A NOTE ON SINGULARITY AND NON-PROPER VALUE SET OF POLYNOMIAL MAPS OF $\mathbb{C}^2$

NGUYEN VAN CHAU

ABSTRACT. Some properties of the relation between the singular point set and the non-proper value curve of polynomial maps of $\mathbb{C}^2$ are expressed in terms of Newton-Puiseux expansions.

1. INTRODUCTION

Recall that the so-called *non-proper value set* A_f of a polynomial map $f = (P,Q) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $P, Q \in \mathbb{C}[x,y]$, is the set of all point $b \in \mathbb{C}^2$ such that there exists a sequence $\mathbb{C}^2 \ni a_i \to \infty$ with $f(a_i) \to b$. The set A_f is empty if and only if f is proper and f has a polynomial inverse if and only if f has not singularity and $A_f = \emptyset$. The mysterious Jacobian conjecture (JC) (See [4] and [8]), posed first by Keller in 1939 and still open, asserts that if f has not singularity, then f has a polynomial inverse. In other words, (JC) shows that the non-proper value set of a non-singular polynomial map of $\mathbb{C}^2$ must be empty. In any way one may think that the knowledge on the relation between the singularity set and the non-proper value set should be useful in pursuit of this conjecture.

Jelonek in [9] observed that for non-constant polynomial map f of $\mathbb{C}^2$ the non-proper value set A_f, if non empty, must be a plane curve such that each of its irreducible components can be parameterized by a non-constant polynomial map from $\mathbb{C}$ into $\mathbb{C}^2$. Following [6], the non-proper value set A_f can be described in term of Newton-Puiseux expansion as follows. Denote by Π the set of all finite fractional power series $\varphi(x, \xi)$ of the form

$$(1.1) \quad \varphi(x, \xi) = \sum_{k=1}^{n_\varphi - 1} a_k x^{1 - \frac{k}{m_\varphi}} + \xi x^{1 - \frac{n_\varphi}{m_\varphi}}, n_\varphi, m_\varphi \in \mathbb{N}, \ \gcd\{k : a_k \neq 0\} = 1,$$

where ξ is a parameter. For convenience, we denote $\psi \prec \varphi$ if $\varphi(x, \xi) = \psi(x, c + $ lower terms in x). We can fix a coordinate (x, y) such that P and Q are monic

in y, i.e $\deg_y P = \deg P$ and $\deg_y Q = \deg Q$. For each $\varphi \in \Pi$ we represent

$$P(x, \varphi(x, \xi)) = p_\varphi(\xi)x^{\frac{a_\varphi}{m_\varphi}} + \text{lower terms in } x, 0 \neq p_\varphi \in \mathbb{C}[\xi]$$

(1.2)

$$Q(x, \varphi(x, \xi)) = q_\varphi(\xi)x^{\frac{b_\varphi}{m_\varphi}} + \text{lower terms in } x, 0 \neq q_\varphi \in \mathbb{C}[\xi]$$

$$J(P, Q)(x, \varphi(x, \xi)) = j_\varphi(\xi)x^{\frac{J_\varphi}{m_\varphi}} + \text{lower terms in } x, 0 \neq j_\varphi \in \mathbb{C}[\xi].$$

Note that a_φ, b_φ and J_φ are integer numbers.

A series $\varphi \in \Pi$ is *a horizontal series* of P (of Q) if $a_\varphi = 0$ and $\deg p_\varphi > 0$ (resp. $b_\varphi = 0$ and $\deg q_\varphi > 0$), φ is a *dicritical series* of $f = (P, Q)$ if φ is a horizontal series of P or Q and $\max\{a_\varphi, b_\varphi\} = 0$ and φ is a *singular series* of f if $\deg j_\varphi > 0$. Note that for every singular series φ of f the equation $J(P, Q)(x, y) = 0$ always has a root $y(x)$ of the form $\varphi(x, c + \text{lower terms in } x)$, which gives a branch curve at infinity of the curve $J(P, Q) = 0$. We have the following relations:

i) If f (resp. P, Q) tends to a finite value along a branch curve at infinity γ, then there is a dicritical series φ of f (resp. a horizontal series φ of P, a horizontal series φ of Q) such that γ can be represented by a Newton-Puiseux of the form $\varphi(x, c + \text{lower terms in } x)$;

ii) If φ is a dicritical series of f and

$$f(x, \varphi(x, \xi)) = f_\varphi(\xi) + \text{lower terms in } x;$$

then $\deg f_\varphi > 0$ and its image is a component of A_f.

