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HILBERT'S FOURTEENTH PROBLEM AND ALGEBRAIC EXTENSIONS WITH AN APPENDIX ON ROBERTS TYPE COUNTEREXAMPLES

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1. Introduction

Let k be a field, $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k for $n \in \mathbf{N}$, and $k(\mathbf{x})$ the field of fractions of $k[\mathbf{x}]$. Then, Hilbert's Fourteenth Problem asks whether the k-subalgebra $L \cap k[\mathbf{x}]$ of $k[\mathbf{x}]$ is finitely generated whenever L is a subfield of $k(\mathbf{x})$ containing k. In 1950's, Zariski [30] showed that $L \cap k[\mathbf{x}]$ is always finitely generated if the transcendence degree $\text{trans.deg}_k L$ of L over k is at most two, while Nagata [26] gave the first counterexample having $\text{trans.deg}_k L = 4$ in case of $n = 32$. In 1990, Roberts [28] constructed a different type of counterexample having $\text{trans.deg}_k L = 6$ when $n = 7$. Following Nagata and Roberts, several new counterexamples have been constructed. Mukai [25] and Steinberg [29] refined Nagata's construction. Kojima-Miyanishi [12] and the author [15] generalized Roberts' counterexample for $n \geqslant 7$, while Freudenburg [9] and Daigle-Freudenburg [1] made use of Roberts' counterexample to obtain ones for $n = 6$ and $n = 5$, respectively.

The author improved Roberts type construction thoroughly, and obtained several remarkable new counterexamples. For example, the answer to Hilbert's Fourteenth Problem is affirmative if $\text{trans.deg}_k L \leqslant 2$ by Zariski [30], while negative if $\text{trans.deg}_k L \geqslant 4$ by Nagata [26]. It have been a great concern whether there exists a counterexample having $\text{trans.deg}_k L = 3$ since 1958's. The author [16] settled this problem in the negative by giving counterexamples in case of $n = 4$. The author [17] also gave the first counterexamples for $n = 3$. Due to Zariski [30], $n = 3$ is the smallest possible dimension where a counterexample can exist, and if L is a counterexample for $n = 3$, then $k(\mathbf{x})/L$ is necessarily an algebraic extension. The counterexamples in [17] are the first counterexamples in the case where $k(\mathbf{x})/L$ is an algebraic extension.

In the present article, we survey recent results on Hilbert's Fourteenth Problem, focusing on the case where $k(\mathbf{x})/L$ is an algebraic extension. We discuss relations between finite generation of the k-algebra $L \cap k[\mathbf{x}]$ and the structure

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of the extension $k(\mathbf{x})/L$ by giving a series of quite simple counterexamples (Section 2). We also sketch an outline of a general construction of counterexamples (Section 3). This construction covers all the counterexamples of Roberts type. In Appendix, we collect some other kind of Roberts type counterexamples which involve the kernels of derivations.

To conclude this section, we remark on two important and well studied problems which are both special cases of Hilbert's Fourteenth Problem - the problem of finite generation of the invariant ring for a group action on $k[\mathbf{x}]$, and that of the kernel of a derivation of $k[\mathbf{x}]$. Hilbert originally studied the former problem, especially in the case where the action is linear. Nagata [26] in particular settled this original problem of Hilbert in the negative. Roberts' counterexample [28] is easily described as the kernel of a derivation (cf. [3]). However, $k(\mathbf{x})/L$ never be an algebraic extension if L is a counterexample in these special cases.

2. Algebraic extensions

First, we recall an important affirmative result on Hilbert's Fourteenth Problem.

Theorem 2.1 (Noether [27]). *Let G be a finite group acting on the k-algebra $k[\mathbf{x}]$. Then, the invariant ring $k[\mathbf{x}]^G$ is finitely generated.*

An action of G on $k[\mathbf{x}]$ naturally extends to that on $k(\mathbf{x})$. Then, we have $k[\mathbf{x}]^G = k(\mathbf{x})^G \cap k[\mathbf{x}]$. Hence, the theorem above is an affirmative answer to Hilbert's Fourteenth Problem. Furthermore, $k(\mathbf{x})/k(\mathbf{x})^G$ is an algebraic extension if G is a finite group.

In what follows, we assume that k is of characteristic zero, and give a series of counterexamples to Hilbert's Fourteenth Problem in the case where $k(\mathbf{x})/L$ is an algebraic extension.

Assume that $n = 3$. Consider the Laurent polynomials

$$f_1 = (1 + x_1 x_2^{d^2}) x_2^{1-d}, \quad f_2 = (1 - x_1 x_2^{d^2}) x_2^{-d}, \quad f_3 = x_2^{-1} + x_3, \tag{2.1}$$

where $d \in \mathbf{N}$.

