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A PROOF OF THE EQUIVALENCE OF THE DIXMIER, JACOBIAN AND POISSON CONJECTURES

PASCAL KOSSIVI ADJAMAGBO AND ARNO VAN DEN ESSEN

1. Notations, definitions and the main results

Throughout this paper R denotes a commutative ring with 1 and n is a positive integer. The polynomial ring in $x_1, \ldots, x_n$ over R is denoted by $R^{[n]}$ or $R[x_1, \ldots, x_n]$. The *n-th Weyl algebra over* R, denoted by $A_n(R)$, is the associative R-algebra with generators $y_1, \ldots, y_{2n}$ and relations

$[y_i, y_{i+n}] = 1$ for all $1 \leqslant i \leqslant n$ and $[y_i, y_j] = 0$ otherwise.

The *n-th canonical Poisson algebra* $P_n(R)$ *over* R is the polynomial ring $R^{[2n]}$ endowed with the canonical Poisson bracket $\{,\}$ defined by

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_{i+n}} - \frac{\partial f}{\partial x_{i+n}}\frac{\partial g}{\partial x_i}\right)$$

An R-endomorphism ϕ of $R^{[2n]}$ such that $\{\phi(f), \phi(g)\} = \{f, g\}$ for all f, g is called an *endomorphism of* $P_n(R)$. Finally let $\mathbb{C}$ be the field of complex numbers. Then we have the following conjectures.

Poisson Conjecture(PC). For every n the following statement holds:
PC(n,$\mathbb{C}$). Every endomorphism of $P_n(\mathbb{C})$ is an automorphism.

Dixmier Conjecture(DC). For every n the following statement holds:
DC(n,$\mathbb{C}$). Every endomorphism of $A_n(\mathbb{C})$ is an automorphism.

Jacobian Conjecture(JC). For every n the following statement holds:
Every $\mathbb{C}$-endomorphism ϕ of $\mathbb{C}^{[n]}$ with $\det J\phi = 1$ is an automorphism, where $J\phi = (\frac{\partial \phi(x_i)}{\partial x_j})_{1 \leqslant i,j \leqslant n}$.

It is well-known that $DC(n, \mathbb{C})$ implies $JC(n, \mathbb{C})$ (see [2] and [6]). Recently it was shown by Tsuchimoto in [8] that conversely $JC(2n, \mathbb{C})$ implies $DC(n, \mathbb{C})$. (see also the preprint [3] of Belov and Kontsevich). His proof uses the theory of p-curvatures. Another proof was given in [1].

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In this paper we give a self-contained, purely algebraic proof of this last implication and use the Poisson Conjecture to show that all three conjectures described above are equivalent (see Theorem 4.1). Our proof is inspired by the paper [3].

The contents of this paper are arranged as follows: in section 2 we study endomorphisms of canonical Poisson algebras over commutative rings. In particular we show that if $n!$ is a unit in R, then every endomorphism of $P_n(R)$ has Jacobian determinant 1. It follows that $JC(2n, \mathbb{C})$ implies $P(n, \mathbb{C})$. Therefore, to obtain the equivalence of the three conjectures, it remains to show that the n-dimensional Poisson Conjecture over $\mathbb{C}$ implies the n-dimensional Dixmier Conjecture over $\mathbb{C}$. This is done in section 4 by reduction modulo a suitable prime number. The machinery to carry out this reduction is developed in section 3, where we study endomorphisms of Weyl algebras over rings of positive characteristic.

2. Endomorphisms of canonical Poisson algebras over commutative rings

Throughout this section R denotes a commutative ring. Let $A = R^{[2n]}$, $(e) := (e_1, \dots, e_{2n})$ the standard basis of the free A-module $E := A^{2n}$ and $(e^*) := (e_1^*, \dots, e_{2n}^*)$ the dual basis of (e). The *canonical symplectic form* on E is the bilinear form given by

$$\omega := \sum_{i=1}^{n} e_i^* \wedge e_{i+n}^*$$

where $e_i^* \wedge e_{i+n}^*$ is the alternating 2-form on E defined as follows: if $p < q$ then $(e_i^* \wedge e_{i+n}^*)(e_p, e_q) = 1$ if $i = p$ and $i + n = q$ and 0 otherwise. If L is an A-linear endomorphism of E, then $L^*\omega$ is by definition equal to $\omega \circ (L, L)$. Furthermore if $L^*\omega = \omega$, then L is called *symplectic*. A polynomial map $F := (F_1, \dots, F_{2n}) \in A^{2n}$ is called *symplectic* if the A-linear map on E defined by the matrix $(JF)^t$ is symplectic.

