]}$, we get a bijection $B_F : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$. Note that, contrary to infinite fields, an endomorphism of $\mathbb{F}_q^{[n]}$ can be non-invertible but induce a bijective map $B_F : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$ (like the map X^3 on $\mathbb{F}_3^{[1]}$).
What was done in the paper [1] is compute which bijections of $\mathbb{F}_q^n$ can be made by tame automorphisms of $\mathbb{F}_q^{[n]}$. It turned out that

- if q is odd, or if $q = 2$, one can make any bijection.
- If $q = 2^m$ where $m \geqslant 2$, then one can only make half of the bijection: any tame automorphism of $\mathbb{F}_q^{[n]}$, seen as a bijection $\mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$ will induce an even permutation of the symmetric group with q^n elements.

The question is thus:

Conjecture. *If $q = 2^m$ where $m \geqslant 2$, then any polynomial automorphism of $\mathbb{F}_q^{[n]}$ induces an even bijection $\mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$.*

Note that answering this question in the negative would imply that one has found a non-tame automorphism, with trivial proof that it is non-tame.

References

[1] S. Maubach, *Polynomial automorphisms over finite fields,* Serdica Math. J. **27** (2001), no. 4, 343–350.

Open Problems and Comments

MASAYOSHI MIYANISHI
School of Science and Technology
Kwansei Gakuin University
2-1 Gakuen, Sanda
Hyogo 669–1337, Japan
E-mail: miyanisi@ksc.kwansei.ac.jp

Let X be a normal affine variety defined over the complex field $\mathbb{C}$ with an effective algebraic action of an algebraic group G. Let $\varphi : X \to X$ be an unramified endomorphism which commutes with the G-action. The following is the equivariant version of the generalized Jacobian conjecture.

Conjecture 1. *With the above settings, φ is a finite morphism.*

If X has the Euler number $\chi(X) = 1$ then the conjecture says that φ is an automorphism.

Suppose that the algebraic quotient $Y := X//G$ exists under the above setting. Then φ induces an endomorphism $\psi : Y \to Y$. We have the following result.

Theorem 2. (cf. [8]) *Let the notations and assumptions be the same as in the above conjecture. Suppose that G is a reductive algebraic group. Then the endomorphism $\psi : Y \to Y$ is unramified.*

If G is a unipotent algebraic group, ψ is not necessarily unramified as shown by the following example.