As a consequence of [19, Theorem 1.1 and Proposition 5.1], we have the following

Theorem 2.2. *Assume that $n = 3$. Then, $[k(\mathbf{x}) : k(f_1, f_2, f_3)] = d$. If $d \geqslant 3$, then $k(\mathbf{x})/k(f_1, f_2, f_3)$ is not a Galois extension and $k(f_1, f_2, f_3) \cap k[\mathbf{x}]$ is not finitely generated.*

Note that $k(f_1, f_2, f_3, x_2) = k(x_1, x_2, x_3)$ and $x_2^d + f_1 f_2^{-1} x_2^{d-1} - 2 f_2^{-1} = 0$.

From Theorems 2.1 and 2.2, one might expect that $L \cap k[\mathbf{x}]$ is finitely generated whenever $k(\mathbf{x})/L$ is a Galois extension, that is, L is the invariant field of a group action on $k(\mathbf{x})$. However, this is not true.

Assume that $n = 4$. The field $k(\mathbf{x})$ is generated by

$$v_1 = x_1^{-1}, \quad v_2 = x_1^6 x_2 - x_1^{-1}, \quad v_3 = x_1^4 x_3 + x_1^{-3}, \quad v_4 = x_4 + x_1^{-1} \tag{2.2}$$

over $k(v_1, v_2, v_3, v_4) = k(\mathbf{x})$. So, we may define an action of $\mathbf{Z}/2\mathbf{Z} = \{1, \sigma\}$ on $k(\mathbf{x})$ by

$$\sigma \cdot v_1 = v_2, \quad \sigma \cdot v_2 = v_1, \quad \sigma \cdot v_3 = v_3, \quad \sigma \cdot v_4 = v_4. \tag{2.3}$$

Then, $k(\mathbf{x})/k(\mathbf{x})^{\mathbf{Z}/2\mathbf{Z}}$ is a Galois extension of degree two, and

$$\begin{aligned} k(\mathbf{x})^{\mathbf{Z}/2\mathbf{Z}} &= k(v_1 + v_2, v_1 v_2, v_3, v_4) \\ &= k(x_1^6 x_2, x_1^{-1}(x_1^6 x_2 - x_1^{-1}), x_1^4 x_3 + x_1^{-3}, x_4 + x_1^{-1}). \end{aligned}$$

As a consequence of the main theorem of [20], we have the following

Theorem 2.3. *The k-algebra $k(\mathbf{x})^{\mathbf{Z}/2\mathbf{Z}} \cap k[\mathbf{x}]$ is not finitely generated.*

By Zariski [30] and Theorems 2.2 and 2.3, we obtain the following table.

Answers to Hilbert's Fourteenth Problem

	$n \leqslant 2$	$n = 3$	$n \geqslant 4$
$[k(\mathbf{x}) : L] = 2$	Affirmative	?	Negative
$[k(\mathbf{x}) : L] \geqslant 3$	Affirmative	Negative	Negative

Problem 2.1. *Assume that $n = 3$. Let L be a subfield of $k(\mathbf{x})$ containing k such that $[k(\mathbf{x}) : L] = 2$. Is the k-subalgebra $L \cap k[\mathbf{x}]$ of $k[\mathbf{x}]$ finitely generated?*

Note that $k(\mathbf{x})/L$ is necessarily a Galois extension if $[k(\mathbf{x}) : L] = 2$, while we do not have a counterexample with $[k(\mathbf{x}) : L] = 2$ when $n = 3$. So, one might expect that $L \cap k[\mathbf{x}]$ is finitely generated whenever $n = 3$ and $k(\mathbf{x})/L$ is a Galois extension. However, this is not true.

Assume that $n = 3$. Let δ_1 and δ_2 be natural numbers with $\delta_1 < \delta_2$ such that δ_2 is not divisible by δ_1, let δ_0 be the greatest common divisor of δ_1 and δ_2, and let ϵ be an integer at least equal to the least common multiple of δ_1 and δ_2. We define Laurent polynomials by

$$g_1 = (x_2^{-1} - x_1 x_2^{\epsilon})^{\delta_1}, \quad g_2 = (x_2^{-1} + x_1 x_2^{\epsilon})^{\delta_2}, \quad g_3 = x_2^{-\delta_0} + x_3. \tag{2.4}$$

The following result is mentioned in [19].

