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HERBIE: A CUBIC LINEAR COUNTEREXAMPLE TO THE DEPENDENCE PROBLEM

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Abstract. We show that there exists a cubic linear counterexample to the (Homogeneous) Dependence Problem in dimension 53. More generally, there exists a power linear counterexample of degree d to the (H)DP (in sufficiently large dimensions), if and only if there exists *any* counterexample of degree d to the (H)DP.

1. Introduction

The study of power linear maps started in 1983, when L.M. Drużkowksi showed that in order to prove the Jacobian Conjecture, one only needs to prove the cubic linear case, see [6]. The Jacobian Conjecture asserts that for a polynomial map

$$F = (F_1, F_2, \ldots, F_n) \in \mathbb{C}[x]^n$$

with $\mathbb{C}[x] = \mathbb{C}[x_1, x_2, \ldots, x_n]$, the so-called Keller condition

$$\det \mathcal{J} F \in \mathbb{C}^*$$

where $\mathcal{J}F$ is the Jacobian of F, implies that F is invertible, i.e. there exists a polynomial map

$$G = (G_1, G_2, \ldots, G_n)$$

such that

$$F(G) = G(F) = x$$

where $x = (x_1, x_2, \ldots, x_n)$ is the identity map.

Now Drużkowski showed that in order to prove this conjecture, one may assume that F is of special cubic linear form, i.e.

$$F = (x_1 + L_1^3, x_2 + L_2^3, \ldots, x_n + L_n^3) = x + L^{*3} \tag{1.1}$$

where $L_i = A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{in}x_n$ is a linear form for each i. On the other hand, if the Jacobian Conjecture turns out not to be true, then one can find a counterexample of the form (1.1) as well.

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A conjecture that is studied in connection to the JC is the so-called *Linear Dependence Problem for (Homogeneous) Jacobians*. This problem is the following: Assume H is a homogeneous map in dimension n of degree $d \geqslant 2$, such that $\mathcal{J}H$ is nilpotent. Is there a vector v over $\mathbb{C}$ such that $v^{\mathrm{t}} \cdot \mathcal{J}H = 0$. Or equivalently: is there a vector v over $\mathbb{C}$ such that $\langle v, H\rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the usual bilinear product. We call this problem $HDP(n, d)$. We get a more general problem if H does not need to be homogeneous, which we call $GDP(n, d)$.

It is known that $GDP(n, d)$ has an affirmative answer, if and only if either $n \leqslant 2$ or $n = 3$ and $d \leqslant 3$, see e.g. [8]. The homogeneous variant of this conjecture has an affirmative answer for $n \leqslant 3$, see [1]. Also, $HDP(4, 2)$ and $HDP(4, 3)$ have affirmative answers, see e.g. [8]. But in [2], it is shown that $HDP(5, d)$ has a negative answer for all even $d \geqslant 6$. Furthermore, $HDP(n, d)$ has a negative answer for all $n \geqslant 6$ and $d \geqslant 4$ and $HDP(n, 3)$ has a negative answer for all $n \geqslant 10$.

So there exists a cubic counterexample to the homogeneous dependence problem in dimension 10. We show that there exists a counterexample of cubic linear form as well. The proof essentially uses the same reduction techniques as Drużkowksi used for the JC, which were later refined by G. Gorni and G. Zampieri to relate other problems to the special cubic linear case as well, see [9] or [8, §6.4].

In section 1, we show that there exists a counterexample to the *power linear dependence problem*, i.e. a counterexample of cubic linear form to the homogeneous dependence problem. In section 2, we study a conjecture on power linear maps of large degree, formulated by He Tong on the conference.

