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ON THE ŁOJASIEWICZ EXPONENT AT INFINITY OF POLYNOMIAL MAPPINGS

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Abstract. This is a survey article on the Łojasiewicz exponent at infinity of polynomial mappings. This exponent is a numerical measure of behaviour of polynomial mappings at infinity. We present properties, formulas and some applications of the Łojasiewicz exponent at infinity.

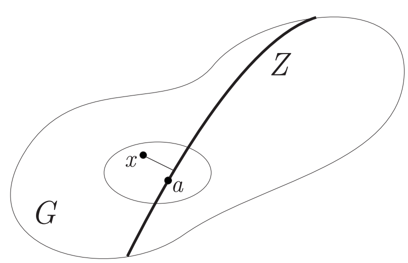
1. Introduction

The origin of the Łojasiewicz inequality (and the Łojasiewicz exponent) lies in the distribution theory. Precisely in the division problem posed by L. Schwartz in [34]. In the solution of this problem (in the full generality by Łojasiewicz [24]), the main difficulty was to explain the structure of real analytic sets (i.e. subsets of $\mathbb{R}^n$ described by systems of real analytic equations). From this description of analytic sets fundamental Łojasiewicz inequality follows, which is the main fact used in solution of the division problem.

Theorem 1.1 (The Łojasiewicz Inequality). *Let $G \subset R^n$ be an open set and let $f : G \to R$ be a real analytic function. Then for any $a \in G$ there exist $\nu \geqslant 0$ and $C > 0$ such that*

$$|f(x)| \geqslant C\rho(x, Z)^\nu$$

for x in a neighbourhood of a, where $Z = V(f)$ is the zero-set of f in G and $\rho(x, Z)$ is the distance of x to Z.



In many branches of mathematics (real analytic geometry, semi and subanalytic geometry, singularity theory, jacobian conjecture, polynomial mappings) there are many useful variants of this inequality. Their common feature is that it

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is an estimation from below of the norm of an analytic mapping f by the distance of an argument to the zero-set of f. The best exponent ν in such an inequality is called the Łojasiewicz exponent. One of these variants is the Łojasiewicz inequality at infinity for polynomial mappings. In this case $|f(z)|$ is the norm of a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ with finite zero-set Z in $\mathbb{C}^n$ and $\rho(z, Z)$ is approximately equal to $|z|$ for large z and a neighbourhood under consideration is a neighbourhood of infinity (i.e. it is the complement of a compact set). So, in this case the Łojasiewicz inequality looks like

$$|F(z)| \geqslant C\,|z|^{\nu} \text{ in a neighbourhood of infinity.}$$

The best exponent ν in the above inequality i.e. the biggest one is the Łojasiewicz exponent of F at infinity. It plays an important role in many problems of polynomial mappings. We present some applications of this exponent.

In Section 2 we give definition of the Łojasiewicz exponent and its elementary properties. In Section 3 we recall the known main formulas for the Łojasiewicz exponent in terms of various numerical invariants of polynomial mappings. In Section 4 we describe applications of the Łojasiewicz exponent to the theory of polynomial mappings and in particular to the jacobian conjecture. Section 5 presents applications to the theory of bifurcation points (critical values at infinity) of polynomials. In Section 6 we pose a few problems.

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2. Definitions and elementary properties

Definition 2.1. Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping. The *Łojasiewicz exponent of F at infinity* is defined as the best exponent ν (the biggest one) for which the following inequality holds

$$|F(z)| \geqslant C\,|z|^{\nu} \tag{2.1}$$

for some constant C and sufficiently large $|z|$. We denote this exponent by $\mathcal{L}_\infty(F)$. Precisely

$$\mathcal{L}_\infty(F) := \sup\left\{\nu \in \mathbb{R} : \underset{C>0}{\exists}\ \underset{R>0}{\exists}\ \underset{z\in\mathbb{C}^n,\ |z|\geqslant R}{\forall} |F(z)| \geqslant C\,|z|^{\nu}\right\}$$

We have $\mathcal{L}_\infty(F) \in \mathbb{R} \cup \{-\infty\}$.

Remark 1. Analogous definition can be given for a real polynomial mapping $F : \mathbb{R}^n \to \mathbb{R}^m$.

Proposition 2.1. *$\mathcal{L}_\infty(F) > -\infty$ if and only if the zero-set of F is isolated (equivalently $\#F^{-1}(0) < +\infty$).*

Proof. The implication "$\Rightarrow$" is obvious, because any non-isolated complex algebraic set is unbounded.