Theorem 2.4. *The k-algebra $k(g_1, g_2, g_3) \cap k[\mathbf{x}]$ is not finitely generated.*

We claim that $k(\mathbf{x})/k(g_1, g_2, g_3)$ is a Galois extension if k contains a primitive δ_ith root ζ_i of unity for $i = 1, 2$. In fact, observe that the field $k(\mathbf{x})$ is generated by $\bar{g}_1 := x_2^{-1} - x_1 x_2^{\epsilon}$, $\bar{g}_2 = x_2^{-1} + x_1 x_2^{\epsilon}$ and g_3 over k. Define $\sigma_1, \sigma_2 \in \mathrm{Aut}_k\, k(\mathbf{x})$ by

$$\sigma_1(\bar{g}_1) = \zeta_1 \bar{g}_1, \sigma_1(\bar{g}_2) = \bar{g}_2, \ \sigma_1(g_3) = g_3$$
$$\sigma_2(\bar{g}_1) = \bar{g}_1, \ \sigma_2(\bar{g}_2) = \zeta_2 \bar{g}_2, \ \sigma_2(g_3) = g_3,$$

and let H be the subgroup of $\mathrm{Aut}_k\, k(\mathbf{x})$ generated by σ_1 and σ_2. Then, we have $k(\mathbf{x})^H = k(g_1, g_2, g_3)$. Therefore, $k(\mathbf{x})/k(g_1, g_2, g_3)$ is a Galois extension. The Galois group H is isomorphic to $(\mathbf{Z}/\delta_1\mathbf{Z}) \times (\mathbf{Z}/\delta_2\mathbf{Z})$. Note that the assumption on δ_1 and δ_2 implies that ζ_1 or ζ_2 cannot be a rational number.

Problem 2.2. *Assume that $n = 3$ and $k = \mathbf{Q}$. Let L be a subfield of $k(\mathbf{x})$ containing k such that $k(\mathbf{x})/L$ is a Galois extension. Is the k-subalgebra $L \cap k[\mathbf{x}]$ of $k[\mathbf{x}]$ always finitely generated?*

In the cases of Theorems 2.3 and 2.4, the Galois groups are abelian. So, one may want to ask for which finite group G does there exist L for which $L \cap k[\mathbf{x}]$ is not finitely generated and $k(\mathbf{x})/L$ is a Galois extension with Galois group isomorphic to G. The answer is *any* finite group $G \neq \{1\}$ as follows.

Since the case where $G = \mathbf{Z}/2\mathbf{Z}$ is treated in Theorem 2.3, let us assume that the number $|G|$ of elements of G is at least three. Assume further that G is acting faithfully and transitively on the subset $\{1, \ldots, n-1\}$ of indices. Since G acts on itself faithfully and transitively by left multiplication, such an action always exists for $n = |G| + 1$. On the other hand, G cannot act faithfully on $\{1, 2\}$, since $|G| \geqslant 3$ by assumption. Accordingly, the assumption above implies that $n \geqslant 4$. The field $k(\mathbf{x})$ is generated by

$$w_1 = (n-2)x_1^{-1}, \quad w_i = x_1^6 x_i - x_1^{-1} \ (i = 2, \ldots, n-1), \quad w_n = x_n + x_1^{-1} \tag{2.5}$$

over k. We define an action of G on $k(\mathbf{x})$ by

$$\sigma \cdot w_i = w_{\sigma \cdot i} \ (i = 1, \ldots, n-1) \quad \text{and} \quad \sigma \cdot w_n = w_n \quad \text{for each } \sigma \in G. \tag{2.6}$$

Since this action is faithful, the Galois group of $k(\mathbf{x})/k(\mathbf{x})^G$ is isomorphic to G.

The following theorem is a consequence of the main result of [20].

Theorem 2.5. *For the action of G on $k(\mathbf{x})$ defined above, $k(\mathbf{x})^G \cap k[\mathbf{x}]$ is not finitely generated.*

3. Construction of counterexamples

In this section, we sketch an outline of the construction of Roberts type counterexamples. First, we illustrate the mechanism which makes a k-subalgebra R of $k[\mathbf{x}]$ not to be finitely generated. Roughly speaking, there are two major factors about it:

- R contains a certain infinite system of polynomials;
- R does not contain certain kind of polynomials.

For instance, let us look at the k-subalgebra

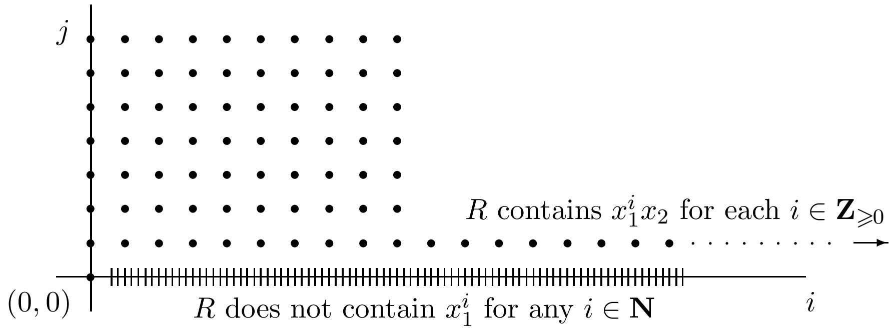
$$R = k[\{x_1^i x_2^j \mid i \in \mathbf{Z}_{\geqslant 0}, j \in \mathbf{N}\}], \tag{3.1}$$

where $\mathbf{Z}_{\geqslant 0}$ denotes the set of nonnegative integers. It is easy to see that R is not finitely generated. The reason is that R contains $x_1^i x_2$ for each $i \in \mathbf{Z}_{\geqslant 0}$, whereas R does not contain x_1^i for any $i \in \mathbf{N}$. The next figure shows $(i, j) \in (\mathbf{Z}_{\geqslant 0})^2$ such that $x_1^i x_2^j$ belongs to R.