2. Power linear counterexamples to the dependence problem

The map

$$H = \begin{pmatrix} 6x_{10}(x_9x_1 - x_{10}x_2) \\ 6x_9(x_9x_1 - x_{10}x_2) \\ 6x_{10}(x_9x_3 - x_{10}x_4) \\ 6x_9(x_9x_3 - x_{10}x_4) \\ 6x_{10}(x_9x_5 - x_{10}x_6) \\ 6x_9(x_9x_5 - x_{10}x_6) \\ 6x_9(x_1x_4 - x_2x_3) \\ 6x_9(x_3x_6 - x_4x_5) \\ 6(x_8(x_1x_4 - x_2x_3) - x_7(x_3x_6 - x_4x_5)) \\ x_9^3 \end{pmatrix}$$

is a cubic homogeneous counterexample to the Linear Dependence Problem ($\mathcal{J}H$ is nilpotent). One can verify that there are 53 cubic linear powers such that each component of H can be written as a $\mathbb{Z}$-linear combination of these powers:

$$\begin{aligned} H_1 \quad = \quad & (x_1 + x_9 + x_{10})^3 - (x_1 + x_9)^3 - (x_1 + x_{10})^3 + x_1^3 - \\ & (x_9 + x_{10})^3 + x_9^3 + x_{10}^3 - (x_2 + x_{10})^3 - (x_2 - x_{10})^3 + 2x_2^3 \end{aligned}$$

$$
\begin{aligned}
H_2 &= -(x_2+x_9+x_{10})^3+(x_2+x_9)^3+(x_2+x_{10})^3-x_2^3+\\
&\quad (x_9+x_{10})^3-x_9^3-x_{10}^3+(x_1+x_9)^3+(x_1-x_9)^3-2x_1^3\\
H_3 &= (x_3+x_9+x_{10})^3-(x_3+x_9)^3-(x_3+x_{10})^3+x_3^3-\\
&\quad (x_9+x_{10})^3+x_9^3+x_{10}^3-(x_4+x_{10})^3-(x_4-x_{10})^3+2x_4^3\\
H_4 &= -(x_4+x_9+x_{10})^3+(x_4+x_9)^3+(x_4+x_{10})^3-x_4^3+\\
&\quad (x_9+x_{10})^3-x_9^3-x_{10}^3+(x_3+x_9)^3+(x_3-x_9)^3-2x_3^3\\
H_5 &= (x_5+x_9+x_{10})^3-(x_5+x_9)^3-(x_5+x_{10})^3+x_5^3-\\
&\quad (x_9+x_{10})^3+x_9^3+x_{10}^3-(x_6+x_{10})^3-(x_6-x_{10})^3+2x_6^3\\
H_6 &= -(x_6+x_9+x_{10})^3+(x_6+x_9)^3+(x_6+x_{10})^3-x_6^3+\\
&\quad (x_9+x_{10})^3-x_9^3-x_{10}^3+(x_5+x_9)^3+(x_5-x_9)^3-2x_5^3\\
H_7 &= (x_1+x_4+x_9)^3-(x_1+x_4)^3-(x_1+x_9)^3-(x_4+x_9)^3+\\
&\quad x_1^3+x_4^3-(x_2+x_3+x_9)^3+(x_2+x_3)^3+(x_2+x_9)^3+\\
&\quad (x_3+x_9)^3-x_2^3-x_3^3\\
H_8 &= (x_3+x_6+x_9)^3-(x_3+x_6)^3-(x_3+x_9)^3-(x_6+x_9)^3+\\
&\quad x_3^3+x_6^3-(x_4+x_5+x_9)^3+(x_4+x_5)^3+(x_4+x_9)^3+\\
&\quad (x_5+x_9)^3-x_4^3-x_5^3\\
H_9 &= (x_1+x_4+x_8)^3-(x_1+x_4)^3-(x_8+x_1)^3-(x_8+x_4)^3+\\
&\quad x_1^3+2x_4^3-(x_2+x_3+x_8)^3+(x_2+x_3)^3+(x_8+x_2)^3+\\
&\quad (x_8+x_3)^3-x_2^3-(x_3+x_6+x_7)^3+(x_3+x_6)^3+\\
&\quad (x_7+x_3)^3+(x_7+x_6)^3-x_6^3-2x_3^3+(x_4+x_5+x_7)^3-\\
&\quad (x_4+x_5)^3-(x_7+x_4)^3-(x_7+x_5)^3+x_5^3\\
H_{10} &= x_9^3
\end{aligned}
$$

Furthermore, one can check that these 53 linear powers are linearly independent. The number 53 is responsible for the dimension of the cubic linear counterexample to the homogeneous dependence problem.