Theorem 2.1. *There is equivalence between*
(i) *F is symplectic.*
(ii) *$F^* : A \to A$, $p \to p(F)$ is an endomorphism of $P_n(R)$.*
Furthermore, if i) or ii) holds and $n!$ is a unit in R, then $detJF = 1$.

The proof of this result is based on the following lemma. If B is a bilinear form on E we write $M_{(e)}(B)$ to denote the matrix $(B(e_i, e_j))_{1 \leqslant i,j \leqslant 2n}$.

Lemma 2.1. *Let $F = (F_1, \dots, F_{2n}) \in A^{2n}$. Then*

$$M_{(e)}(((JF)^t)^*\omega) = (\{F_i, F_j\})_{1 \leqslant i,j \leqslant 2n}.$$

Proof. Let $1 \leqslant p, q \leqslant 2n$. Then $(((JF)^t)^*\omega)(e_p, e_q) = \omega((JF)^t e_p, (JF)^t e_q)$. Observe $(JF)^t e_p = \sum_{j=1}^{2n} \frac{\partial F_p}{\partial x_j} e_j$. So

$$(e_i^* \wedge e_{i+n}^*)((JF)^t e_p, (JF)^t e_q) = \frac{\partial F_p}{\partial x_i} \frac{\partial F_q}{\partial x_{i+n}} - \frac{\partial F_p}{\partial x_{i+n}} \frac{\partial F_q}{\partial x_i},$$

which gives the desired result. □

Corollary 2.1. *Let $F = (F_1, \ldots, F_{2n}) \in A^{2n}$. There is equivalence between*
i) *F is symplectic i.e. $((JF)^t)^*\omega = \omega$.*
ii) *$\{F_i, F_j\} = \{x_i, x_j\}$ for all $1 \leqslant i, j \leqslant 2n$.*
iii) *F^* is an endomorphism of $P_n(R)$.*

Proof. By taking $F = (x_1, \ldots, x_{2n})$ in the above lemma we see that $M_{(e)}(\omega) = (\{x_i, x_j\})_{1\leqslant i,j\leqslant 2n}$. Furthermore, since two bilinear forms B_1 and B_2 are equal if and only if their matrices $M_{(e)}(B_1)$ and $M_{(e)}(B_2)$ are equal, the equivalence of i) and ii) follows from the lemma above. Finally, using that F^* is a homomorphism with $F^*(x_i) = F_i$ for all i, the implication iii)→ii) is obvious and the implication ii)→iii) follows from the fact that the Poisson bracket is bilinear, antisymmetric and satisfies Leibniz' rule i.e. $\{a, bc\} = \{a, b\}c + \{a, c\}b$ for all $a, b, c \in A$. □

Proof of Theorem 2.1. The first statement follows from Corollary 2.1. So let F be symplectic and let as before L be the A-linear map of $E = A^{2n}$ defined by the matrix $(JF)^t$. Put $v := e_1^* \wedge \ldots \wedge e_{2n}^*$ the standard volume form on E. Since $\omega^n = n!(-1)^{\frac{n(n-1)}{2}} v$ (see for example [7], Exemple 1.4, page 123) and $n!$ is a unit in R it follows that ω^n is a volume form on E. Furthermore, since F is symplectic $L^*\omega = \omega$ and hence $L^*(\omega^n) = (L^*(\omega))^n = \omega^n$. On the other hand it is well-known that since ω^n is a volume form we have that $L^*(\omega^n) = (detL)\omega^n$ (see Exercise on page 21 of [7]). Since $\{\omega^n\}$ forms a basis of the A-module of all $2n$-forms on E, it follows that $det(L) = 1$. Since $det(L) = det(JF)^t = det(JF)$ we get $det(JF) = 1$, as desired. □

3. Endomorphisms of Weyl algebras over rings of positive characteristic

Throughout this section R denotes a non-zero commutative ring and p is a prime number such that $p.1_R = 0$. It follows that

(*) If $0 \neq r \in R$ and $m.r = 0$, with $m \in \mathbb{Z}$, then $m \in p\mathbb{Z}$.