Now, consider the two conditions for a k-subalgebra R of $k[\mathbf{x}]$:

(a) There exists $g \in k[\mathbf{x}] \setminus k$ such that R contains a polynomial of the form $g x_n^l +$ (terms of lower degree in x_n) for each $l \in \mathbf{N}$.

(b) No polynomial in which the monomial x_n^l appears with nonzero coefficient is contained in R for any $l \in \mathbf{N}$.



Proposition 3.1 (cf. [19, Lemma 2.1])**.** *If a k-subalgebra R of $k[\mathbf{x}]$ satisfies the conditions* (a) *and* (b)*, then R is not finitely generated.*

Next, we discuss how to construct a field L which will be a Roberts type counterexample. Let $\Phi : \mathbf{Z}^m \times \mathbf{Z}^r \to \mathbf{Q}^n$ be an injective homomorphism of groups for $m, r \in \mathbf{Z}_{\geqslant 0}$, and A a k-subalgebra of the polynomial ring $k[\mathbf{y}] = k[y_1, \ldots, y_m]$ in m variables over k. We define a k-subalgebra A^Φ of $k(\mathbf{x})$ as follows:

Let $k[\mathbf{y}^{\pm 1}]$ and $k[\mathbf{z}^{\pm 1}]$ be the Laurent polynomial rings in $y_1, \ldots, y_m$ and $z_1, \ldots, z_r$ over k, respectively. We denote $\mathbf{y}^\alpha = y_1^{\alpha_1} \cdots y_m^{\alpha_m}$, $\mathbf{z}^\beta = z_1^{\beta_1} \cdots z_r^{\beta_r}$ and $\mathbf{x}^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ for each $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_r)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$. Let $k(\Phi^{-1}(\mathbf{Z}^n))$ be the field generated by $\mathbf{y}^\alpha \otimes \mathbf{z}^\beta$ for $(\alpha, \beta) \in \Phi^{-1}(\mathbf{Z}^n)$ over k. Since Φ is injective, a homomorphism $\Phi_* : k(\Phi^{-1}(\mathbf{Z}^n)) \to k(\mathbf{x})$ is defined by $\Phi_*(\mathbf{y}^\alpha \otimes \mathbf{z}^\beta) = \mathbf{x}^{\Phi((\alpha,\beta))}$ for each $(\alpha, \beta) \in \Phi^{-1}(\mathbf{Z}^n)$. Then, we define

$$A^\Phi = \Phi_*((A \otimes_k k[\mathbf{z}^{\pm 1}]) \cap k(\Phi^{-1}(\mathbf{Z}^n))). \tag{3.2}$$

By definition, A^Φ is contained in the Laurent polynomial ring in $x_1, \ldots, x_n$ over k, but is not contained in $k[\mathbf{x}]$ in general. The important fact is that there exist a great many choices of A and Φ for which $R = A^\Phi \cap k[\mathbf{x}]$ satisfies the conditions (a) and (b). By Proposition 3.1, a field L is a counterexample to Hilbert's Fourteenth Problem if $L \cap k[\mathbf{x}] = A^\Phi \cap k[\mathbf{x}]$ for such A and Φ.

Proposition 3.2. *Let $\Phi : \mathbf{Z}^m \times \mathbf{Z}^r \to \mathbf{Q}^n$ be an injective homomorphism of groups, and A a k-subalgebra of $k[\mathbf{y}]$. If $k(A) \cap k[\mathbf{y}^{\pm 1}] \subset A$, then $k(A^\Phi) \cap k[\mathbf{x}] = A^\Phi \cap k[\mathbf{x}]$. Here, $k(R)$ denotes the field of fractions of R for each integral domain R.*

Proposition 3.2 is proved as follows. It suffices to show that $k(A^\Phi) \cap k[\mathbf{x}] \subset A^\Phi$. For each $f \in k(A^\Phi) \cap k[\mathbf{x}]$, there exist $g_1, g_2 \in (A \otimes_k k[\mathbf{z}^{\pm 1}]) \cap k(\Phi^{-1}(\mathbf{Z}^n))$ such that $\Phi_*(g_1/g_2) = \Phi_*(g_1)/\Phi_*(g_2) = f$. By definition, $\Phi_*^{-1}(k[\mathbf{x}]) \subset k[\mathbf{y}^{\pm 1}] \otimes_k k[\mathbf{z}^{\pm 1}]$. So, g_1/g_2 belongs to $k[\mathbf{y}^{\pm 1}] \otimes_k k[\mathbf{z}^{\pm 1}]$. Write $g_1/g_2 = \sum_{\beta \in \mathbf{Z}^r} a_\beta \otimes \mathbf{z}^\beta$, where $a_\beta \in k[\mathbf{y}^{\pm 1}]$ for each β. To conclude that $f \in A^\Phi$, it remains only to verify that a_β is in A for each β. Suppose the contrary. Then, $g_1' = g_1 - g_2 \sum_{\beta \in \mathbf{Z}^r} (a_\beta - a'_\beta) \otimes \mathbf{z}^\beta$ is a nonzero element of $A \otimes_k k[\mathbf{z}^{\pm 1}]$, and $g_1'/g_2 = \sum_{\beta \in \mathbf{Z}^r} a'_\beta \otimes \mathbf{z}^\beta \neq 0$. Here,

