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A NOTE ON A NOTE BY MUTSUO OKA

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1. Introduction

Consider a polynomial mapping (f, g) from $\mathbb{C}^2$ to $\mathbb{C}^2$, where $f(x, y)$, $g(x, y)$ are polynomials in two variables with coefficients in $\mathbb{C}$. The Jacobian conjecture asserts that if the jacobian of f and g is a non zero constant then the map (f, g) is an automorphism.

In "Note on boundary obstruction to jacobian conjecture of two variables" [4], M. Oka suggests a strategy to prove the jacobian conjecture in two variables. He proved

Theorem 1.1. *Assume that f is a strictly reduced polynomial which has a jacobian partner polynomial g ($J(f, g) = 1$). Then the following conditions are necessary.*

1. $f : \mathbb{C}^2 \to \mathbb{C}$ *has no critical point.*
2. $\Delta(f; x, y)$ *is convenient.*
3. $\Delta(f; x, y)$ *has no boundary obstruction.*
4. *The outside boundary multiplicity* $mult_\infty(f)$ *is strictly greater than* 1.

He suggests to prove that there is no polynomial f which satisfies the four conditions of the theorem.

In this article we will prove that indeed such polynomials exist. The simplest polynomial we found has degree 18.

Some years ago, Kaliman [3] suggested that to prove the jacobian conjecture one could try to prove that there do not exist polynomials with no critical point and whose fibers are all irreducible. Such polynomials exist [1]. Then mixing the two suggestions one can ask if there exist strictly reduced polynomials whose all fibers are irreducible and satisfying the four above conditions. The polynomial of degree 18 has one reducible fiber. But we prove that they do exist. The simplest example we found has degree 27.

In the first part of the article, we will recall the definitions we need to understand Oka's theorem, and in the second part we will describe our polynomials satisfying all the conditions of the theorem.

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2. Definitions

A polynomial $h(x,y)$ is called a *weighted homogeneous polynomial* of degree d with respect to the weight vector $P=(a,b)$, if it satisfies the equality $h(xt^a, yt^b) = t^d h(x,y)$. We call d the degree of h with respect to the weight P. A rational function $h(x,y) = h_1(x,y)/h_2(x,y)$ is called a *weighted homogeneous rational function* of degree d if h_1, h_2 are weighted homogeneous polynomials of degree d_1, d_2 with respect to a weight P, and $d = d_1 - d_2$.

Let $f(x,y) = \sum_{\nu=(\nu_1,\nu_2)} c_\nu x^{\nu_1} y^{\nu_2}$ be a polynomial. The *Newton diagram* $\Delta(f;x,y)$ is the convex hull of integral points $\nu = (\nu_1,\nu_2)$, such that $c_\nu \neq 0$. A face $\Xi \in \partial\Delta(f;x,y)$ is called an *outside face* if the line supporting Ξ does not pass through the origin and $\Delta(f;x,y)$ and the origin are in the same half plane.

Let Ξ be an outside face of $\Delta(f;x,y)$ and let $f_\Xi(x,y) = \sum_{\nu\in\Xi} c_\nu x^{\nu_1} y^{\nu_2}$. Then f_Ξ is a weighted homogeneous polynomial for a weight vector $P=(a,b)$ associated to Ξ. We say that the face Ξ is *strictly positive* if a and b are strictly positive, that it is *horizontal* (resp. *vertical*) if $P=(0,1)$ (resp. $P=(1,0)$). An *elementary horizontal face* is a face Ξ such that $f_\Xi(x,y) = ey^q(x+c)^p$ with $c \neq 0$, $p \neq q$, $e \neq 0$.

We say that f is *strictly reduced* if f is not a linear function, and $\Delta(f;x,y)$ has neither strictly positive, nor elementary horizontal, nor elementary vertical outside faces.

The Newton diagram $\Delta(f;x,y)$ is *convenient* if $f(x,0)$, $f(0,y)$ are non constant polynomials.

One says that $\Delta(f;x,y)$ has *no boundary obstructions* if for any outside face Ξ with a weight vector P there exists a weighted homogeneous rational function $\phi(x,y)$ with weight P, such that $J(f_\Xi,\phi) = 1$.

Let Ξ be an outside face of $\Delta(f;x,y)$, one can factorize $f_\Xi(x,y)$ as

$$f_\Xi(x,y) = cx^p y^q \prod_{i=1}^{i=m} (x^b + c_i y^a)^{\nu_i}$$

if Ξ is strictly positive,

$$f_\Xi(x,y) = cx^p y^q \prod_{i=1}^{i=m} (x^b y^{-a} + c_i)^{\nu_i}$$

if $a \leqslant 0 < b$. The *face multiplicity* $m(f,\Xi)$ is the greatest common divisor of the integers $p, q, \nu_1, \cdots, \nu_m$. The *outside boundary multiplicity* $m_\infty(f)$ is defined by the greatest common divisor of $m(f,\Xi)$ for all outside face Ξ of $\Delta(f;x,y)$.