It is not known whether there exists a quadratic counterexample to the HDP. But there does exist a quadratic counterexample to the GDP, namely the map

$$
\hat{H} = \begin{pmatrix} x_2 \\ x_1^2 - x_3 \\ 2x_1x_2 - x_4 \\ x_2^2 \end{pmatrix}
$$

is a quadratic counterexample to $GDP(4,2)$. Hence the map

$$
H = \begin{pmatrix} x_2 \\ x_1^2 - x_3 + 1 \\ 2x_1x_2 - x_4 - 2 \\ x_2^2 + 1 \end{pmatrix}
$$

is a quadratic counterexample to the GDP as well.

Now $H = B \cdot G$, where

$$G = \begin{pmatrix} (x_2+1)^2 \\ (x_2-1)^2 \\ (x_1+1)^2 \\ (x_1+x_2)^2 \\ (2x_1+x_3+1)^2 \\ (2x_1+x_3-1)^2 \\ (x_3+x_4+1)^2 \\ (x_3+x_4-1)^2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -1 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition 2.1. We call a map *quadratic affine*, if and only if it is of the form $(L_1^2, L_2^2, \ldots, L_n^2)$ with $\deg L_i \leqslant 1$ for all i.

$G = (L_1^2, L_2^2, \ldots, L_8^2)$ is quadratic affine with

$$L = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_7 \\ L_8 \end{pmatrix} = \begin{pmatrix} x_2+1 \\ x_2-1 \\ x_1+1 \\ x_1+x_2 \\ 2x_1+x_3+1 \\ 2x_1+x_3-1 \\ x_3+x_4+1 \\ x_3+x_4-1 \end{pmatrix}$$

Theorem 2.1. *If $\mathcal{J}(BG)$ is nilpotent, then $\mathcal{J}(G(Bx))$ is nilpotent as well.*

Proof. Assume $(\mathcal{J}(BG))^n = 0$. Then

$$\begin{aligned} \mathcal{J}(G(Bx))^{n+1} &= ((\mathcal{J}G)_{x=Bx} \cdot B)^{n+1} \\ &= (\mathcal{J}G)_{x=Bx} \cdot (B \cdot (\mathcal{J}G)_{x=Bx})^n \cdot B \\ &= (\mathcal{J}G)_{x=Bx} \cdot (\mathcal{J}(BG))^n_{x=Bx} \cdot B \\ &= 0 \end{aligned}$$

whence $\mathcal{J}(G(Bx))$ is nilpotent. □

Since $\mathcal{J}(BG) = \mathcal{J}H$ is nilpotent, so is $\mathcal{J}(G(Bx))$, so $G(Bx)$ is a candidate counterexample to the GDP.

The following properties of $G = (L_1^2, L_2^2, \ldots, L_8^2)$ and B hold:

- $L_1^2, L_2^2, \ldots, L_8^2$ are linearly independent over $\mathbb{C}$,
- $L_i \in \mathbb{C}[x_1, x_2, x_3, x_4]$ for all i,
- $(x_1, x_2, x_3, x_4) \mapsto B(x_1, x_2, x_3, x_4, 0, 0, 0, 0)$ is invertible,

The above properties imply that $G(Bx)$ is a quadratic affine counterexample to the GDP.

The following properties of H hold for all $i : 1 \leqslant i \leqslant 8$:

$$H_i = \sum_{j=1}^{n} B_{ij} L_i^d$$

where B_{ij} is the j-th entry of the i-th row of B.