$a'_\beta = a_\beta$ if $a_\beta \notin A$, and $a'_\beta = 0$ otherwise for each β. Regard g'_1/g_2, g'_1 and g_2 as Laurent polynomials in $z_1, \ldots, z_r$ over $k[\mathbf{y}^{\pm 1}]$. By taking the leading terms for some term ordering, we get an equality $(a'_{\beta_0} \otimes \mathbf{z}^{\beta_0})(c \otimes \mathbf{z}^{\beta_2}) = b \otimes \mathbf{z}^{\beta_1}$ from the equality $(g'_1/g_2)g_2 = g'_1$, where $\beta_0, \beta_1, \beta_2 \in \mathbf{Z}^r$ and $b, c \in A \setminus \{0\}$. Then, $a'_{\beta_0} = b/c$. Hence, a'_{β_0} belongs to $k(A)$, and thus belongs to A by the assumption that $k(A) \cap k[\mathbf{y}^{\pm 1}] \subset A$. This contradicts that a'_{β_0} is not an element of A. Therefore, a_β is in A for each β, thereby completing the proof.

We note that every counterexample of Roberts type has the form $k(A^\Phi)$ for some A and Φ. Of course, we need to choose Φ and A carefully so that $R = A^\Phi \cap k[\mathbf{x}]$ satisfies (a) and (b), but we omit the detailed discussion on this subject here.

Finally, we list A and Φ used to realize the counterexamples given in Section 2.

To construct $k(f_1, f_2, f_3)$, we take $(m, r) = (3, 0)$ and

$$A = k[(1 + y_1)y_2^{d-1}, (1 - y_1)y_2^d, y_2 + y_3],$$

and define $\Phi : \mathbf{Z}^3 \to \mathbf{Q}^3$ by $\Phi((\alpha_1, \alpha_2, \alpha_3)) = (\alpha_1, d^2\alpha_1 - \alpha_2, \alpha_3)$.

To construct $k(\mathbf{x})^{\mathbf{Z}/2\mathbf{Z}}$, we take $(m, r) = (4, 0)$ and

$$A = k[y_2, y_1(y_2 - y_1), y_3 + y_1^3, y_4 + y_1],$$

and define $\Phi : \mathbf{Z}^4 \to \mathbf{Q}^4$ by $\Phi((\alpha_1, \alpha_2, \alpha_3, \alpha_4)) = (-\alpha_1 + 6\alpha_2 + 4\alpha_3, \alpha_2, \alpha_3, \alpha_4)$.

To construct $k(g_1, g_2, g_3)$, we take $(m, r) = (3, 0)$ and

$$A = k[(y_2 - y_1)^{\delta_1}, (y_2 + y_1)^{\delta_2}, y_2^{-\delta_0} + y_3],$$

and define $\Phi : \mathbf{Z}^3 \to \mathbf{Q}^3$ by $\Phi((\alpha_1, \alpha_2, \alpha_3)) = (\alpha_1, \epsilon\alpha_1 - \alpha_2, \alpha_3)$.

To construct $k(\mathbf{x})^G$, we take $m = n$, $r = 0$ and $A = k[w'_1, \ldots, w'_{n-1}]^G[y_n + y_1]$. Here, $w'_1 = (n-2)y_1$ and $w'_i = y_i - y_1$ for $i = 2, \ldots, n-1$, and the action of G on $k[w'_1, \ldots, w'_{n-1}]$ is defined by $\sigma \cdot w'_i = w'_{\sigma \cdot i}$ for each i. We define $\Phi : \mathbf{Z}^n \to \mathbf{Q}^n$ by

$$\Phi((\alpha_1, \ldots, \alpha_n)) = (-\alpha_1 + 6(\alpha_2 + \alpha_3 + \cdots + \alpha_{n-1}), \alpha_2, \alpha_3, \ldots, \alpha_n).$$

We note that, in the cases of the counterexamples of Roberts [28], Freudenburg [9], and Daigle-Freudenburg [1], we take (m, r) to be $(4, 3)$, $(4, 2)$ and $(4, 1)$, respectively.

4. Appendix

In this appendix, we collect some typical counterexamples of Roberts type which are obtained as the kernels of derivations.