Example 3. Let X be an affine smooth surface with an $\mathbb{A}^1$-fibration $\rho : X \to C$ such that $C \cong \mathbb{A}^1$ and ρ has two irreducible multiple fibers of multiplicity 2 (cf. [4, Example 2.2.7]). Let $2F_0, 2F_1$ be the multiple fibers of ρ. Let $P_0 = \rho(F_0)$ and $P_1 = \rho(F_1)$. Let $\nu : C' \to C$ be the double covering which ramifies over the point P_0 and the point at infinity P_∞. Let $\nu^{-1}(P_1) = \{Q_1, Q_2\}$. Let $\widetilde{X}$ be the normalization of the fiber product $X \times_C C'$ and let $\widetilde{\rho} : \widetilde{X} \to C'$ be the natural $\mathbb{A}^1$-fibration induced by ρ. Then $\widetilde{\rho}$ has two multiple fibers of multiplicity 2 lying over the points Q_1, Q_2 and has one reduced, reducible fiber $\widetilde{\rho}^*(Q_0) = G_1 + G_2$, where $\nu^{-1}(P_0) = \{Q_0\}$. Let $X' := \widetilde{X} - G_2$. Then X' is isomorphic to X and the covering morphism $\widetilde{\varphi} : \widetilde{X} \to X$ restricted onto X' is a non-finite unramified endomorphism of degree 2. Since the $\mathbb{A}^1$-fibration ρ is trivial on the open set $U = C - \{P_0, P_1\}$, write $\rho^{-1}(U) = U \times \mathbb{A}^1$. Similarly, the $\mathbb{A}^1$-fibration $\rho' : X' \to C'$, which is induced by $\widetilde{\rho}$, is trivial on the subset $U' := \nu^{-1}(U)$. If we take a fiber parameter t on $\rho^{-1}(U) = U \times \mathbb{A}^1 = U \times \operatorname{Spec} \mathbb{C}[t]$, then t is also a fiber parameter on ${\rho'}^{-1}(U') = U' \times \mathbb{A}^1$. Since X is affine, the partial derivative $\partial/\partial t$ multiplied by a suitable regular function a on C gives rise to a locally nilpotent derivation δ on the coordinate ring $B = \Gamma(X, \mathcal{O}_X)$. Then δ defines a non-trivial G_a action σ on X such that ρ is the quotient morphism. Since the covering morphism $\widetilde{X} \to X$ is a finite étale morphism, the G_a-action σ lifts uniquely to a G_a-action $\widetilde{\sigma}$ on $\widetilde{X}$ which stabilizes the component G_2. Hence the G_a-action σ lifts up uniquely to a G_a-action σ' on X' such that $\varphi \cdot \sigma' = \sigma \cdot \varphi$. In other terms, the locally nilpotent derivation δ lifts up uniquely to a locally nilpotent derivation δ' on the coordinate ring B' of X' such that $\varphi^* \cdot \delta = \delta' \cdot \varphi^*$. Then the algebraic quotients $X//G_a$ and $X'//G_a$ are C and C' respectively, and the induced morphism $\psi : C' \to C$ coincides with ν. However, by the construction, ν is ramified.

Notwithstanding, with the assumption that $\Gamma(X, \mathcal{O}_X)$ is factorial if G is unipotent, one can hope that ψ is still unramified. If $\dim X = 2$, the factoriality of $\Gamma(X, \mathcal{O}_X)$ implies that the quotient morphism $\mu : X \to Y$ is a smooth morphism. The following result entails the unramifiedness of ψ when $\dim X = 2$.

Lemma 4. *Let $\mu : X \to Y$ and $\mu' : X' \to Y'$ be two $\mathbb{A}^1$-fibrations between affine varieties, and let $\varphi : X' \to X$ be an étale morphism. Let $\psi : Y' \to Y$ be a morphism such that $\psi \cdot \mu' = \mu \cdot \varphi$. Then there exists an exact sequence of $\mathcal{O}_{X'}$-Modules*

$$\mu'^* \Omega^1_{Y'/Y} \longrightarrow \varphi^* \Omega^1_{X/Y} \longrightarrow \Omega^1_{X'/Y'} \longrightarrow 0 \, .$$

If μ' and μ are smooth morphisms, then ψ is an étale morphism if and only if $\varphi^* \Omega^1_{X/Y} \xrightarrow{\sim} \Omega^1_{X'/Y'}$.

The equivariant Jacobian conjecture can be decomposed into the following conjectures.

Conjecture 5. *Let X be a normal affine variety with an effective algebraic group action of G. Suppose that the algebraic quotient $Y = X//G$ exists. Let $\varphi : X \to X$ be an unramified endomorphism which commutes with the G-action on X. Let $\psi : Y \to Y$ be the induced endomorphism. Then ψ is a finite morphism.*

Conjecture 6. *Let the notations and assumptions be the same as in Conjecture 5. Suppose that the Euler number $\chi(X) = 1$ and that the induced endomorphism ψ is an automorphism. Then the endomorphism φ is an automorphism.*