The implication "$\Leftarrow$" follows from the Hilbert Nullstellensatz. Indeed, if the set $F^{-1}(0)$ is isolated (then finite since $F^{-1}(0)$ is an algebraic set) then for any $i \in \{1, \ldots, n\}$ there exists a polynomial $P_i \in \mathbb{C}[z_i]$ in one variable z_i such that $V(P_i) \supset V(F)$. Hence by the Hilbert Nullstellensatz there exist n_i and polynomials A_j^i in variables $z_1, ..., z_n$ such that

$$P_i(z_i)^{n_i} = A_1^i(z)F_1(z) + ... + A_m^i(z)F_m(z).$$

Hence for sufficiently large $|z_i|$

$$\begin{aligned} C\,|z_i|^{n_i \deg P_i} &\leqslant |P_i(z_i)^{n_i}| \leqslant \left|A_1^i(z)\right| |F_1(z)| + \cdots + \left|A_m^i(z)\right| |F_m(z)| \\ &\leqslant C'\,|z|^{\max\limits_j (\deg A_j^i)}\,|F(z)|\,, \end{aligned}$$

for constants $C, C' > 0$. Then for sufficiently large $|z|$

$$\frac{C}{C'}\,|z|^{\min\limits_i (n_i \deg P_i) - \max\limits_{i,j} (\deg A_j^i)} \leqslant |F(z)|\,.$$

which gives that $\mathcal{L}_\infty(F) > -\infty$. $\square$

By this property we will assume in the sequel, that $\#F^{-1}(0) < +\infty$.

Remark 2. For the Łojasiewicz inequality in general case of non-isolated algebraic sets see [18], [19], [20], [12].

Simple examples

Example 2.1. 1. For $F(x, y) = (x, xy - 1) : \mathbb{C}^2 \to \mathbb{C}^2$ we have $\mathcal{L}_\infty(F) = -1$.

2. For $F(x, y) = (x, y + x^2) : \mathbb{C}^2 \to \mathbb{C}^2$ we have $\mathcal{L}_\infty(F) = \frac{1}{2}$.

More sophisticated examples [6], [7], [41].

Example 2.2. 1. Take $p, q \in \mathbb{N}$, $0 < q$, $1 < p$ and define

$$\begin{aligned} f(x, y) &:= y^p + (x + y^q)^p : \mathbb{C}^2 \to \mathbb{C} \\ F(x, y) &:= \operatorname{grad} f = (p(x + y^q)^{p-1}, py^{p-1} + pqy^{q-1}(x + y^q)^{p-1}) : \mathbb{C}^2 \to \mathbb{C}^2. \end{aligned}$$

Then

$$\mathcal{L}_\infty(F) = -1 + \frac{p}{q}.$$

2. Take $p, q \in \mathbb{N}$, $0 < q < p$ and define

$$\begin{aligned} f(x, y) &:= y + y^{1+q}x^{p-q} : \mathbb{C}^2 \to \mathbb{C} \\ F(x, y) &:= \operatorname{grad} f = ((p - q)y^{1+q}x^{p-q-1}, 1 + (1 + q)y^q x^{p-q}) : \mathbb{C}^2 \to \mathbb{C}^2. \end{aligned}$$

Then

$$\mathcal{L}_\infty(F) = -\frac{p}{q}.$$

The examples show that any rational number is a Łojasiewicz exponent at infinity.

In the sequel we will use a generalization of the Łojasiewicz exponent at infinity on unbounded sets.

Definition 2.2. Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping and $S \subset \mathbb{C}^n$ be an unbounded set. *The Łojasiewicz exponent of F at infinity on S* is defined as the best exponent ν for which the following inequality holds

$$|F(z)| \geqslant C\,|z|^{\nu}$$

for some constant C and sufficiently large z in S. We denote this exponent by $\mathcal{L}_\infty(F|S)$. Precisely

$$\mathcal{L}_\infty(F|S) := \sup\left\{\nu \in \mathbb{R} : \underset{C>0}{\exists}\ \underset{R>0}{\exists}\ \underset{z\in S,\ |z|\geqslant R}{\forall} |F(z)| \geqslant C\,|z|^{\nu}\right\}$$

Example 2.3. For $F(x, y) = (x, xy - 1) : \mathbb{C}^2 \to \mathbb{C}^2$ we have $\mathcal{L}_\infty(F) = -1$. If $S := \{y = 0\}$, then

$$\mathcal{L}_\infty(F|S) = 0.$$

Now, we will give the basic, elementary properties of the Łojasiewicz exponent.