The following proposition is well-known. Therefore we only give the proof for the case $n = 1$. The general case is done by induction on n.

Let $Z(A_n(R))$ denote the center of $A_n(R)$ i.e. the set of all elements of $A_n(R)$ which commute with all elements of $A_n(R)$.

Proposition 3.1. $Z(A_n(R)) = R[y_1^p, \ldots, y_{2n}^p]$.

Proof. (in case $n = 1$). Let $Z = Z(A_1(R))$ and $z = \sum a_i(y_2)y_1^i \in Z$. Then $[y_2, z] = 0$ implies that $\sum ia_i(y_2)y_1^{i-1} = 0$. So by (*) we get

$$z = \sum a_{pj}(y_2)y_1^{pj}.$$

Using $[z, y_1] = 0$ we also find that $a_{pj}(y_2) \in R[y_2^p]$ for all j. Consequently the center Z is contained in the ring $R[y_1^p, y_2^p]$. Conversely y_1^p commutes with y_1 and

y_2, whence belongs to the center. Similarly y_2^p belongs to the center. So the ring $R[y_1^p, y_2^p]$ is contained in the center Z. Hence $Z = R[y_1^p, y_2^p]$, as desired. □

Now let ϕ be an R-endomorphism of $A_n(R)$. Put $P_i = \phi(y_i)$ and as before $Z = Z(A_n(R))$. The main result of this section is

Theorem 3.1. *$A_n(R) = \oplus_{0 \leqslant \alpha_i < p} Z P_1^{\alpha_1} \dots P_{2n}^{\alpha_{2n}}$.*

Before we prove this result we deduce some fundamental consequences.

Proposition 3.2. i) *$\phi(Z) \subset Z$.*
ii) *If $\phi(Z) = Z$, then $\phi(A_n(R)) = A_n(R)$.*
iii) *If $\phi_{|Z} : Z \to Z$ is injective, then $\phi : A_n(R) \to A_n(R)$ is injective.*

Proof. i) Let $z \in Z$. Then $\phi(z)$ commutes with each $\phi(y_i) = P_i$, hence with each element $P_1^{\alpha_1} \dots P_{2n}^{\alpha_{2n}}$. Also $\phi(z)$ commutes with Z. So by Theorem 3.1 $\phi(z)$ commutes with $A_n(R)$ i.e. $\phi(z) \in Z$.
ii) Let $a \in A_n(R)$. By Theorem 3.1 we can write a in the form

$$a = \sum z_\alpha P_1^{\alpha_1} \dots P_{2n}^{\alpha_{2n}},\ z_\alpha \in Z.$$

Since $\phi(Z) = Z$, there exist $z_\alpha^* \in Z$ such that $z_\alpha = \phi(z_\alpha^*)$. Also $P_i = \phi(y_i)$. Hence

$$a = \phi(\sum z_\alpha^* y_1^{\alpha_1} \dots y_{2n}^{\alpha_{2n}}) \in \phi(A_n(R)).$$

So $A_n(R) = \phi(A_n(R))$.
iii) Let $a \in A_n(R)$ with $\phi(a) = 0$. Observe that by Proposition 3.1 a can be written in the form $a = \sum z_\alpha y_1^{\alpha_1} \dots y_{2n}^{\alpha_{2n}}$, with $z_\alpha \in Z$ and all $0 \leqslant \alpha_i < p$. Then $\phi(a) = 0$ implies that $\sum \phi(z_\alpha) P_1^{\alpha_1} \dots P_{2n}^{\alpha_{2n}} = 0$. Since by i) each $\phi(z_\alpha)$ belongs to Z, it follows from the direct sum decomposition in Theorem 3.1 that all $\phi(z_\alpha)$ are zero. From the injectivity of $\phi_{|Z}$ it then follows that all z_α are zero, which gives that a is zero. □

Before we turn to the proof of Theorem 3.1 we give the following useful lemma concerning the degree of an R-endomorphism ϕ of $A_n(R)$. The degree of ϕ, denoted by $\deg_y \phi$, is by definition the maximun of the degrees of the elements $\phi(y_i)$, where the degree of a non-zero element a of $A_n(R)$ is the maximum of the degrees of the monomials cy^α appearing in a. By Proposition 3.2 i) and Proposition 3.1 it follows that ϕ induces an R-endomorphism of the polynomial ring $Z(A_n(R)) = R[y_1^p, \dots, y_{2n}^p]$, which we denote by ϕ_{pol}.