Note that the equivariant generalized Jacobian conjecture is the ordinary generalized Jacobian conjecture when G is trivial. The Conjecture 6 was treated in [6] without assuming that φ is unramified (see [1] also). By Gurjar [2], the two-dimensional quotient of $\mathbb{A}^n$ under an algebraic group action of a reductive group is isomorphic to $\mathbb{A}^2/\Gamma$, where Γ is a small finite subgroup of $\mathrm{GL}\,(2,\mathbb{C})$. Thus, if one notes that $\chi(\mathbb{A}^2/\Gamma) = 1$, the Conjecture 5 in this case is reduced to ask whether an unramified endomorphism of $\mathbb{A}^2/\Gamma$ is an automorphism. Indeed, the generalized Jacobian conjecture for $\mathbb{A}^2/\Gamma$ is equivalent to the following (cf. [9]):

Conjecture 7. *Let φ be an unramified endomorphism of $\mathbb{A}^2$ which commutes with a linear action of a small finite subgroup Γ of $\mathrm{GL}\,(2,\mathbb{C})$. Then φ is an automorphism.*

There are several known results (see [8]).

Lemma 8. *The following assertions hold.*

(1) *Suppose that G is a reductive algebraic group. Suppose further that ψ is an automorphism and that a general fiber of the quotient morphism $\mu : X \to Y$ contains a dense orbit. Then φ is an automorphism.*
(2) *The generalized Jacobian conjecture holds if $\dim X = 1$.*
(3) *Let G be a reductive algebraic group. Suppose that $\dim Y = 1$ and that a general fiber of μ contains a dense orbit. Suppose further that $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$. Then φ is an automorphism.*
(4) *Let G be a unipotent algebraic group. Suppose that the induced unramified endomorphism $\psi : Y \to Y$ is an automorphism. Then φ is an automorphism.*

Lemma 9. *Suppose that the multiplicative group G_m acts linearly and effectively on the affine space $\mathbb{A}^n$. Write the G_m-action as*

$$^t(x_1, \ldots, x_n) = (t^{\alpha_1}x_1, \ldots, t^{\alpha_n}x_n)$$

with $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n$. Suppose further that $\alpha_1 \geqslant 0$. Let m be the dimension of the G_m-fixed point locus. If the Jacobian conjecture for $\mathbb{A}^m$ holds, then $\varphi : \mathbb{A}^n \to \mathbb{A}^n$, which is a G_m-equivariant unramified endomorphism, is an automorphism.

Theorem 10. (cf. [7]) *Let an algebraic group G of positive dimension act effectively on a normal affine surface X and let $\varphi : X \to X$ be a G-equivariant unramified endomorphism. Suppose that $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$ and that $\Gamma(X, \mathcal{O}_X)$ is factorial if G is unipotent. Suppose further that X is not elliptic-ruled* [1]. *Then φ is an automorphism.*

If one uses results obtained in [3, 5, 11, 12], one can obtain the following result.

Theorem 11. *The following assertions hold.*

(1) *Let $\varphi : \mathbb{A}^3 \to \mathbb{A}^3$ be an unramified endomorphism which commutes with an effective G_m-action. If the Jacobian conjecture for $\mathbb{A}^2$ and the Conjecture 7 hold, then φ is an automorphism.*

[1] X is said to be elliptic-ruled if X is birational to a $\mathbb{P}^1$-bundle over an elliptic curve

(2) *Let $\varphi : \mathbb{A}^3 \to \mathbb{A}^3$ be an unramified endomorphism which commutes with a fixed-point free G_a-action. If the Jacobian conjecture for $\mathbb{A}^2$ holds, φ is an automorphism.*