Let R be a commutative k-domain. A k-linear map $D : R \to R$ is called a *derivation* if $D(fg) = D(f)g + fD(g)$ for each $f, g \in R$. Then, the kernel

$$R^D = \{f \in R \mid D(f) = 0\}$$

of D becomes a k-subalgebra of R. Note that D extends naturally to a derivation of the field K of fractions of R, and $R^D = K^D \cap R$ by definition. Therefore, the problem of finite generation of $k[\mathbf{x}]^D$ is a special case of Hilbert's Fourteenth

Problem if D is a derivation of $k[\mathbf{x}]$. Since k is of characteristic zero, we get $\text{trans.deg}_k\, k(\mathbf{x})^D < \text{trans.deg}_k\, k(\mathbf{x}) = n$ if $D \neq 0$. Consequently, $k[\mathbf{x}]^D$ is always finitely generated if $n \leqslant 3$ by Zariski [30]. On the other hand, there exists D such that $k[\mathbf{x}]^D$ is not finitely generated if $n = 4$. For instance, consider a derivation of $k[\mathbf{x}]$ defined by

$$\begin{aligned} D(x_1) &= tx_1x_3^{t+1} + tx_1x_2^{t+1} + (1-t)x_1^{t+2} \\ D(x_2) &= tx_2x_1^{t+1} + tx_2x_3^{t+1} + (1-t)x_2^{t+2} \\ D(x_3) &= tx_3x_2^{t+1} + tx_3x_1^{t+1} + (1-t)x_3^{t+2} \\ D(x_4) &= x_1^t x_2^t x_3^t, \end{aligned}$$

where $t \in \mathbf{Z}_{\geqslant 0}$. Then, we have the following

Theorem 4.1 (cf. [18])**.** *Assume that $n = 4$. Then, the kernel of the derivation D defined as in* (4.1) *is not finitely generated if $t \geqslant 3$.*

A derivation D of $k[\mathbf{x}]$ is said to be *locally nilpotent* if, for each $f \in k[\mathbf{x}]$, there exists $l \in \mathbf{N}$ such that $D^l(f) = 0$. We say that D is *triangular* if $D(x_i)$ belongs to $k[x_1, \dots, x_{i-1}]$ for $i = 1, \dots, n$, and that D is *nice* if $D^2(x_i) = 0$ for $i = 1, \dots, n$. A triangular derivation and a nice derivation are both locally nilpotent.

It is known that $k[\mathbf{x}]^D$ is always finitely generated if D is a locally nilpotent derivation of $k[\mathbf{x}]$ such that $D(f) = 1$ for some $f \in k[\mathbf{x}]$ (see for example [6, Corollary 1.3.23]). Van den Essen [4, Conjecture 6.12] conjectured that there exists a derivation D of $k[\mathbf{x}]$ with $D(f) = 1$ for some $f \in k[\mathbf{x}]$ such that $k[\mathbf{x}]^D$ is not finitely generated in case D is not locally nilpotent.

This conjecture was solved in the affirmative by the author.

Theorem 4.2 ([23])**.** *Assume that $n \geqslant 5$ and $t \geqslant 3$. Let $\tilde{D}$ be the extension of the derivation D defined as in* (4.1) *obtained by setting $\tilde{D}(x_5) = 1$ and $\tilde{D}(x_i) = 0$ for $i = 6, \dots, n$. Then, $k[\mathbf{x}]^{\tilde{D}}$ is not finitely generated.*

We note that the derivation D defined as in (4.1) is not locally nilpotent. The following problem remains unsettled.

Problem 4.1. *Assume that $n = 4$. Is $k[\mathbf{x}]^D$ finitely generated whenever D is a locally nilpotent derivation of $k[\mathbf{x}]$?*

Daigle-Freudenburg [1] gave a triangular derivation whose kernel is not finitely generated in case of $n = 5$, while Daigle-Freudenburg [2] showed that the kernel of a triangular derivation is always finitely generated if $n = 4$ (see also [24]). It is known that a triangular derivation D of $k[\mathbf{x}]$ necessarily satisfies the following condition if $n \geqslant 2$:

(†) There exists $f_1, \dots, f_n \in k[\mathbf{x}]$ with $D(f_1) = 0$ such that $k[f, f_2, \dots, f_n] = k[\mathbf{x}]$.

In case $n \geqslant 2$, it is difficult to construct a locally nilpotent derivation of $k[\mathbf{x}]$ not satisfying (†). The first example of such a locally nilpotent derivation was given by Freudenburg [8] as follows (see also [7], [10]).