Writing x_i instead of y_i^p for all i, the degree of ϕ_{pol}, denoted by $\deg_x \phi_{pol}$, is the maximum of all degrees of the polynomials $\phi(x_i)$ in $R[x_1, \dots, x_{2n}]$.

Lemma 3.1. *$deg_y \phi = deg_x \phi_{pol}$.*

Proof. Since $\phi(y_i^p) \in R[y_1^p, \dots, y_{2n}^p]$ we get that

$$\phi(y_i^p) = \sum c_j (y_1^p)^{j_1} \dots (y_{2n}^p)^{j_{2n}}$$

for some $c_j \in R$. So

$$\phi_{pol}(x_i) = \sum c_j x_1^{j_1} \dots x_{2n}^{j_{2n}}$$

whence $\deg_x \phi_{pol}(x_i) = \max_{c_j \neq 0}(j_1 + \dots + j_{2n})$. Consequently

$$\deg_y \phi(y_i^p) = p \deg_x \phi_{pol}(x_i).$$

Finally, since $\deg_y \phi(y_i^p) = \deg_y \phi(y_i)^p = p\deg_y \phi(y_i)$, the desired result follows. $\square$

Proof of Theorem 3.1. Consider the Z-module

$$N = \sum_{0 \leqslant \alpha_i < p} Z P_1^{\alpha_1} \dots P_{2n}^{\alpha_{2n}}.$$

Using the inner derivations $[P_i, -]$ on $A_n(R)$ one readily obtains that N is a free Z-module of rank p^{2n}. From Proposition 3.1 we obtain that

$$M := A_n(R) = \sum_{0 \leqslant \alpha_i < p} Z y_1^{\alpha_1} \dots y_{2n}^{\alpha_{2n}}$$

and, using the derivations $[y_i, -]$, it follows that M is also a free Z-module of rank p^{2n}.

First assume that R is a *domain* and put $S = Z \backslash \{0\}$. Since $N \subset M$ we get that $S^{-1}N \subset S^{-1}M$. Furthermore both modules are $S^{-1}Z$-vector spaces of dimension p^{2n}, hence they are equal. So there exists a non-zero element u in Z with $uM \subset N$ i.e. for each $m \in M$ there exist $u_\alpha \in Z$ such that

$$um = \sum_{0 \leqslant \alpha_i < p} u_\alpha P_1^{\alpha_1} \dots P_{2n}^{\alpha_{2n}}.$$

From Lemma 3.2 below we obtain that $u | u_\alpha$ in Z for each α. Since $A_n(R)$ is a domain, it follows that $m \in N$, whence $M \subset N$.So $A_n(R) = N$, as desired. $\square$

Lemma 3.2. *If $m \in M$ satisfies*

$$um = \sum_{0 \leqslant \alpha_i < p} u_\alpha P_1^{\alpha_1} \dots P_{2n}^{\alpha_{2n}} \tag{3.1}$$

then $u | u_\alpha$ in Z for each α.

Proof. Let $m \neq 0$ and use induction on $d := \max\{\, |\alpha| \,,\, |u_\alpha| \neq 0\}$. If $d = 0$, then $um = u_0$, $u_0 \in Z$. Write $m = \sum z_\alpha y^\alpha$ on the free Z-basis y^α, $0 \leqslant \alpha_i < p$. Then $u_0 = um = \sum (u z_\alpha) y^\alpha$. So $u_0 = u z_0$ i.e. $u | u_0$ in Z.