Lemma 2.1. *Let f be a homogeneous polynomial of degree d over $\mathbb{C}$. Then f can be written as a sum of d-th powers L_i^d of linear forms.*

Proof. Since each polynomial is a sum of monomials, we may assume that f is a monomial. Assume first that $f = x_1^r x_2^{d-r}$. We show that f can be written as a sum of $x_1^d, (x_1 + x_2)^d, (x_1 + 2x_2)^d, \ldots, (x_1 + dx_2)^d$. So assume that this is not the case. The space of homogeneous polynomials in x_1 and x_2 of degree d is $(d + 1)$-dimensional, for it is generated by the $d + 1$ polynomials $x_1^d, x_1^{d-1}x_2, \ldots, x_1 x_2^{d-1}, x_2^d$. Since the $d + 1$ linear powers $x_1^d, (x_1 + x_2)^d, (x_1 + 2x_2)^d, \ldots, (x_1 + dx_2)^d$ do not generate all homogeneous polynomials in x_1 and x_2 of degree d, they are linearly dependent, say that

$$\lambda_0 x_1^d + \lambda_1(x_1 + x_2)^d + \lambda_2(x_1 + 2x_2)^d + \cdots + \lambda_d(x_1 + dx_2)^d = 0 \tag{2.1}$$

with $\lambda_i \neq 0$ for some i. We show by induction on d that this is impossible. Differentiating (2.1) to x_2 gives

$$\lambda_1(x_1 + x_2)^{d-1} + 2\lambda_2(x_1 + 2x_2)^{d-1} + \cdots + d\lambda_d(x_1 + dx_2)^{d-1} = 0.$$

Substituting $x_1 = x_1 - x_2$ and writing $\lambda_i' := (i + 1)\lambda_{i+1}$ gives

$$\lambda_0' x_1^{d-1} + \lambda_1'(x_1 + x_2)^{d-1} + \cdots + \lambda_{d-1}'(x_1 + (d - 1)x_2)^{d-1} = 0$$

and $\lambda_i = 0$ for all i follows by induction on d.

Assume next that f is a monomial in $m > 2$ indeterminates and that every monomial of degree d in less than m indeterminates can be written as a sum of linear d-th powers. Then $f = gh$, where g is a bivariate monomial of degree t and h is a monomial of degree $d - t$ in $m - 2$ indeterminates. Since g is bivariate, we can write g as a linear combination of t-th powers of linear forms, say that

$$g = L_1^t + L_2^t + \cdots + L_r^t.$$

Then $f = hL_1^t + hL_2^t + \cdots + hL_r^t$, so it suffices to write hL_i^t as a sum of linear d-th powers. For that purpose, we first write hy^t as a sum of linear d-th powers. This is possible, since hy^t only has $m - 2 + 1 < m$ indeterminates. Next, we substitute $y = L_i$. ∎

Corollary 2.1. *For each degree $d \geqslant 3$, there exists a power linear counterexample of degree d to the HDP.*

Proof. Let H be a counterexample of degree d that is not necessarily power linear (such H exist, see [2]). According to Lemma 2.1, there exists a set of, say, N linear forms L_i, such that each component of H can be written as a linear combination of the components of $G := (L_1^d, L_2^d, \ldots, L_N^d)$, i.e. $H = BG$ for some matrix B.

The construction of a power linear counterexample of degree d to the HDP is similar as that of the quadratic affine map given above. □

Corollary 2.2. *There exists a quadratic linear counterexample to the HDP, if and only if there exists a quadratic counterexample to the HDP.*

3. Power linear Keller maps of large degree

Definition 3.1. We write $(Ax)^{*d}$ for the map $((A_1x)^d, (A_2x)^d, \ldots, (A_nx)^d)$.