Proposition 2.2. *$\mathcal{L}_\infty(F)$ is an invariant of linear (not affine) automorphisms of $\mathbb{C}^n$ and $\mathbb{C}^m$.*

Proof. It follows from the fact that for any linear automorphism

$$L : \mathbb{C}^k \to \mathbb{C}^k$$

there exist constants $C, C' > 0$ such that

$$C'\,|z| \leqslant |L(z)| \leqslant C\,|z|\,, \quad z \in \mathbb{C}^k.$$

□

Proposition 2.3. *$\mathcal{L}_\infty(F)$ is not an invariant of polynomial automorphisms of the domain $\mathbb{C}^n$ and the codomain $\mathbb{C}^m$.*

Proof. 1. Triangular automorphisms of $\mathbb{C}^n$ may change $\mathcal{L}_\infty(F)$. For example $\mathcal{L}_\infty(x, y) = 1$ and $\mathcal{L}_\infty(x, y + x^2) = \frac{1}{2}$.

2. Translations in $\mathbb{C}^m$ may change $\mathcal{L}_\infty(F)$. For example $\mathcal{L}_\infty(x, xy - 1) = -1$ and $\mathcal{L}_\infty(x, xy) = -\infty$. □

Proposition 2.4. $\mathcal{L}_\infty(F) \leqslant \deg F$, *where* $\deg F := \max(\deg F_1, ..., \deg F_m)$.

Proof. It follows from the elementary inequality for polynomial mappings

$$|F(z)| \leqslant C\,|z|^{\deg F} \quad \text{for} \quad |z| \gg 0.$$

□

Proposition 2.5. *$\mathcal{L}_\infty(F)$ is attained on a meromorphic curve at infinity i.e.*

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|\Gamma),$$

where Γ is a meromorphic curve at infinity.

By a *meromorphic curve* in $\mathbb{C}^n$ we mean a set $\Gamma \subset \mathbb{C}^n$ for which there exists a holomorphic mapping (called a *parametrization of* Γ)

$$\Phi = (\varphi_1, ..., \varphi_m) : \{t \in \mathbb{C} : |t| > R\} \to \mathbb{C}^n$$

such that $\operatorname{Im}\Phi = \Gamma$ and Φ has a pole at $\infty \in \bar{\mathbb{C}}$ (it means that each φ_i has at most a pole at ∞ and at least one of φ_i has a pole at ∞ – this implies that $|\Phi(t)| \to \infty$ when $t \to \infty$). If we define $\deg\Phi := \max(\deg\varphi_1, ..., \deg\varphi_m)$ ($\deg\varphi_i$ are well defined because each $\varphi_i \not\equiv 0$ has an expansion in a Laurent series at ∞

$$\varphi_i(t) = \sum_{k=k_0}^{-\infty} a_k t^k,\ a_{k_0} \neq 0,\ k_0 \in \mathbb{Z}$$

and then we put $\deg\varphi_i := k_0$), then it is easy to show that

$$\mathcal{L}_\infty(F|\Gamma) = \frac{\deg(F \circ \Phi)}{\deg\Phi}.$$

It may be also written

$$|F \circ \Phi(t)| \sim |\Phi(t)|^{\mathcal{L}_\infty(F|\Gamma)} \sim |t|^{\mathcal{L}_\infty(F|\Gamma)\deg\Phi} \qquad \text{for} \quad t \to \infty.$$

Proof of Proposition 2.5. It follows from the standard Curve Selection Lemma at infinity (see [25] Lemma 2, [9] Prop. 1). It suffices to note that the minimum set of $|F(z)|$ on all spheres $S_R \subset \mathbb{C}^n$ for $R > 0$ is an semialgebraic and unbounded set. □

Proposition 2.6. *$\mathcal{L}_\infty(F)$ is a rational number.*

Proof. It follows from the above Proposition 2.5 because

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|\Gamma) = \frac{\deg(F \circ \Phi)}{\deg\Phi},$$

where Φ is a parametrization of an appropriate meromorphic curve Γ at infinity. □

Proposition 2.7. *$\mathcal{L}_\infty(F)$ is attained i.e. for the number $\mathcal{L}_\infty(F)$ the Łojasiewicz inequality holds*

$$|F(z)| \geqslant C\,|z|^{\mathcal{L}_\infty(F)} \qquad \textit{for} \ \ |z| \gg 0\ .$$

Proof. It follows from the Proposition 2.5 because then

$$|F(z)| \sim |z|^{\mathcal{L}_\infty(F)} \quad \text{on} \ \ \Gamma \ \ \text{for} \ \ |z| \gg 0\ .$$

□

3. Formulas for the Łojasiewicz exponent at infinity

In this section we will give some known formulas and estimations of the Łojasiewicz exponent at infinity.