Now let $d > 0$ and $u_\alpha \neq 0$,with $\alpha_j > 0$ for some $1 \leqslant j \leqslant n$. Applying the inner derivation $[P_{j+n}, -]$ to equation (3.1) gives

$$u[P_{j+n}, m] = \sum u_\alpha \alpha_j {P_1}^{\alpha_1} \dots {P_j}^{\alpha_j - 1} \dots {P_{2n}}^{\alpha_{2n}}.$$

Observe that $0 < \alpha_j < p$ and $u_\alpha \neq 0$, so by (*) $u_\alpha \alpha_j \neq 0$. The induction hypothesis implies that $u | \alpha_j u_\alpha$, whence $u | u_\alpha$ in Z. Repeating this argument we obtain that $u | u_\alpha$ for all α with $|\alpha| > 0$ and $u_\alpha \neq 0$. Say $u_\alpha = u v_\alpha$, $v_\alpha \in Z$. Then

$$um = u_0 + \sum_{|\alpha|>0, 0\leqslant \alpha_i < p} u v_\alpha P^\alpha.$$

So $u(m - \sum v_\alpha P^\alpha) = u_0$. Hence by the case $d = 0$ above, it follows that $u | u_0$ in Z. So $u | u_\alpha$ in Z for all α, as desired.

i) Let R be an arbitrary commutative ring. Replacing R by the finitely generated $\mathbb{Z}$-algebra of R generated by the coefficients of the P_i with respect to the monomials y^α, we may assume that R is a finitely generated $\mathbb{Z}$-algebra. In particular R is noetherian and hence its nilradical is a finite intersection of prime ideals. Since a finite power of the nilradical equals the zero ideal we obtain that the zero ideal is a finite product of prime ideals, say $(0) = \mathfrak{p}_1 \mathfrak{p}_2 \ldots \mathfrak{p}_s$.

ii) Let $\mathfrak{p}$ be a prime ideal in R. Then $\overline{R} := R/\mathfrak{p}$ is a domain. So, as shown above

$$\begin{aligned} A_n(\overline{R}) &\subset \sum_{0\leqslant \alpha_i < p} Z(A_n(\overline{R})) \overline{P}_1^{\,\alpha_1} \ldots \overline{P}_{2n}^{\,\alpha_{2n}} \\ &= \sum_{0\leqslant \alpha_i < p} \overline{R}[y_1^p, \ldots, y_{2n}^p] \overline{P}_1^{\,\alpha_1} \ldots \overline{P}_{2n}^{\,\alpha_{2n}}. \end{aligned}$$

Hence

$$A_n(R) \subset \sum_{0\leqslant \alpha_i < p} R[y_1^p, \ldots, y_{2n}^p] P_1^{\alpha_1} \ldots P_{2n}^{\alpha_{2n}} + \mathfrak{p} A_n(R).$$

So by Proposition 3.1 we get $A_n(R) \subset N + \mathfrak{p} A_n(R)$.

iii) Finally we use that $\mathfrak{p}_1 \mathfrak{p}_2 \ldots \mathfrak{p}_s = (0)$ and that $\mathfrak{p}_i \subset R \subset Z$. Namely by ii) we get that $A_n(R) \subset N + \mathfrak{p}_1 A_n(R)$ and $A_n(R) \subset N + \mathfrak{p}_2 A_n(R)$, whence $A_n(R) \subset N + \mathfrak{p}_1 N + \mathfrak{p}_1 \mathfrak{p}_2 A_n(R) \subset N + \mathfrak{p}_1 \mathfrak{p}_2 A_n(R)$. Repeating this argument gives

$$A_n(R) \subset N + \mathfrak{p}_1 \ldots \mathfrak{p}_s A_n(R) = N.$$

So $A_n(R) = N$, as desired. □

4. The proof of the main theorem

The main result of this paper is

Theorem 4.1. *For each $n \geqslant 1$ we have the following implications: $DC(n, \mathbb{C}) \to JC(n, \mathbb{C})$, $JC(2n, \mathbb{C}) \to P(n, \mathbb{C})$ and $P(n, \mathbb{C}) \to D(n, \mathbb{C})$ i.e. the Dixmier, Jacobian and Poisson Conjectures are equivalent.*