On the conference "Polynomial automorphisms and related topics", He Tong formulated the following conjecture:

Conjecture 3.1 (He Tong). *There exists a function $E : \mathbb{N} \to \mathbb{N}$, such that for special power linear Keller maps in dimension n of the form $F = x + (Ax)^{*d}$, $d \geqslant E(\mathrm{corank} A)$ implies that F is "ditto triangularizable", i.e. there is a $T \in \mathrm{GL}_n(\mathbb{C})$ such that*

1. *$\mathcal{J}(T^{-1}FT)$ is triangular (i.e. F is linearly triangularizable),*
2. *$T^{-1}FT$ is special power linear as well (i.e. $T^{-1}FT = x + (Bx)^{*d}$ for some matrix B).*

The following theorem shows that this conjecture has an affirmative answer.

Theorem 3.1. *Let $F = x + (Ax)^{*d}$ be a special power linear Keller map from $\mathbb{C}^n$ to $\mathbb{C}^n$ and $c = \mathrm{corank} A$. Assume that $d \geqslant (2^{c+1} - 1)^2$. Then there is a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}FT$ is of the form*

$$T^{-1}FT = x + (Bx)^{*d}$$

such that B is a triangular matrix.

Before we prove this theorem, we need a lemma:

Lemma 3.1. *Let R be a nonzero relation with $\deg_{y_i} R \leqslant 1$ such that*

$$R\Big(x_1^d, x_2^d, \ldots, x_r^d, (\lambda_{1,1}x_1 + \lambda_{1,2}x_2 + \cdots + \lambda_{1,r}x_r)^d, (\lambda_{2,1}x_1 + \lambda_{2,2}x_2 + \cdots + \lambda_{2,r}x_r)^d, \ldots, (\lambda_{c,1}x_1 + \lambda_{c,2}x_2 + \cdots + \lambda_{c,r}x_r)^d\Big) = 0. \tag{3.1}$$

If $d \geqslant 2^{c+1}(2^{c+1}-2)$, then two of the arguments of R above are linearly dependent.

Proof. Let m be the degree of R. Without loss of generality, we may assume that $R_{y_1} \neq 0$. Assume furthermore that $\lambda_{1,1}\lambda_{2,1}\cdots\lambda_{c',1} \neq 0$ and $\lambda_{c'+1,1} = \cdots = \lambda_{c,1} = 0$. Then there exists a relation $\tilde{R} \in K[z_1, z_2, \ldots, z_{c'+1}]$ with $\deg \tilde{R} \leqslant m$ and $\deg_{z_i} \tilde{R}(z) \leqslant 1$ for all i, such that

$$\tilde{R}\Big(x_1^d, (\lambda_{1,1}x_1 + \cdots + \lambda_{1,r}x_r)^d, \ldots, (\lambda_{c',1}x_1 + \cdots + \lambda_{c',r}x_r)^d\Big) = 0 \tag{3.2}$$

where $K = \mathbb{C}(x_2, x_3, \ldots, x_r)$. Now let $t_1, t_2, \ldots, t_k$ be the terms of $\tilde{R}$ and define

$$g_i := t_i(x_1, \lambda_{1,1}x_1 + \cdots + \lambda_{1,r}x_r, \ldots, \lambda_{c',1}x_1 + \cdots + \lambda_{c',r}x_r)$$

for all i. Then (3.2) comes down to

$$g_1^d + g_2^d + \cdots + g_k^d = 0.$$

Assume that $d \geqslant 2^{c+1}(2^{c+1} - 2)$. Since $\tilde{R}$ has degree $\leqslant 1$ with respect to each of its $c' + 1$ arguments, it has

$$k \leqslant 2^{c'+1} \leqslant 2^{c+1}$$

terms, whence $d \geqslant k(k-2)$.

From [3, Cor. 3.2], using Lefschetz' principle on the algebraic closure $\bar{K}$ of K, it follows that there are $i \neq j$ such that g_i and g_j are linearly dependent over $\bar{K}$ and hence over K as well.