The most complete results on the Łojasiewicz exponent at infinity are in two-dimensional case i.e. for $n = m = 2$ (the case $n = 1$ is trivial since in this case $\mathcal{L}_\infty(F) = \deg F$). So, we will start with this case. For the convenience, we will denote in this case: $F = (f, g)$, $z = (x, y)$. Formulas for $\mathcal{L}_\infty(F)$ were given in various terms. The first one was a formula for $\mathcal{L}_\infty(F)$ in terms of parametrizations of the branches of the zero-sets of f and g at infinity.

Theorem 3.1. ([5], Main Theorem) *If $F \neq const.$, then*

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|V(f) \cup V(g))$$

where $V(f) := f^{-1}(0)$ and $V(g) := g^{-1}(0)$ are zero-sets of f and g.

To reformulate this theorem in terms of parametrizations we notice that $V(f)$ and $V(g)$ are complex algebraic curves in $\mathbb{C}^2$ (provided f, g are not constants). Then in a neighbourhood of infinity in $\mathbb{C}^2$ they are finite unions of meromorphic curves (called branches of f and g at infinity) (see [5], Proposition 3.1). If we denote by Φ_i, $i = 1, ..., r$ (resp. Ψ_j, $j = 1, ..., s$) parametrizations of the branches of f (resp. g) at infinity then we may give the equivalent form of the above theorem.

Theorem 3.2. *If $F \neq const.$, then*

$$\mathcal{L}_\infty(F) = \min_{i,j} \left(\frac{\deg(F \circ \Phi_i)}{\deg \Phi_i}, \frac{\deg(F \circ \Psi_j)}{\deg \Psi_j} \right). \tag{3.1}$$

Example 3.1. Let $F(x, y) = (x, xy - 1)$. Then $f(x, y) = x$ has one branch at infinity with a parametrization $\Phi_1(t) = (0, t)$, $t \in \mathbb{C}$ and g has two branches at infinity with parametrizations $\Psi_1(t) = (t, 1/t)$, $|t| > 1$ and $\Psi_2(t) = (1/t, t)$, $|t| > 1$. Then by the formula (3.1) we easily obtain

$$\mathcal{L}_\infty(F) = -1.$$

The next formula for $\mathcal{L}_\infty(F)$ is more effective (since in the above formula (3.1) it is not easy in general to determine explicitly parametrizations of the branches at infinity of a polynomial). It is expressed in terms of resultant of suitable polynomials.

Let us assume, without loss of generality (see Proposition 2.2) that $F = (f, g)$ satisfies the following assumptions

$$0 < \deg f = \deg_y f, \quad 0 < \deg g = \deg_y g.$$

Let $\mathrm{Res}_y(f(x, y) - u, g(x, y) - v) = Q_0(u, v)x^N + \cdots + Q_N(u, v)$, $Q_0 \neq 0$, be the resultant of $f(x, y) - u$ and $g(x, y) - v$ with respect to y (u, v are new variables). Then

Theorem 3.3. ([41], Thm 1.4.5, [6], Thm 3.1, [7], Thm 3.1)

$$\mathcal{L}_\infty(F) = \begin{cases} \left[\max\limits_{1\leqslant i\leqslant N} \frac{\deg Q_i}{i}\right]^{-1} & \textit{if} \quad Q_0 = const. \\ 0 & \textit{if} \quad Q_0 \neq const.,\ Q_0(0,0) \neq 0 \\ -\left[\min\limits_{0\leqslant i\leqslant r} \frac{\operatorname{ord}_{(0,0)} Q_i}{r+1-i}\right]^{-1} & \textit{if} \quad \begin{array}{l} Q_0(0,0) = \dots = Q_r(0,0) = 0, \\ Q_{r+1}(0,0) \neq 0 \end{array} \\ -\infty & \textit{if} \quad Q_0(0,0) = \dots = Q_N(0,0) = 0. \end{cases}$$

Let us pass to the n-dimensional case. So far, there are no explicit formulas for the Łojasiewicz exponent at infinity of $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ (besides in some particular classes of polynomial mappings - for instance, for proper polynomial mappings [30], Cor. 2.6.). However, there are some results. The first one is a generalization of Theorem 3.1.

Theorem 3.4. ([9], Thm 1) *If $F \neq const.$, then*

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|V(F_1) \cup ... \cup V(F_m)).$$

Of course, it is impossible to generalize Theorem 3.2, since the algebraic sets $V(F_i)$ are $(n-1)$-dimensional (provided they are not empty) and in general not parametrizable.