As already observed in the introduction it remains to prove the last implication. Therefore we first consider the statement $P(n, \mathbb{C})$. More precisely, for every $d \geqslant 1$ we analyse the statement $P(n, \mathbb{C}, d)$, which asserts that the Poisson Conjecture holds for all endomorphisms of $P_n(\mathbb{C})$ of degree $\leqslant d$. This means that if $F = (F_1, \ldots, F_{2n})$ satisfies deg $F_i \leqslant d$ and $\{F_i, F_j\} = \{x_i, x_j\}$ for all i, j , then F has an inverse of degree $\leqslant d^{2n-1}$, according to [6], Proposition 2.3.1. To rewrite this statement in terms of polynomial equations we consider the universal polynomial map of degree d in $2n$ variables $x = (x_1, \ldots, x_{2n})$ i.e.

$$F_i^U := (F_1^U, \ldots, F_{2n}^U)$$

where each $F_i^U := \sum A_\alpha^{(i)} x^\alpha$ with $|\alpha| \leqslant d$ and all $A_\alpha^{(i)}$ are different variables. Let $A := (\ldots, A_\alpha^{(i)}, \ldots)$ and denote by $\mathbb{C}[A]$ the polynomial ring in A over $\mathbb{C}$. Consider the polynomials $P_{ij} := \{F_i^U, F_j^U\} - \{x_i, x_j\}$ for all $1 \leqslant i < j \leqslant 2n$ and let $J := (g_1(A), \ldots, g_r(A))$ be the ideal in $\mathbb{C}[A]$ generated by the coefficients of all monomials x^α appearing in all P_{ij}. Then by Theorem 2.1 and Corollary 2.1 the canonical image of F^U in $(\mathbb{C}[A]/J)[x]^{2n}$ has Jacobian determinant 1 and hence, by the formal inverse function theorem, it has a formal inverse in $(\mathbb{C}[A]/J)[[x]]^{2n}$, represented by some $G(A)$ in $\mathbb{C}[A][[x]]^{2n}$. Let I be the ideal in $\mathbb{C}[A]$ generated by the coefficients in $G(A)$ of all x^α with $|\alpha| > D := d^{2n-1}$ and let $h_1(A), \ldots, h_t(A)$ be a system of generators of I.

Proposition 4.1. *If $P(n, \mathbb{C}, d)$ holds, there exist $b_j^{(i)}$ in $\mathbb{C}[A]$ and a positive integer ρ such that*

$$h_i(A)^\rho = \sum_j b_j^{(i)}(A) g_j(A) \text{ for all } 1 \leqslant i \leqslant t. \tag{4.1}$$

Proof. Let $a \in \mathbb{C}^A$ be a zero of J. Then, since $P(n, \mathbb{C}, d)$ holds, the map $F^U(A = a)$ is invertible and its inverse is equal to $G(A = a)$. So by [6], Proposition 2.3.1 deg $G(A = a) \leqslant d^{2n-1} = D$. Hence a is a zero of I. So every zero of J is a zero of I. Then (4.1) follows from the Nullstellensatz. □

As an immediate consequence of Proposition 4.1 we get

Corollary 4.1. *Let R be a subring of $\mathbb{C}$ containing the coefficients of the polynomials $h_i(A), b_j^{(i)}(A)$ and $g_j(A)$. If $\mathfrak{a}$ is a proper ideal of R and f an endomorphism of $P_n(R/\mathfrak{a})$ of degree $\leqslant d$, then f has an inverse of degree $\leqslant D(= d^{2n-1})$.*

Next we study the statement $D(n, \mathbb{C})$, i.e. the statement that every $\mathbb{C}$- endomorphism of $A_n(\mathbb{C})$ is an automorphism. Since $A_n(\mathbb{C})$ is a simple ring and the kernel of a $\mathbb{C}$-endomorphism is a two-sided ideal which is not the whole ring, it follows that every $\mathbb{C}$-endomorphism of $A_n(\mathbb{C})$ is automatically injective. Hence the Dixmier Conjecture is equivalent to saying that every $\mathbb{C}$-endomorphism of $A_n(\mathbb{C})$ is surjective.