iii) (Lemma 4 in [6])

$$A_f = \bigcup_{\varphi \text{ is a dicritical series of } f} f_\varphi(\mathbb{C}).$$

This note is to present the following relation between the singularity set of f and the non-proper value set A_f in terms of Newton-Puiseux expansion.

Theorem 1.1. *Suppose* $\psi \in \Pi$, $a_\psi > 0$ *and* $b_\psi > 0$, $(a_\psi, b_\psi) = (Md, Me)$, $M \in \mathbb{N}$, $\gcd(d, e) = 1$. *Assume that* $\varphi \in \Pi$ *is a dicritical series of* f *such that* $\psi \prec \varphi$. *If* ψ *is not a singular series of* f, *then*

(i) $(\deg p_\psi, \deg q_\psi) = (Nd, Ne)$ *for some* $N \in \mathbb{N}$,
(ii) $a_\varphi = b_\varphi = 0$ *and*

(1.3)

$$p_\varphi(\xi) = Lcoeff(p_\psi)C^d\xi^{Dd} + \dots$$
$$q_\varphi(\xi) = Lcoeff(q_\psi)C^e\xi^{De} + \dots$$

for some $C \in \mathbb{C}^*$ *and* $D \in \mathbb{N}$.

Here, $Lcoeff(h)$ indicates the coefficient of the leading term of $h(\xi) \in \mathbb{C}[\xi]$.

Theorem 1.1 does not say anything about the existence of dicritical series φ, but only shows some properties of pair $\psi \prec \varphi$. Such analogous observations for the case of non-zero constant Jacobian polynomial map f was obtained earlier in [7].

For the case when $J(P,Q) \equiv const. \neq 0$, from Theorem 1.1 (ii) it follows that if $A_f \neq \emptyset$, then every irreducible components of A_f can be parameterized by polynomial maps $\xi \mapsto (p(\xi), q(\xi))$ with

$$(1.4) \qquad\qquad \deg p / \deg q = \deg P / \deg Q.$$

This fact was presented in [6] and can be reduced from [3]. The estimation (1.4) together with the Abhyankar-Moh Theorem on embedding of the line to the plane in [1] allows us to obtain that *a non-constant polynomial map f of $\mathbb{C}^2$ must have singularities if its non-proper value set A_f has an irreducible component isomorphic to the line.* In fact, if A_f has a component l isomorphic to $\mathbb{C}$, by Abhyankar-Moh Theorem one can choose a suitable coordinate so that l is the line $v = 0$. Then, every dicritical series φ with $f_\varphi(\mathbb{C}) = l$ must satisfy $a_\varphi = 0$ and $b_\varphi < 0$. For this situation we have

Theorem 1.2. *Suppose φ is a dicritical series φ of f with $a_\varphi = 0$ and $b_\varphi < 0$. Then, either φ is a singular series of f or there is a horizontal series ψ of Q such that ψ is a singular series of f and $\psi \prec \varphi$.*

The proof of Theorem 1.1 presented in the next sections 2- 4 is based on those in [7]. The proof of Theorem 1.2 will be presented in Section 5.

2. ASSOCIATED SEQUENCE OF PAIR $\psi \prec \varphi$.

From now on, $f = (P,Q) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is a given polynomial map, $P, Q \in \mathbb{C}[x,y]$. The coordinate (x,y) is chosen so that P and Q are polynomials monic in y, i.e. $\deg_y P = \deg P$ and $\deg_y Q = \deg Q$. Let $\psi, \varphi \in \Pi$ be given. In this section and the two next sections 3-4 we always assume that ψ is not a singular series of f, φ is a dicritical series of f and $\psi \prec \varphi$.