Recall that, for each $(n-1)$-tuple $\mathbf{g} = (g_2, \dots, g_n)$ of elements of $k[\mathbf{x}]$, a derivation $\Delta_{\mathbf{g}}$ of $k[\mathbf{x}]$ is defined by $\Delta_{\mathbf{g}}(f) = J(f, g_2, \dots, g_n)$ for each $f \in k[\mathbf{x}]$. Here, $J(h_1, \dots, h_n)$ denotes the determinant of the n by n matrix $(\partial h_i/\partial x_j)_{i,j}$ for $h_1, \dots, h_n \in k[\mathbf{x}]$.

Theorem 4.3 (Freudenburg [8])**.** *Assume that $n = 3$. The derivation $\Delta_{(F,G)}$ of $k[\mathbf{x}]$ is locally nilpotent and does not satisfy (†) for $F = x_1x_3 + x_2^2$ and $G = x_3F^2 + 2x_1^2x_2F - x_1^5$.*

Using $\Delta_{(F,G)}$ above, we define a derivation D of $k[\mathbf{x}]$ for each $n \geqslant 5$ by

(4.1)
$$D(x_i) = FG^3\Delta_{(F,G)}(x_i)\ (i = 1, 2, 3),\quad D(x_4) = F^2x_1^{12}RS,\quad D(x_5) = G^3x_1RS$$

and $D(x_i) = FG^3x_{i-1}^2R$ $(i = 6, \dots, n)$, where $R = x_1^3 - Fx_2$ and $S = Fx_3 + 2x_1^2x_2$.

Then, we have the following

Theorem 4.4 (cf. [22, Theorem 1.1])**.** *Assume that $n \geqslant 5$. Then, the derivation D of $k[\mathbf{x}]$ defined as above is locally nilpotent and does not satisfy (†). The kernel $k[\mathbf{x}]^D$ is not finitely generated.*

On what follows, we consider derivations of the polynomial ring

$$k[\mathbf{x}, \mathbf{y}] = k[x_1, \dots, x_m, y_1, \dots, y_n]$$

in $m + n$ variables over k for $m, n \in \mathbf{Z}_{\geqslant 0}$ satisfying $D(x_i) = 0$ for each i.

We say that D is *elementary* if $D(y_i)$ belongs to $k[x_1, \dots, x_m]$ for $i = 1, \dots, n$. It is easy to see that an elementary derivation is nice, and hence locally nilpotent. Van den Essen-Janssen [5] showed that $k[\mathbf{x}, \mathbf{y}]^D$ is finitely generated if $m \leqslant 2$ or $n \leqslant 2$ in case D is elementary. Khoury [13] showed that $k[\mathbf{x}, \mathbf{y}]^D$ is finitely generated if $n = 3$, D is elementary and $D(y_i)$ is a monomial for each i.

Assume that $n = m + 1$ and $t \in \mathbf{Z}_{\geqslant 0}$. Kojima-Miyanishi [12] studied the elementary derivation $D_{t,m}$ of $k[\mathbf{x}, \mathbf{y}]$ defined by

$$D(y_i) = x_i^{t+1} \text{ for } i = 1, \dots, m \text{ and } D(y_n) = (x_1 \cdots x_m)^t. \tag{4.2}$$

Theorem 4.5 (Kojima-Miyanishi [12])**.** *If $m \geqslant 3$ and $t \geqslant 2$, then the kernel of $D_{t,m}$ is not finitely generated.*

Roberts' counterexample [28] is obtained as the special case where $m = 3$.

If $t = 0$, then $D_{t,m}(y_n) = 1$, and so $k[\mathbf{x}, \mathbf{y}]^{D_{t,m}}$ is finitely generated for any $m \geqslant 0$ as mentioned. If $m \leqslant 2$, then $k[\mathbf{x}, \mathbf{y}]^{D_{t,m}}$ is finitely generated for any $t \geqslant 0$ due to van den Essen-Janssen [5] or Khoury [13]. Kurano [14] showed that the kernel of $D_{1,3}$ is generated by twelve elements. Finally, we showed the following

Theorem 4.6 ([15, Corollary 1.5])**.** *If $m \geqslant 4$, $t \geqslant 1$ or $m = 3$, $t \geqslant 2$, then the kernel of $D_{t,m}$ is not finitely generated.*

Thereby, finite generation of $k[\mathbf{x}, \mathbf{y}]^{D_{t,m}}$ was completely determined.

Finite generation of $k[\mathbf{x},\mathbf{y}]^{D_{t,m}}$

	$m \leqslant 2$	$m = 3$	$m \geqslant 4$
$t = 1$	Finitely generated	Finitely generated	Not finitely generated
$t \geqslant 2$	Finitely generated	Not finitely generated	Not finitely generated

Next, we consider the elementary derivation of $k[\mathbf{x},\mathbf{y}]$ defined by $D(y_i) = \mathbf{x}^{\delta_i}$ for $i = 1, \dots, n$, where $\delta_i \in (\mathbf{Z}_{\geqslant 0})^m$ for each i. Let $\epsilon^i_{i,j}$ be the ith component of $\delta_i - \delta_j$ for each i and j.