We will show below that $P(n, \mathbb{C})$ implies a more refined statement. To describe it observe that a $\mathbb{C}$-endomorphism ϕ of $A_n(\mathbb{C})$ is surjective if and only if there exist $\psi_1, \ldots, \psi_{2n}$ in $A_n(\mathbb{C})$ such that $\phi(\psi_i) = y_i$ for all i. If all ψ_i have degree $\leqslant N$, for some N, we say that ϕ is *N-surjective.*

Proposition 4.2. *Let $d \geqslant 1$. Then $P(n, \mathbb{C})$ implies*

(4.2) *Every $\mathbb{C}$-endomorphism of $A_n(\mathbb{C})$ of degree $\leqslant d$ is $D := d^{2n-1}$-surjective.*

Proof. i) By contradiction. So let ϕ be a $\mathbb{C}$-endomorphism of $A_n(\mathbb{C})$, say of degree d, which is not D-surjective. This means that there do *not* exist $\psi_1, \ldots, \psi_{2n}$ of degree $\leqslant D$ in $A_n(\mathbb{C})$ such that

$$\phi(\psi_1) - y_1 = 0, \ldots, \phi(\psi_{2n}) - y_{2n} = 0.$$

So if we consider the *universal Weyl algebra elements* of degree D, i.e. the expressions $\psi_i^U = \sum_{|\alpha| \leqslant D} c_\alpha^{(i)} y^\alpha$, where all $c_\alpha^{(i)}$ are different variables and consider the formal expressions

$$\phi(\psi_i^U) - y_i := \sum_{|\alpha| \leqslant D} c_\alpha^{(i)} \phi(y_1)^{\alpha_1} \ldots \phi(y_{2n})^{\alpha_{2n}} - y_i$$

then the coefficients $P_1(C), \ldots, P_s(C)$ of all monomials y^α appearing in these expressions (each $P_i(C)$ is a polynomial in the polynomial ring $\mathbb{C}[C]$, which is generated over $\mathbb{C}$ by all coefficients $c_\alpha^{(i)}$), have no common zero in $\mathbb{C}^C$. So by the Nullstellensatz there exist $Q_1(C), \ldots, Q_s(C)$ in $\mathbb{C}[C]$ such that

$$1 = \sum Q_j(C) P_j(C). \tag{4.3}$$

ii) Now let R be the $\mathbb{Z}$-subalgebra of $\mathbb{C}$ generated by $\frac{1}{n!}$, all coefficients of the monomials y^α appearing in the $\phi(y_i)$, all coefficients appearing in the h_i, g_j and the $b_j^{(i)}$ and all coefficients appearing in all Q_j and P_j. Then ϕ is an R-endomorphism of $A_n(R)$ and R is a finitely generated $\mathbb{Z}$-algebra contained in $\mathbb{C}$. Let $\mathfrak{m}$ be a maximal ideal in R. Then by [4], V, section 3, no.4, theorem 3, $R/\mathfrak{m}$ is a finite field, say of characteristic $p > 0$. So $p \in \mathfrak{m}$. Hence p is not a unit in R, whence pR is a proper ideal in R. Reducing the equations in (4.3) modulo pR we deduce that the endomorphism $\bar{\phi}$ of $A_n(R/pR)$, obtained by reducing the coefficients of ϕ mod pR, is *not* D-surjective.

iii) On the other hand by Proposition 3.2 i) the endomorphism $\bar{\phi}$ of $A_n(R/pR)$ induces an endomorphism $\bar{\phi}_{pol}$ on the polynomial ring $Z(A_n(R/pR))$. From Theorem 4.2 below and Lemma 3.1 we deduce that $\bar{\phi}_{pol}$ is an endomorphism of $P_n(R/pR)$ of degree $\leqslant d$. Then by Corollary 4.1 (applied to $\mathfrak{a} = pR$) we deduce that $\bar{\phi}_{pol}$ has an inverse of degree $\leqslant D = d^{2n-1}$. It follows from Proposition 3.2 ii), iii) that $\bar{\phi}$ is an automorphism of $A_n(R/pR)$. Let τ be its inverse. Then the restriction of τ to the center of $A_n(R/pR)$, denoted by τ_{pol}, is equal to $\bar{\phi}_{pol}^{-1}$ and hence, as observed before, has degree $\leqslant D$. Since $\deg_x \tau_{pol} = \deg_y \tau$ (by Lemma 3.1), we deduce that τ has degree $\leqslant D$. So $\bar{\phi}$ is D-surjective, a contradiction with ii). $\square$

Theorem 4.2. *Let ϕ be an endomorphism of $A_n(R)$ and $\bar{\phi}$ the induced endomorphism of $A_n(R/pR)$. Then the restriction of $\bar{\phi}$ to $Z(A_n(R/pR))$, denoted by $\bar{\phi}_{pol}$, is an endomorphism of $P_n(R/pR)$.*

To prove this theorem we give another description of the Poisson bracket on $P_n(R/pR)$. Put $W := R[y_1^p, \ldots, y_{2n}{}^p]$ and $x_i := y_i^p$ for each i.