In many cases one needs only estimations (mainly from below) of the Łojasiewicz exponent at infinity in terms of the other numerical invariants related to F. First consider the case of equal dimensions i.e. $n = m$. Since $\#F^{-1}(0) < +\infty$ by our general assumption then we may define the geometric degree $d(F)$ of F by

$$d(F) := \sum_{P \in F^{-1}(0)} \mu_P(F),$$

where $\mu_P(F)$ means the multiplicity of F at P. Then we have

Theorem 3.5. ([4], Thm 1 - for $n = m = 2$, [30], Thm 1.10 - for proper mappings and [12], Thm 7.3 - in general).

$$\mathcal{L}_\infty(F) \geqslant d(F) - \prod_{i=1}^{n} \deg F_i + \min_{i=1}^{n}(\deg F_i).$$

In the case $n \neq m$ one may consult [37], where this case is reduced to the considered above case $n = m$ by composition of F with a general projection. Another estimations one may find in [2], [19], [10], [31], [33] and generalizations to an arbitrary ideal in $\mathbb{C}[z_1, ..., z_n]$ (instead of the mapping F) in [20].

4. Applications of the Łojasiewicz exponent at infinity to polynomial automorphisms

In this Section we present applications of the Łojasiewicz exponent at infinity to polynomial automorphisms and in particular to the Jacobian Conjecture. First application is based on the following easy observation.

Proposition 4.1. *Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping. Then F is a proper mapping if and only if $\mathcal{L}_\infty(F) > 0$.*

Recall that by definition F is *proper* if and only if for any compact set $K \subset \mathbb{C}^m$ the preimage $f^{-1}(K)$ is also compact (it is equivalent to the property that for any sequence $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^n$ such that $|z_n| \to \infty$ we have $|F(z_n)| \to \infty$).

Proof. The implication "$\Rightarrow$ " follows from Proposition 2.5. The implication "$\Leftarrow$ " is obvious by the Łojasiewicz inequality

$$|F(z)| \geqslant C\,|z|^{\mathcal{L}_\infty(F)} \quad \text{for} \ \ |z| \gg 0$$

□

Then the famous

Jacobian Conjecture. *If $F = (F_1, ..., F_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial mapping and $\operatorname{Jac} F \equiv 1$ in $\mathbb{C}^n$, then F is a polynomial automorphism,*

can be reduced to a problem on the Łojasiewicz exponent by the following proposition.

Proposition 4.2. *If $F = (F_1, ..., F_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial mapping and $\operatorname{Jac} F \equiv 1$ then F is a polynomial automorphism of $\mathbb{C}^n$ if and only if $\mathcal{L}_\infty(F) > 0$.*

Proof. The implication "$\Rightarrow$" is obvious because polynomial automorphims are proper mappings. Conversely, if $\mathcal{L}_\infty(F) > 0$ then F is proper. Hence F being a proper, local biholomorphism is an unbranched covering of $\mathbb{C}^n$. By the Monodromy Theorem F is a homeomorphism, hence biholomorphism. From the Łojasiewicz inequality

$$|F(z)| \geqslant C\,|z|^{\mathcal{L}_\infty(F)} \quad \text{for} \ \ z \gg 0$$

we obtain

$$\left|F^{-1}(w)\right| \leqslant C\,|w|^{1/\mathcal{L}_\infty(F)} \quad \text{for} \ \ w \gg 0.$$

Hence F^{-1} is also a polynomial mapping. □

Hence to prove Jacobian Conjecture it is enough to check if F is proper. In particular case for $n = 2$, by Theorem 3.1, it suffices to check properness of $F = (f, g)$ on $V(f) \cup V(g)$.

Proposition 4.3. *If $F = (f, g)$ and $\operatorname{Jac} F = 1$ then F is a polynomial automorphism of $\mathbb{C}^2$ if and only if $F|V(f) \cup V(g)$ is proper.*

It is interesting that it suffices to check the last condition only on one of the sets $V(f)$ or $V(g)$.

Proposition 4.4. *([8]) If $F = (f, g)$ and $\operatorname{Jac} F = 1$ then F is a polynomial automorphism of $\mathbb{C}^2$ if and only if $g|V(f)$ or $f|V(g)$ is proper (or equivalently $\mathcal{L}_\infty(g|V(f)) > 0$ or $\mathcal{L}_\infty(f|V(g)) > 0$).*

By Proposition 4.2 if F is a polynomial automorphism *of* $\mathbb{C}^n$ then $\mathcal{L}_\infty(F) > 0$. What positive rational number is it in this case? The answer was given by Płoski in [30].