Let us represent

$$(2.1) \qquad \varphi(x,\xi) = \psi(x,0) + \sum_{k=0}^{K-1} c_k x^{1 - \frac{n_k}{m_k}} + \xi x^{1 - \frac{n_K}{m_K}},$$

where $\frac{n_\psi}{m_\psi} = \frac{n_0}{m_0} < \frac{n_1}{m_1} < \cdots < \frac{n_{K-1}}{m_{K-1}} < \frac{n_K}{m_K} = \frac{n_\varphi}{m_\varphi}$ and $c_k \in \mathbb{C}$ may be the zero, so that the sequence of series $\{\varphi_i\}_{i=0,1\ldots,K}$ defined by

$$(2.2) \qquad \varphi_i(x,\xi) := \psi(x,0) + \sum_{k=0}^{i-1} c_k x^{1 - \frac{n_k}{m_k}} + \xi x^{1 - \frac{n_i}{m_i}}, i = 0, 1, \ldots, K - 1,$$

and $\varphi_K := \varphi$ satisfies the following properties:

S1) $m_{\varphi_i} = m_i$.

S2) For every $i < K$ at least one of polynomials p_{φ_i} and q_{φ_i} has a zero point different from the zero.

S3) For every $\phi(x,\xi) = \varphi_i(x,c_i) + \xi x^{1-\alpha}$, $\frac{n_i}{m_i} < \alpha < \frac{n_{i+1}}{m_{i+1}}$, each of the polynomials p_ϕ and q_ϕ is either constant or a monomial of ξ.

The representation (2.1) of φ is thus the longest representation such that for each index i there is a Newton-Puiseux root $y(x)$ of $P = 0$ or $Q = 0$ such that

$y(x) = \varphi_i(x, c + \text{ lower terms in } x)$, $c \neq 0$ if $c_i = 0$. This representation and the associated sequence $\varphi_0 \prec \varphi_1 \prec \cdots \prec \varphi_K = \varphi$ is well defined and unique. Further, $\varphi_0 = \psi$.

For simplicity in notations, below we shall use lower indices "i" instead of the lower indices "φ_i".

For each associated series φ_i, $i = 0 \ldots, K$, let us represent

(2.3)
$$P(x, \varphi_i(x, \xi)) = p_i(\xi) x^{\frac{a_i}{m_i}} + \text{lower terms in } x$$
$$Q(x, \varphi_i(x, \xi)) = q_i(\xi) x^{\frac{b_i}{m_i}} + \text{lower terms in } x,$$

where $p_i, q_i \in \mathbb{C}[\xi] - \{0\}$, $a_i, b_i \in \mathbb{Z}$ and $m_i := \text{mult}(\varphi_i)$.

The property that P and Q are polynomials monic in y ensures that the Newton-Puiseux roots at infinity $y(x)$ of each equations $P(x, y) = 0$ and $Q(x, y) = 0$ are fractional power series of the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^{1 - \frac{k}{m}}, \ m \in \mathbb{N}, \ \gcd\{k : c_k \neq 0\} = 1,$$

for which the map $\tau \mapsto (\tau^m, y(\tau^m))$ is meromorphic and injective for τ large enough . Let $\{u_i(x), i = 1, \ldots \deg P\}$ and $\{v_j(x), j = 1, \ldots \deg Q\}$ be the collections of the Newton-Puiseux roots of $P = 0$ and $Q = 0$, respectively. In view of the Newton theorem we can represent

(2.4)
$$P(x, y) = A \prod_{i=1}^{\deg P} (y - u_i(x)), \quad Q(x, y) = B \prod_{j=1}^{\deg Q} (y - v_i(x)).$$

We refer the readers to [2] and [5] for the Newton theorem and the Newton-Puiseux roots.