Lemma 4.1. *Let $A \in W$ and $B \in A_n(R)$. Then $[A, B] \in pA_n(R)$.*

Proof. Since $[\,,\,]$ is R-bilinear we may assume that $A = (y_1^p)^{\alpha_1} \dots (y_{2n}{}^p)^{\alpha_{2n}}$ and $B = y_1{}^{\beta_1} \dots y_{2n}{}^{\beta_{2n}}$. Using Leibniz' rule we may even assume that $A = y_i{}^p$ and $B = y_j$, in which case the result is clear. □

Proposition 4.3. *Let $a, b \in R/pR[x_1 \dots, x_{2n}]$ and $A, B \in W$ such that $a = A(mod\, pA_n(R))$ and $b = B(mod\, pA_n(R))$. Then $\frac{1}{p}[A,B]$ is a well-defined element of $A_n(R)$ and*

$$\{a, b\} = \frac{1}{p}[A, B](mod\, pA_n(R)).$$

Proof. Since R has no $\mathbb{Z}$-torsion the first statement follows from Lemma 4.1. To prove the formula, observe that both $\{\,,\,\}$ and $[\,,\,]$ are bilinear, antisymmetric and satisfy Leibniz' rule. Therefore it suffices to show that

$$\{x_i, x_j\} = \frac{1}{p}[y_i^p, y_j{}^p](mod\, pA_n(R))$$

for all $i < j$. If $j \neq i + n$ both sides are zero. So assume $j = i + n$. Then the result follows from the following formula

$$\frac{1}{p}[y_{i+n}^p, y_i^p] = \frac{1}{p}\sum_{k=0}^{p-1} \frac{(p!)^2}{(k!)^2(p-k)!} y_i^k y_{i+n}^k = -1(mod\, p\mathbb{Z}[y_i, y_{i+n}]) \tag{4.4}$$

which can be found in [5]. □

Proof of Theorem 4.2. To prove that $\bar{\phi}_{pol}$ is an endomorphism of $P_n(R/pR)$ it suffices to show that $\{\bar{\phi}_{pol}(x_i), \bar{\phi}_{pol}(x_j)\} = \{x_i, x_j\}$ for all $i < j$. Since $y_i^p \in Z$ it follows that $\phi(y_i^p) = A + pA'$ with A in W and A' in $A_n(R)$. Similarly $\phi(y_j^p) = B + pB'$ with B in W and B' in $A_n(R)$. Then by Proposition 4.3

$$\{\bar{\phi}_{pol}(x_i), \bar{\phi}_{pol}(x_j)\} = \frac{1}{p}[A, B](mod\, pA_n(R)).$$

Since by Lemma 4.1 both $[A, B']$ and $[A', B]$ belong to $pA_n(R)$ we get

$$\frac{1}{p}[\phi(y_i^p), \phi(y_j^p)] = \frac{1}{p}[A, B] + [A, B'] + [A', B] + p[A', B'] = \frac{1}{p}[A, B](mod\, pA_n(R)).$$

So $\{\bar{\phi}_{pol}(x_i), \bar{\phi}_{pol}(x_j)\} = \frac{1}{p}[\phi(y_i^p), \phi(y_j^p)](mod\, pA_n(R))$.

If $j \neq i+n$ the righthand side is equal to zero and hence equals $\{x_i, x_j\}$. Finally if $j = i+n$ the righthand side equals $\phi(\frac{1}{p}[y_i^p, y_{i+n}^p])(mod\, pA_n(R))$, which by (4.4) is equal to 1 and hence equals $\{x_i, x_{i+n}\}$. □

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Université Paris 6
UFR 929, 4 Place Jussieu
75252 Paris, France.

Department of Mathematics
Radboud University Nijmegen
Toernooiveld, 6525 ED Nijmegen
The Netherlands.