References

[1] J. Ax, *The elementary theory of finite fields*, Ann. of Math. **88** (1968), 239–271.
[2] R.V. Gurjar, *Two dimensional quotients of $\mathbb{C}^n$ are isomorphic to $\mathbb{C}^2/\Gamma$*, preprint.
[3] Sh. Kaliman, *Free $\mathbb{C}_+$-actions on $\mathbb{C}^3$ are translations*, Invention. Math. **156** (2004), 163–173.
[4] T. Kambayashi and M. Miyanishi, *On two recent views of the Jacobian Conjecture*, Contemporary Mathematics **369** (2005), 113–138.
[5] M. Koras and P. Russell, *Contractible threefolds and $\mathbb{C}^*$-actions on $\mathbb{C}^3$*, J. Algebraic Geom. **6** (1997), no. 4, 671–695.
[6] K. Masuda, *G-endomorphisms of affine G-varieties which induce automorphisms of the invariant subrings of the coordinate rings*, J. Algebra, to appear.
[7] K. Masuda and M. Miyanishi, *Étale endomorphisms of algebraic surfaces with G_m-actions*, Math. Ann. **319** (2001), 493–516.
[8] K. Masuda and M. Miyanishi, *Equivariant Jacobian conjectures*, preprint.
[9] M. Miyanishi and K. Masuda, *Generalized Jacobian conjecture and related topics*, the Proceedings of International Colloquium on Algebra, Arithmetic and Geometry, 427–466, Tata Institute of Fundamental Research and International Mathematical Union, January, 2000.
[10] M. Miyanishi, *Curves on rational and unirational surfaces*, Tata Institute of Fundamental Research, Lectures on Mathematics and Physics **60**, Springer Verlag, 1978.
[11] M. Miyanishi, *Normal affine subalgebras of a polynomial ring*, Algebraic and Topological Theories - to the memory of Dr. Takehiko MIYATA, Kinokuniya, 1985, 37-51.
[12] M. Miyanishi, *Open algebraic surfaces*, Centre de Recherches Mathématiques, Vol **12**, Université de Montréal, Amer. Math. Soc., 2001.

The integer case of the plane Jacobian conjecture as a problem on integer points in plane curve

NGUYEN VAN CHAU
Institute of Mathematics
18 Hoang Quoc Viet Road
10307 Hanoi, Vietnam
E-mail: nvchau@math.ac.vn

Given $F = (P, Q) \in \mathbb{Z}[x, y]^2$, a polynomial map with integer coefficients. The mysterious Jacobian conjecture (JC), posed first by Keller in 1939 asserts that such a map F is invertible and has a polynomial inverse with integer coefficients if Jacobian $JF := PxQy - PyQx \equiv 1$. It was observed in [C] that *if $JF \equiv 1$ and if the complex plane curve $P = 0$ has infinitely many integer points, then such a map F has a polynomial inverse with integer coefficients*. This observation reduces the integer case of (JC) to a question of the algebra-arithmetic geometry.

Question 1. (Integer case of (JC)) *Whether the Jacobian condition $JF \equiv 1$ ensures that the curve $P = 0$ has infinitely many integer points ?*

In fact, the proof in [C] shows that if F has not polynomial inverse, then the numbers of integer points in curves $P = k$, $k \in \mathbb{Z}[i]$, must be uniformly bounded. In view of Siegel's theorem [Abh. Deutsch. Akad. Wiss. Berlin Kl. Phys.-Mat. 1929, no. 1] such a curve $P = 0$ with infinitely many integer points must be a rational curve.

Question 2. (Rational case of (JC)). *Whether a polynomial map $f = (p, q) \in \mathbb{C}[x, y]^2$ with $Jf \equiv c \in \mathbb{C}^*$ is invertible if the curve $p = 0$ is a rational curve ?.*

Note that such a map f is invertible if the curve $p = 0$ has an irreducible component homeomorphic to $\mathbb{C}$ or if all fibres of p are irreducible and the generic fiber of p is a rational curve (see [R], [LW] and [NN]).

References

[C] Nguyen Van Chau, *Integer points on a curve and the plane Jacobian problem*, Ann. Polon. Math. **88** (2006), 53-58.

[LW] Le Dung Trang and C. Weber, *Polynómes fibres rationnelles et conjecture jacobienne 2 variables.* C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), 581–584.

[NN] W.D. Neumann and P. Norbury, *Nontrivial rational polynomials in two variables have reducible fibres.* Bull. Austral. Math. Soc. **58** (1998), 501–503.