Theorem 4.1. *If F is a polynomial automorphism of $\mathbb{C}^n$ then*

$$\mathcal{L}_\infty(F) = \frac{1}{\deg F^{-1}}.$$

In particular case $n = 2$ the above theorem can be considerable strengthen and inversed

Theorem 4.2. ([30])

1. If F is a polynomial automorphism of $\mathbb{C}^2$, then $\deg F^{-1} = \deg F$. Hence

$$\mathcal{L}_\infty(F) = \frac{1}{\deg F}.$$

2. An arbitrary (without assumption on the jacobian) polynomial mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ is an automorphism of $\mathbb{C}^2$ if and only if $\mathcal{L}_\infty(F) = \frac{1}{\deg F}$.

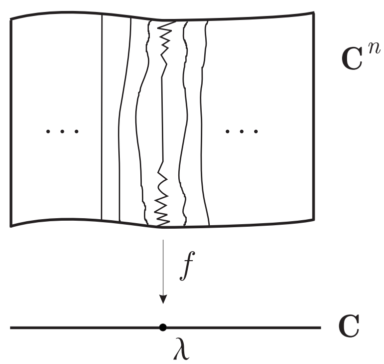
The Łojasiewicz exponent can also be used for characterization not only polynomial automorphisms but also coordinates of polynomial automorphisms. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial. The gradient $\nabla f := (\frac{\partial f}{\partial z_1}, ..., \frac{\partial f}{\partial z_n}) : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial mapping.

Theorem 4.3. *([6], [7]) A polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ is a coordinate of a polynomial automorphism of $\mathbb{C}^2$ (i.e. there exists a polynomial g such that (f, g) is a polynomial automorphism of $\mathbb{C}^2$) if and only if the gradient ∇f does not vanish and $\mathcal{L}_\infty(\nabla f) > -1$.*

Remark 3. The both conditions in the last theorem are effective and can be checked algorithmically.

5. The Łojasiewicz exponent at infinity of the gradient of a polynomial

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial and $\nabla f := (\frac{\partial f}{\partial z_1}, ..., \frac{\partial f}{\partial z_n}) : \mathbb{C}^n \to \mathbb{C}^n$ its gradient. Of course, the all theorems concerning the Łojasiewicz exponent at infinity, which were given in the previous sections, may be applied to ∇f. However, in this case it is very important to know the behaviour of ∇f at infinity on the fibres $f^{-1}(\lambda)$ and near these fibres. Namely, this behaviour is closely related to the notion of bifurcation points of f. By definition a point $\lambda \in \mathbb{C}$ is *a bifurcation point of f* if there exists no neighbourhood U of λ such that $f|f^{-1}(U) : f^{-1}(U) \to U$ is a trivial C^∞-bundle. It is known that the set of bifurcation points of f is finite [29], [40], [39], Cor. 1.2.13. It contains critical values of f i.e. the set $\{\lambda : \exists_{z\in\mathbb{C}^n}\ \lambda = f(z),\ \nabla f(z) = 0\}$ and, so called, critical values of f at infinity, which are defined similarly as bifurcation points except the condition that $f|f^{-1}(U)$ is a trivial C^∞-bundle outside a compact set in $\mathbb{C}^n$. Precisely, $\lambda \in \mathbb{C}$ is a *critical value of f at infinity* if there exists no neighbourhood U of λ and a compact set $K \subset \mathbb{C}^n$ such that $f|(f^{-1}(U)\setminus K) : f^{-1}(U)\setminus K \to U$ is a trivial C^∞-bundle. In the figure λ is a critical value of f at infinity.



We will denote by $\Lambda(f)$ the set of critical values of f at infinity. See [13] for other equivalent definitions of critical values of f at infinity in the case $n = 2$.

It turns out that the Łojasiewicz exponent at infinity characterizes critical value of f at infinity (completely in case $n = 2$). First we define the appropriate notion of the Łojasiewicz exponent at infinity near a fibre $f^{-1}(\lambda)$. Let $\lambda \in \mathbb{C}$ and K_δ be a disc of radius δ with the centre at λ. Then we define the notion of *the Łojasiewicz exponent at infinity near a fibre* $f^{-1}(\lambda)$ by

$$\mathcal{L}_{\infty,\lambda}(f) := \lim_{\delta\to 0^+} \mathcal{L}_\infty(\nabla f|f^{-1}(K_\delta)) \tag{5.1}$$

Notice that the function $(0, +\infty) \ni \delta \mapsto \mathcal{L}_\infty(\nabla f|f^{-1}(K_\delta))$ is decreasing, so the limit in (5.1) always exists. One can define $\mathcal{L}_{\infty,\lambda}(f)$ in an equivalent, more geometric way.