Since t_i and t_j are *not* linearly dependent over K (otherwise they would not have been separated as individual terms), two arguments of $\tilde{R}$ must be linearly dependent over K. Since the leading coefficients with respect to x_1 of both arguments of $\tilde{R}$ are constants in $\mathbb{C}$ these arguments of $\tilde{R}$ are linearly dependent over $\mathbb{C}$. The arguments of $\tilde{R}$ are a subset of those of R, so the desired result follows. □

Proof of Theorem 3.1. If there is a permutation P such that $P^{-1}AP$ is triangular, then $P^{-1}FP$ is of the form $x + (Bx)^{*d}$ such that B is a triangular matrix and we are done. So assume that a permutation P as above does not exist. Then it follows from [7, lemma 1.2] (see also [8, prop. 6.3.9]) that A has a principal minor determinant that does not vanish, say that some principal minor of size $(m \times m)$ has a nonzero determinant.

Since $\mathcal{J}(Ax)^{*d}$ is nilpotent, the sum of the determinants of its principal minors of size $(m \times m)$ is zero. It follows that

$$\sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} M_{i_1,i_2,\ldots,i_m} (A_{i_1}x)^{d-1} (A_{i_2}x)^{d-1} \cdots (A_{i_m}x)^{d-1} = 0 \tag{3.3}$$

where A_j is the j-th row of A and $M_{i_1,i_2,\ldots,i_m}$ is the principal minor determinant of A that corresponds to the row and columns with indices $i_1, i_2, \ldots, i_m$.

Since not all $M_{i_1,i_2,\ldots,i_m}$ are zero, it follows from (3.3) that

$$R = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} M_{i_1,i_2,\ldots,i_m} y_{i_1} y_{i_2} \cdots y_{i_m}$$

is a nontrivial relation between $(A_1x)^{d-1}, (A_2x)^{d-1}, \ldots, (A_nx)^{d-1}$. Clearly, $\deg_{y_i} R \leqslant 1$ for all i. It follows from 3.1, noticing that $d - 1 \geqslant 2^{c+1}(2^{c+1} - 2)$, that

$$(A_{i_p}x)^{d-1} = \lambda (A_{j_q}x)^{d-1}$$

for some $\lambda \in \mathbb{C}$ and $i_p \neq j_q$, furthermore $M_{i_1,i_2,\ldots,i_m} \neq 0$ for certain $i_1 < i_2 < \cdots < i_m$.

Since $M_{i_1,i_2,\ldots,i_m} \neq 0$, it follows that $(A_{i_p}x)^{d-1} \neq 0$. Subtracting row j_q of A λ times from row i_p of A results in a matrix with more zero rows than A. So if T^{-1} is the elementary matrix that subtracts row j_q λ times from row i_p, then $T^{-1}FT$ is of the form $x + (Bx)^{*d}$ such that B has more zero rows than A. Now the result follows by induction on the number of nonzero rows of A. □

At the very end, we present a table on what is known about the HDP for power linear Keller maps of several degrees d in several dimensions n. We know that the HDP has an affirmative answer for large degrees compared to the dimension and a negative answer if the degree $d \geqslant 3$ is fixed and the dimension is sufficiently large.

↑ d

⋮	√√√	√√√	√√√	√√√	√√√	√√√	√√?	√√?	√√?	√√?	√√?	√√?	√√?	⋮⋮⋮
10	√	√	√	√	√	√	?	?	?	?	?	?	?	?××
9	√	√	√	√	√	√	?	?	?	?	?	?	?	?××
8	√	√	√	√	√	√	?	?	?	?	?	?	?	?××
7	√	√	√	√	√	√	?	?	?	?	?	?	?	?××
6	√	√	√	√	√	√	?	?	?	?	?	?	?	?××
5	√	√	√	√	√	√	?	?	?	?	?	?	?	?××
4	√	√	√	√	√	√	?	?	?	?	?	?	?	?××
3	√	√	√	√	√	√	√	?	?	?	?	?	?	?××
2	√	√	√	√	√	√	?	?	?	?	?	?	?	???
1	√	√	√	√	√	√	√	√	√	√	√	√	√	√√√
	1	2	3	4	5	6	7	8	9	10	11	12	13	···

$n \rightarrow$

Power Linear Dependence Problem

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