For each $i = 0 \ldots, K$, let us define

- $S_i := \{k : 1 \leqslant k \leqslant \deg P : u_k(x) = \varphi_i(x, a_{ik} + \text{ lower terms in } x), a_{ik} \in \mathbb{C}\}$;
- $T_i := \{k : 1 \leqslant k \leqslant \deg Q : v_k(x) = \varphi_i(x, b_{ik} + \text{ lower terms in } x), b_{ik} \in \mathbb{C}\}$;
- $S_i^0 := \{k \in S_i : a_{ik} = c_i\}$;
- $T_i^0 := \{k \in T_i : b_{ik} = c_i\}$.

Represent

$$p_i(\xi) = A_i \bar{p}_i(\xi)(\xi - c_i)^{\#S_i^0}, \bar{p}_i(\xi) := \prod_{k \in S_i \backslash S_i^0} (\xi - a_{ik}),$$

and

$$q_i(\xi) = B_i \bar{q}_i(\xi)(\xi - c_i)^{\#T_i^0}, \bar{q}_i(\xi) := \prod_{k \in T_i \backslash T_i^0} (\xi - b_{ik}).$$

Note that $A_i = Lcoeff(p_i)$ and $B_i = Lcoeff(q_i)$.

Lemma 2.1. *For $i = 1, \ldots, K$*

$$A_i = A_{i-1}\bar{p}_{i-1}(c_{i-1}), \deg p_i = \#S_i = \#S_{i-1}^0,$$

$$\frac{a_i}{m_i} = \frac{a_{i-1}}{m_{i-1}} + \#S_{i-1}^0\left(\frac{n_{i-1}}{m_{i-1}} - \frac{n_i}{m_i}\right),$$

$$B_i = B_{i-1}\bar{q}_{i-1}(c_{i-1}), \deg q_i = \#T_i = \#T_{i-1}^0,$$

$$\frac{b_i}{m_i} = \frac{b_{i-1}}{m_{i-1}} + \#T_{i-1}^0\left(\frac{n_{i-1}}{m_{i-1}} - \frac{n_i}{m_i}\right).$$

Proof. Note that $\varphi_0(x, \xi) = \psi(x, \xi)$ and $\varphi_i(x, \xi) = \varphi_{i-1}(x, c_{i-1}) + \xi x^{1-\frac{n_i}{m_i}}$ for $i > 0$. Then, substituting $y = \varphi_i(x, \xi)$, $i = 0, 1, \ldots, K$, into the Newton factorizations of $P(x, y)$ and $Q(x, y)$ in (2.4) one can easy verify the conclusions. □

3. Polynomials $j_i(\xi)$

Let $\{\varphi_i\}$ be the associated series of the pair $\psi \prec \varphi$. Denote

$$\Delta_i(\xi) := a_i p_i(\xi)\dot{q}_i(\xi) - b_i \dot{p}_i(\xi)q_i(\xi).$$

As assumed, ψ is not a singular series of f. So, we have

$$J(P, Q)(x, \psi(x, \xi)) = j_\psi x^{\frac{J_\psi}{m_\psi}} + \text{lower terms in } x, j_\psi \equiv const. \in \mathbb{C}^*$$

and

$$J(P, Q)(x, \varphi_i(x, \xi)) = j_i x^{\frac{J_i}{m_i}} + \text{lower terms in } x, j_i \equiv const. \in \mathbb{C}^*$$

for $i = 0, \ldots, K$.

Lemma 3.1. *Let $0 \leqslant i < K$. If $a_i > 0$ and $b_i > 0$, then*

$$\Delta_i(\xi) \equiv \begin{cases} -m_i j_i & \text{if } a_i + b_i = 2m_i - n_i + J_i, \\ 0 & \text{if } a_i + b_i > 2m_i - n_i + J_i. \end{cases}$$