In the notation above, we have the following

Theorem 4.7 ([15, Theorem 1.4])**.** *Assume that $m \geqslant 3$, $n = 4$ and $\epsilon^i_{i,j} > 0$ for each $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 4$ with $i \neq j$. If*

$$\frac{\epsilon^1_{1,4}}{\min\{\epsilon^1_{1,2}, \epsilon^1_{1,3}\}} + \frac{\epsilon^2_{2,4}}{\min\{\epsilon^2_{2,3}, \epsilon^2_{2,1}\}} + \frac{\epsilon^3_{3,4}}{\min\{\epsilon^3_{3,1}, \epsilon^3_{3,2}\}} \leqslant 1, \tag{4.3}$$

then $k[\mathbf{x},\mathbf{y}]^D$ is not finitely generated.

The following conjecture remains unsolved.

Conjecture 4.1 ([15, Conjecture 4.8])**.** *Assume that $m = 3$, $n = 4$, and $\epsilon^i_{i,j} > 0$ for any $i \neq j$. If the inequality* (4.3) *is not satisfied, then $k[\mathbf{x},\mathbf{y}]^D$ is finitely generated.*

In case of $(m, n) = (4, 2)$, the kernel of an elementary derivation is always finitely generated as mentioned, while Freudenburg [11] gave a nice derivation of $k[\mathbf{x},\mathbf{y}]$ whose kernel is not finitely generated. We generalize the derivation of Freudenburg as follows.

For nonnegative integers u, v and α_i, β_i, w_i for $i = 1, 2$, set

$$P = x_2^{\alpha_2} y_1 - x_1^{\alpha_1} y_2, \quad Q_1 = x_1^{\alpha_1} y_3 - y_1 P^{u+1}, \quad Q_2 = x_2^{\alpha_2} y_3 - y_2 P^{u+1}.$$

We define a derivation D of $k[\mathbf{x},\mathbf{y}]$ for $(m, n) = (2, 4)$ by $D(x_i) = 0$ for $i = 1, 2$, and

$$D(y_i) = x_i^{\alpha_i} \text{ for } i = 1, 2, \quad D(y_3) = P^{u+1}, \quad D(y_4) = x_1^{\beta_1} x_2^{\beta_2} P^v Q_1^{w_1} Q_2^{w_2}. \tag{4.4}$$

Then, it is readily verified that $D(P) = 0$ and $D(Q_i) = 0$ for $i = 1, 2$. This implies that $D(y_i)^2 = 0$ for each i, and hence D is a nice derivation.

Theorem 4.8 (cf. [21])**.** *Assume that $(m, n) = (2, 4)$. Then, the kernel of the derivation D defined as in* (4.4) *is not finitely generated if $\alpha_i \geqslant 1$, $\beta_i \geqslant 1$ and $w_i \geqslant 0$ for $i = 1, 2$, and*

$$0 \leqslant v \leqslant u \quad \text{and} \quad \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \leqslant 1.$$

The nice derivation of Freudenburg [11] is the special case where $\alpha_i = t + 1$, $\beta_i = t$, $w_i = 0$ for $i = 1, 2$ and $u = v = t$ with $t \geqslant 2$. At present, it is not known whether the kernel of a nice derivation of $k[\mathbf{x},\mathbf{y}]$ is always finitely generated if $(m,n)=(1,4)$. On the other hand, Daigle-Freudenburg [1] gave a triangular derivation whose kernel is not finitely generated when $(m,n)=(1,4)$. We generalize it as follows.

Let t_i be a nonnegative integers for $i=1,2,3,4$. We define a derivation D of $k[\mathbf{x},\mathbf{y}]$ for $(m,n)=(1,4)$ by $D(x_1)=0$, and

$$D(y_1)=x_1^{t_1},\quad D(y_2)=x_1^{t_2}y_1,\quad D(y_3)=x_1^{t_3}y_2,\quad D(y_4)=x_1^{t_4}. \tag{4.5}$$

Theorem 4.9 (cf. [21])**.** *Assume that $(m,n)=(1,4)$ and D is the derivation of $k[\mathbf{x},\mathbf{y}]$ defined as in* (4.5)*. If $t_4<t_1$ and $4t_1+t_2+\max\{t_2,t_3\}\leqslant 6t_4$, then $k[\mathbf{x},\mathbf{y}]^D$ is not finitely generated.*

The triangular derivation of Daigle-Freudenburg [1] is the special case where $t_1=t+1$, $t_2=t_3=0$ and $t_4=t$ with $t\geqslant 2$.

The following conjecture also remains unsolved.

Conjecture 4.2. *Assume that $(m,n)=(1,4)$ and D is a derivation of $k[\mathbf{x},\mathbf{y}]$ defined as in* (4.5)*. If $t_4<t_1$ and $4t_1+t_2+\max\{t_2,t_3\}>6t_4$, then $k[\mathbf{x},\mathbf{y}]^D$ is finitely generated.*

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