3. The first example

We start with the following polynomial:

$$f = (x^2y + x)^2 + (x^2y + x) + xy.$$

This polynomial has two critical points $(0, -1)$ and $(-1, 1)$. The equation of the line going through these points is $-2x - y = 1$.

Make the change of variables $X = -2x - y - 1$. Let $f_1(X, y) = f(-(X + 1 + y)/2, y)$. The two critical points are on the line $X = 0$.

Now, as in [2], consider $f_2(v, w) = f_1(1/v, 2v+v^2w) \in \mathbb{C}[v, w]$. This polynomial has no critical point.

Its equation is

$$f_2 = \frac{1}{16}[v^6(vw + 2)^6 + w^2(v^3w + 1)^4 - v^{12}w^6 + g]$$

where

$$\begin{aligned} g \;=\; & 4v^{10}w^5 + (12v^7 + 42v^8 + 40v^9)w^4 + \\ & (12v^4 + 48v^5 + 92v^6 + 176v^7 + 160v^8)w^3 + \\ & (4v + 18v^2 + 60v^3 + 137v^4 + 264v^5 + 368v^6 + 320v^7)w^2 + \\ & (8 + 20v + 88v^2 + 164v^3 + 336v^4 + 384v^5 + 320v^6)w - \\ & 4 + 32v + 68v^2 + 160v^3 + 160v^4 + 128v^5. \end{aligned}$$

The Newton polygon of a generic fiber has vertices $O = (0, 0), A = (0, 2), B = (12, 6), C = (6, 0)$. The Newton polygon is convenient, and f_2 is strictly reduced. There are two outside faces and the corresponding face polynomials are $v^6(vw + 2)^6$ and $w^2(v^3w + 1)^4$. So the outside boundary multiplicity is 2. Now we look at the boundary obstructions. From Lemma 14 in [4], the boundary obstruction is satisfied for the weight $(1, -1)$. One has to check the obstruction for the other face. We have

$$J(w^2(v^3w + 1)^4, -1/2vw^{-1}(v^3w + 1)^{-3}) = 1.$$

Then the boundary obstructions are satisfied.

4. Example with irreducible fibers

We start with the following polynomial

$$f = (x^2y + x/4) + xy - y/2 + y^2.$$

This polynomial has two critical points $(-1, 1/4)$ with Milnor number 2 and $(1/2, -1/8)$ with Milnor number 1.

Let $f_1(X, y) = f(X + y, y)$ and $f_2(v, w) = f_1(1/v, -1/4v + v^2w)$. Then f_2 has also two critical points $(8/5, 55/112), (-4/5, 5/64)$. The equation of the line going through them is $-5/36v + 512/45w - 1 = 0$. Then we consider $f_3(V, w) =$

$f_2(-(V+1-512/45w)36/5, w)$ and finally $f_4(x,y) = f_3(1/x, -5/144x + x^2y) \in \mathbb{C}[x,y]$. The polynomial f_4 has no critical point.

We have

$$f_4 = 2^{66}/5^{12}y^3(x^3y - 45/512)^6 + 2^{66}/5^{12}x^9(xy-5/144)^9 - 2^{66}/5^{12}x^{18}y^9 + g$$

where g is a polynomial whose Newton polygon is strictly contained in the Newton polygon of f_4.

The Newton polygon of the generic fiber of f_4 has vertices $O = (0,0), A = (0,3), B = (18,9), C = (9,0)$.The Newton polygon is convenient, and f_4 is strictly reduced. There are two outside faces and the corresponding face polynomials are $x^9(xy-5/144)^9$ and $y^3(x^3y-45/512)^6$. So the outside boundary multiplicity is 3. Now we look at the boundary obstructions. Again the boundary obstruction is satisfied for the weight $(1,-1)$. One has to check the obstruction for the other face. We have

$$J(y^3(x^3y-45/512)^6, 512/135xy^{-2}(x^3y-45/512)^{-5}) = 1.$$

Then the boundary obstructions are satisfied.

The polynomial f has all its fibers irreducible, then also the polynomial f_1 does. Then the polynomial f_2 could only have $v = 0$ as a component of a fiber. It is easy to check that it is not possible. Then the polynomial f_2 has all its fibers irreducible, and f_3 as well. Then the only possible component of f_4 is $x = 0$ and again one checks that this is not possible. Then all fibers of f_4 are irreducible.

References

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