Further, $\Delta_i(\xi) \equiv 0$ if and only if $p_i(\xi)$ and $q_i(\xi)$ have a common zero point. In this case

$$p_i(\xi)^{b_i} = Cq_i(\xi)^{a_i}, \ C \in \mathbb{C}^*.$$

Proof. Since $a_i > 0$ and $b_i > 0$, taking differentiation of $Df(t^{-m_i}, \varphi_i(t^{-m_i}, \xi))$, we have that

$$m_i j_i t^{-J_i + n_i - 2m_i - 1} + \text{ higher terms in } t = -\Delta_i(\xi)t^{-a_i - b_i - 1} + \text{ higher terms in } t.$$

Comparing two sides of it we can get the first conclusion. The remains are left to the readers as an elementary exercise. □

4. PROOF OF THEOREM 1.1

Consider the associated sequence $\{\varphi_i\}_{i=1}^K$ of the pair $\psi \prec \varphi$. Since φ is a dicritical series of f and $a_\psi = a_0 > 0, b_\psi = b_0 > 0$, we can see that

$$\deg p_0 > 0, \deg q_0 > 0.$$

Represent $(a_0, b_0) = (Md, Me)$ with $\gcd(d, e) = 1$. Without loss of generality we can assume that

$$\deg p_K > 0, \; a_K = 0 \text{ and } b_K \leqslant 0.$$

Then, from the construction of the sequence φ_i it follows that

(4.1)
$$\begin{cases} p_i(c_i) = 0 \text{ and } a_i > 0, & i = 0, 1, \ldots, K-1 \\ q_i(c_i) = 0 & \text{if } b_i > 0 \end{cases}$$

Then, by induction using Lemma 2.1, Lemma 3.1 and (4.1) we can obtain without difficulty the following.

Lemma 4.1. *For $i = 0, 1, \ldots, K-1$ we have*

$$a_i > 0, b_i > 0, \tag{a}$$

$$\frac{a_i}{b_i} = \frac{\#S_i}{\#T_i} = \frac{d}{e} \tag{b}$$

and

$$\frac{\#S_i^0}{\#T_i^0} = \frac{d}{e}, \bar{p}_i(\xi)^e = \bar{q}_i(\xi)^d. \tag{c}$$

Now, we are ready to complete the proof.

First note that $\deg p_\psi = \#S_0$ and $\deg q_\psi = \#T_0$. Then, from Lemma 4.1 (c) it follows that

$$(\deg p_\psi, \deg q_\psi) = (Nd, Ne)$$

for $N = \gcd(\deg p_\psi, \deg q_\psi) \in \mathbb{N}$. Thus, we get Conclusion (i).

Next, we will show $b_K = 0$. Indeed, by Lemma 2.1 (iii) and Lemma 4.1 (b-c) we have

$$\begin{aligned} \frac{b_K}{m_K} &= \frac{b_{K-1}}{m_{K-1}} + \#T_{K-1}^0 \left(\frac{n_{K-1}}{m_{K-1}} - \frac{n_K}{m_K} \right) \\ &= \frac{e}{d} \left[\frac{a_{K-1}}{m_{K-1}} + \#S_{K-1}^0 \left(\frac{n_{K-1}}{m_{K-1}} \frac{n_K}{m_K} \right) \right] \\ &= \frac{e}{d} \frac{a_K}{m_K} \\ &= 0, \end{aligned}$$

as $a_K = 0$. Thus, we get

$$a_K = b_K = 0.$$

Now, we detect the form of polynomials $p_K(\xi)$ and $q_K(\xi)$. Using Lemma 2.1 (ii-iii) to compute the leading coefficients A_K and B_K we can get

$$A_K = A_0\left(\prod_{k\leqslant K-1}\bar{p}_k(c_k)\right), B_K = B_0\left(\prod_{k\leqslant K-1}\bar{q}_k(c_k)\right).$$

Let C be a $d-$radical of $(\prod_{k\leqslant K-1}\bar{p}_k(c_k))$. Then, by Lemma 2.1 (ii) and Lemma 4.1 (c) we have that

$$A_K = A_0 C^d, B_K = B_0 C^e.$$

Let $D := \gcd(\#S^0_{K-1}, \#T^0_{K-1})$. Then, by Lemma 4.1 (b-c) we get

$$\deg p_K = \#S^0_{K-1} = Dd, \deg q_K = \#T^0_{K-1} = De.$$

Thus,

$$p_K(\xi) = A_0 C^d \xi^{Dd} + \dots$$
$$q_K(\xi) = B_0 C^e \xi^{De} + \dots$$

This proves Conclusion (ii). □

5. Proof of Theorem 1.2

Suppose φ is a dicritical series φ of f with $a_\varphi = 0$ and $b_\varphi < 0$. Since $b_\varphi < 0$, there is a horizontal series ψ of Q such that $\psi \prec \varphi$. We will show that ψ is a singular series of f.