[R] M. Razar, *Polynomial maps with constant Jacobian.* Israel J. Math. **32** (1979), 97–106.

Is $\mathrm{SAut}R[t][x_1, \ldots, x_n] \to \mathrm{SAut}R[t]/(t^m)[x_1, \ldots, x_n]$ surjective?

VÉNÉREAU STÉPHANE
Mathematical Institute
University Basel
Rheinsprung 21, CH-4051 Basel, Schweit
E-mail: stephane.venereau@unibas.ch

The ring R is commutative with unity and t is an indeterminate. The notation SAut stands for the Special Automorphism Group, i.e. the group of automorphisms with Jacobian determinant equal to one. The application in question in the title is the morphism of groups induced by the canonical epimorphism $R[t] \longrightarrow R[t]/(t^m)$. Note that for $m = 1$ the answer is trivially yes. This question has positive answer for any ring containing $\mathbb{Q}$ and any $m, n \geqslant 1$ (see [EMV]). When R has positive characteristic p, or more generally when $pr = 0$ for some $r \in R/\sqrt{(0)}$ and for $n = 1$ then the automorphism $\alpha : x_1 \mapsto x_1 + rtx_1^p \in \mathrm{SAut}R[t]/(t^m)[x_1]$ furnishes a negative answer $\forall m \geqslant 2$.

So the remaining cases are : $n \geqslant 2$ and $\mathbb{Q} \not\subset R$. The motivation for the question comes from [V] where the non- surjectivity case given above is at the origin of the construction of some "bad" objects (see the references.)

References

[EMV] A. van den Essen, S. Maubach and S. Vénéreau, *The special automorphism group of $R[t]/(t^m)[x_1, \ldots, x_n]$ and the coordinates of a subring of $R[t][x_1, \ldots, x_n]$*, J. Pure Appl. Algebra **210** (2007), 141–146.

[V] S. Vénéreau, *New bad lines in $R[x, y]$ and optimization of the Epimorphism Theorem*, J. Algebra **302** (2006), 729-749.

Stable Tameness

DAVID WRIGHT
Department of Mathematics
Washington University in St. Louis
St. Louis, MO 63130, USA

Let $R = k^{[n]}$, the polynomial ring in n variables over a field k. Let $\mathrm{GA}_2(R)$ denote the automorphisms of $\mathbb{A}^2_R$.

Problem 1. *Are all elements of* $\mathrm{GA}_2(R)$ *stably tame?*

Remark. The *length* of an element of $\mathrm{GA}_2(R)$ is defined the minimal number of elementary automorphisms in a factorization of it in $\mathrm{GA}_2(K)$, where K is the field of fractions of R. This question is answered affirmatively for elements of length $\leqslant 3$ in [1]. Sooraj Kuttykrishnan has now resolved the length 4 case. These results assume only that R is a UFD, with Kutttykrishnan's result requiring a further mild condition.

Problem 2. *What is the structure of* $\mathrm{GA}_2(R)$?

Remark. Actually it is proved in [2] and [3] that $\mathrm{GA}_2(R)$ has the structure of an amalgamated free product

$$\mathrm{Af}_2(k) *_{\mathrm{Bf}_2(k)} W,$$

where $\mathrm{Af}_2(k)$ is the affine group over k, $\mathrm{Bf}_2(k)$ is the lower triangular affine group, and W is an obscure group which is a bit difficult to define (see Theorem 1 of [2]). We would like to have a better understanding of W.

References

[1] E. Edo, *Totally stably tame variables*, J. Algebra **287** (2005), 15-31.
[2] D. Wright, *The amalgamated free product structure of $GL_2(k[X, Y])$ and the weak Jacobian theorem for two variables*, J. Pure Appl. Algebra **12** (1978), 235-251.
[3] D. Wright, *Normal forms and the Jacobian conjecture*, Automorphisms of Affine Spaces (A. van den Essen, ed.), Kluwer Academic Publishers, 1995, 145-156.