Proposition 5.1. ([11], Thm 5.1 - for $n = 2$, [35] - in general)

$$\mathcal{L}_{\infty,\lambda}(f) = \inf_{\Gamma\in\boldsymbol{\Theta}} \mathcal{L}_\infty(\nabla f|\Gamma)$$

where Θ is the set of all meromorphic curves approximating $f^{-1}(\lambda)$ (it means that

$$(f|\Gamma)(z) \underset{z\to\infty}{\longrightarrow} \lambda.$$

Then we have the following complete characterization of critical values of f at infinity in the case $n = 2$.

Theorem 5.1. ([41], Thm 1.3.2, [22], Section 3, [11], Cor. 3.5) *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a non-constant polynomial. A point $\lambda \in \mathbb{C}$ is a critical value of f at infinity if and only if $\mathcal{L}_{\infty,\lambda}(f) < -1$ (it is also equivalent to the condition $\mathcal{L}_{\infty,\lambda}(f) < 0$).*

In n-dimensional case we have only

Theorem 5.2. ([27], Thm 1.4) *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial.*

1. If $\lambda \in \mathbb{C}$ is a critical value of f at infinity then $\mathcal{L}_{\infty,\lambda}(f) \leqslant -1$.

2. If $\mathcal{L}_{\infty,\lambda}(f) \leqslant -1$ and additionally f has only isolated singularities at infinity then λ is a critical value of f at infinity.

One can no hope that the condition $\mathcal{L}_{\infty,\lambda}(f) \leqslant -1$ is equivalent for λ to be a critical value of f at infinity. In [28] there were studied polynomials

$$f_{n,q}(x, y, z) = x - 3x^{2n+1}y^{2q} + 2x^{3n+1}y^{3q} + yz, \quad n, q \in \mathbb{N}.$$

They are coordinates of polynomial automorphisms which implies that they have no bifurcation points. But we have

$$\mathcal{L}_\infty(\nabla f) = \mathcal{L}_{\infty,0}(f_{n,q}) = -\frac{n}{q}.$$

Let us pass to formulas and properties of $\mathcal{L}_{\infty,\lambda}(f)$. As in the above considerations the complete results are in two-dimensional case.

Assume that a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ is of the form

$$f(x, y) = y^n + a_1(x)y^{n-1} + ... + a_n(x), \quad \deg a_i \leqslant i, \ i = 1, ..., n, \ n \geqslant 1.$$

and consider the following resultant

$$\mathrm{Res}_y(f(x, y) - \lambda, f'_y(x, y) - u) = Q_0(\lambda, u)x^N + \cdots + Q_N(\lambda, u), \quad Q_0 \neq 0,$$

of $f(x, y) - \lambda$ and $f'_y(x, y) - u$ with respect to y where λ and u are new variables.

Theorem 5.3. ([11], Thms 4.1, 4.6 and 4.7) *For $\lambda_0 \in \mathbb{C}$*

$$\mathcal{L}_{\infty,\lambda_0}(f) = \begin{cases} \left[\max\limits_{1\leqslant i\leqslant N} \frac{\deg_u Q_i}{i}\right]^{-1} & \text{if} \quad Q_0(\lambda_0, 0) \neq 0 \quad \text{and} \ \deg_u Q_0 = 0 \\ 0 & \text{if} \quad Q_0(\lambda_0, 0) \neq 0 \quad \text{and} \ \deg_u Q_0 > 0 \\ -\left[\min\limits_{0\leqslant i\leqslant r} \frac{\mathrm{ord}_{(0,0)} Q_i}{r+1-i}\right]^{-1} & \text{if} \quad \begin{array}{l} Q_0(\lambda_0, 0) = ... = Q_r(\lambda_0, 0) = 0, \\ Q_{r+1}(\lambda_0, 0) \neq 0 \end{array} \\ -\infty & \text{if} \quad Q_0(\lambda_0, 0) = ... = Q_N(\lambda_0, 0) = 0 \end{cases}$$

The formulae for the Łojasiewicz exponent at infinity of ∇f on the fibers $f^{-1}(\lambda)$ are similar. For any $\lambda_0 \in \mathbb{C}$ consider the resultant

$$\mathrm{Res}_y(f(x, y) - \lambda_0, f'_y(x, y) - u) = R_0(u)x^M + \cdots + R_M(u), \quad R_0 \neq 0.$$

Then

Theorem 5.4. ([32], Prop. 2.4)

$$\mathcal{L}_\infty(\nabla f|f^{-1}(\lambda_0)) = \begin{cases} \left[\max\limits_{1\leqslant i\leqslant M} \frac{\deg R_i}{i}\right]^{-1} & if \quad R_0 = const. \\ 0 & if \quad R_0 \neq const.,\ R_0(0)\neq 0 \\ -\left[\min\limits_{0\leqslant i\leqslant r} \frac{\operatorname{ord}_0 R_i}{r+1-i}\right]^{-1} & if \quad \begin{array}{l} R_0(0)=\ldots=R_r(0)=0, \\ R_{r+1}(0)\neq 0 \end{array} \\ -\infty & if \quad R_0(0)=\ldots=R_M(0)=0 \end{cases}$$