Observe that φ is a horizontal series of P since $a_\varphi = 0$. Hence, $\deg p_\psi > 0$, since $\psi \prec \varphi$. Represent

$$P(x, \psi(x, \xi)) = p_\psi(\xi)x^{\frac{a_\psi}{m_\psi}} + \text{lower terms in } x,$$
$$Q(x, \psi(x, \xi)) = q_\psi(\xi) + \text{lower terms in } x,$$
$$J(P, Q)(x, \psi(x, \xi)) = j_\psi(\xi)x^{\frac{J_\psi}{m_\psi}} + \text{lower terms in } x.$$

Since $a_\psi > 0$ and $b_\psi = 0$, taking differentiation of $Df(t^{-m_\psi}, \psi(t^{-m_\psi}, \xi))$ we have that

$$m_\psi j_\psi(\xi)t^{J_\psi+n_\psi-2m_\psi-1} + \text{ h.terms in } t = -a_\psi p_\psi(\xi)\dot{q}_\psi(\xi)t^{-a_\psi-1} + \text{ h.terms in } t.$$

Comparing two sides of it we get that

$$m_\psi j_\psi(\xi) = -a_\psi p_\psi(\xi)\dot{q}_\psi(\xi).$$

As $\deg p_\psi > 0$, we get $\deg j_\psi(\xi) > 0$, i.e. ψ is a singular series of f. □

6. Last comment

To conclude the paper we want to note that instead of the polynomial maps $f = (P, Q)$ we may consider pairs $f = (P, Q) \in k((x))[y]^2$, where k is an algebraically closed field of zero characteristic and $k((x))$ is the ring of formal Laurent series in variable x^{-1} with finite positive power terms. Then, in view of the Newton theorem the polynomial $P(y)$ and $Q(y)$ can be factorized into linear factors in $k((x))[y]$. And the notions of *horizontal series*, *dicritical series* and *singular*

series can be introduced in an analogous way. In this situation the statements of Theorem 1.1 and Theorem 1.2 are still valid and can be proved in the same way as in sections 2-5.

Acknowledgments

References

[1] S. S. Abhyankar and T. T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. **276** (1975), 148-166.

[2] S. S. Abhyankar, *Expansion Techniques in Algebraic Geometry*, Tata Institute of Fundamental Research, 1977.

[3] E. B. Bartolo, Ph. Cassou-Nogues and H. Maugendre, *Quotients jacobiens d'applications polynomiales (Jacobian quotients of polynomial mappings)*, Annales de l'Institut Fourier **53** (2) (2003), 399-428

[4] H. Bass, E. Connell and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 287-330.

[5] E. Brieskorn and H. Knorrer, *Ebene algebraische Kurven*, Birkhauser, Basel-Boston-Stuttgart, 1981.

[6] Nguyen Van Chau, *Non-zero constant Jacobian polynomial maps of C^2*, Ann. Polon. Math. **71** (1999), 287-310.

[7] Nguyen Van Chau, *Note on the Jacobian condition and the non-proper value set*, Ann. Polon. Math. **84** (2004), 203-210.

[8] Van den Essen, Arno, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics 190, Birkhauser, Basel, 2000.

[9] Z. Jelonek, *The set of points at which a polynomial map is not proper,* Ann. Polon. Math. **58** (1993), 259-266.

Institute of Mathematics
18 Hoang Quoc Viet
10307 Hanoi, Vietnam

E-mail address: nvchau@math.ac.vn