The formulas given in the above theorems imply the generic properties of the functions

$$\begin{aligned} \mathbb{C}\ni\lambda &\mapsto \mathcal{L}_{\infty,\lambda}(f), \\ \mathbb{C}\ni\lambda &\mapsto \mathcal{L}_\infty(\nabla f|f^{-1}(\lambda)). \end{aligned}$$

Theorem 5.5. ([11], Thm 4.9) *The above functions are constants ($\geqslant 0$) and identical for $\lambda \notin \Lambda(f)$. For $\lambda \in \Lambda(f)$*

1. $\mathcal{L}_{\infty,\lambda}(f) = \mathcal{L}_\infty(\nabla f|f^{-1}(\lambda)) = -\infty$ *if the polynomial $f-\lambda$ has a multiple factors,*
2. $\mathcal{L}_{\infty,\lambda}(f) < \mathcal{L}_\infty(\nabla f|f^{-1}(\lambda)) - 1$ *otherwise.*

Now we give a theorem which relates the Łojasiewicz exponents at infinity near the fibers $\mathcal{L}_{\infty,\lambda}(f)$ to the global Łojasiewicz exponent $\mathcal{L}_\infty(\nabla f)$ of the gradient of f at infinity. Namely, they are closely connected if $\Lambda(f)\neq\emptyset$.

Theorem 5.6. ([41], Thm 1.4.3, [11], Cor. 3.6) *If $f:\mathbb{C}^2\to\mathbb{C}$ is a non-constant polynomial and $\Lambda(f)\neq\emptyset$ then*

$$\mathcal{L}_\infty(\nabla f) = \min_{\lambda\in\Lambda(f)} \mathcal{L}_{\infty,\lambda}(f).$$

In the considered case of the gradient of a polynomial there are known another formulae for $\mathcal{L}_\infty(\nabla f)$. In [3] the authors express $\mathcal{L}_\infty(\nabla f)$ in terms of the Eisenbud-Neumann diagrams of the curves $V(f-\lambda)$, $\lambda\in\mathbb{C}$, at infinity. By passing to projective space $\mathbb{P}^2$ one can express the Łojasiewicz exponent at infinity in some local invariants of the curves . Namely, in [32] and [15] the exponents $\mathcal{L}_\infty(\nabla f)$ and $\mathcal{L}_{\infty,\lambda}(f)$ are given in terms of polar invariants of the curves $V(f-\lambda)$ at the points of the line at infinity in $\mathbb{P}^2$. Formulas for $\mathcal{L}_\infty(\nabla f)$ under additional nondegenerate conditions of f at infinity in terms of the Newton diagrams at infinity one can find in [23], [1], [42].

Remark 4. Many of the above facts have local counterparts i.e. properties of the local Łojasiewicz exponent $\mathcal{L}_0(F)$ for holomorphic mappings with an isolated critical point at 0. In particular case of the gradient $\mathcal{L}_0(\nabla f)$ is a very important invariant of a singularity f.

Remark 5. There are also results on the Łojasiewicz exponent at infinity in real case i.e. for real polynomial mappings $F = (F_1, ..., F_m) : \mathbb{R}^n \to \mathbb{R}^m$ (see [14], [21], [36], [1]).

Remark 6. There are other characterizations of some characteristic points of a polynomial in terms of the Łojasiewicz exponent at infinity. One may consult [11], Section 6, [16], [39] for the Malgrange and Fedoryuk points of a polynomial. See also [26].

Remark 7. *Other types of the Łojasiewicz inequality at infinity for polynomial mappings and for the gradient of a polynomial one may find in* [38]*,* [16]*,* [33]*,* [17]*.*

6. Problems

Problem 6.1. *Find effective formulas for the Łojasiewicz exponent at infinity $\mathcal{L}_\infty(F)$ for polynomial mappings F in n-dimensional case.*

Problem 6.2. *Find effective formulas for the Łojasiewicz exponent at infinity $\mathcal{L}_\infty(\nabla f)$ of the gradient of a polynomial f in n-dimensional case.*

Problem 6.3. *Find effective formulas for the Łojasiewicz exponent at infinity of the gradient $\mathcal{L}_\infty(\nabla f)$ of a polynomial f non-degenerate at infinity in terms of its Newton polyhedron at infinity.*

Problem 6.4. *Characterize critical values at infinity of a polynomial f in terms of the Łojasiewicz exponent in n-dimensional case.*

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