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SOME FAMILIES OF POLYNOMIAL AUTOMORPHISMS II.

ERIC EDO

Abstract. We work with families of polynomial automorphisms of the complex affine plane whose generic length is 2. We obtain two new results which show some differences between the behavior of these families and the family of generic length 3 studied in [2].

1. Introduction

We first recall some notations from [2]. We denote by G the group of polynomial automorphisms of the complex plane $\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[X,Y])$. An element $\sigma \in G$ is defined by a pair of polynomials $(f,g) \in \mathbb{C}[X,Y]^2$ such that: $\mathbb{C}[f,g] = \mathbb{C}[X,Y]$ and we set $\sigma = (f,g)$. We define the degree of $\sigma \in G$ by: $\deg(\sigma) = \max\{\deg(f), \deg(g)\}$. We denote by A the subgroup of *affine* automorphisms (of degree 1) and by B the subgroup of *triangular* automorphisms (of the form $(aX + P(Y), bY + c)$ with $a, b \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $P \in \mathbb{C}[Y]$). By the Jung-van der Kulk theorem (cf. [5] and [6]), G is the amalgamated product of A and B along $A \cap B$. This property allows us to define the *multidegree* of $\sigma \in G$ as the sequence of the degrees of the triangular automorphism in a decomposition of σ as a product of affine and triangular automorphisms (cf. [3]). We denote by G_d the set of all automorphisms of G whose multidegree is $d = (d_l, \ldots, d_1)$ where $d_1, \ldots, d_l \geqslant 2$ are integers. For example, for all $d_1, d_2 \geqslant 2$, we have

$$G_{(d_2,d_1)} = \{a_3 b_2 a_2 b_1 a_1\,;\ a_1, a_3 \in A\,,\ a_2 \in A \smallsetminus B\,,\ b_1, b_2 \in B\,,\ \deg(b_i) = d_i\}$$

We denote by $G(\mathbb{C}[Z])$ the group of polynomial $\mathbb{C}[Z]$-automorphisms of $\mathbb{A}^2_{\mathbb{C}[Z]} =$ Spec $\mathbb{C}[Z][X,Y]$. For $\sigma \in G(\mathbb{C}[Z])$ and $z \in \mathbb{C}$, we denote by $\sigma_{Z\to z} \in G$ the automorphism obtained by substituting $Z = z$ in σ. For almost all values $z \in \mathbb{C}$ the multidegree of $\sigma_{Z\to z}$ equals the multidegree of σ considered as a $\mathbb{C}(Z)$-automorphisms of $\mathbb{A}^2_{\mathbb{C}(Z)} =$ Spec $\mathbb{C}(Z)[X,Y]$. We call this multidegree the *generic multidegree* of σ.

The group G can be endowed with the structure of an infinite-dimensional algebraic variety (cf. [11], see also [7], [8] and [9] for a recent development of the theory of ind-affine varieties). If $H \subset G$, we denote by $\overline{H}$ the closure of H in G for the Zariski topology associated with this structure. The most important

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thing to remember about this topology is the following: For all $\sigma \in G(\mathbb{C}[Z])$, the map $z \mapsto \sigma_{Z\to z}$ is a continuous map from $\mathbb{C}$ to G. For example, the set $\overline{G_{(d_2,d_1)}}$ contains all elements $\sigma_{Z\to 0}$ where the generic multidegree of $\sigma \in G(\mathbb{C}[Z])$ is (d_2, d_1).

This is the main result of [2] (see [2] Theorem 1.2 and his corollary):

Theorem 1.1. *Let $a, b \geqslant 2$ and $c \geqslant 1$ be integers. We have*

$$G_{(a+c(ab-1))} \cap \overline{G_{(a+(c-1)(ab-1),b,a)}} \neq \emptyset,$$

but if $(a, b) \neq (2, 2)$, then

$$G_{(a+c(ab-1))} \not\subset \overline{G_{(a+(c-1)(ab-1),b,a)}}.$$

Examining at the proof of Theorem 1.1 in [2], we can set $b = 1$. With $d = a-1$ and $e = cd$ we obtain

Theorem 1.2. *Let $d, e \geqslant 1$ be integers. If $e \in d\,\mathbb{N}$ then*

$$G_{(d+e+1)} \cap \overline{G_{(e+1,d+1)}} \neq \emptyset.$$

As an example (for $d = e = 1$), we can consider the following automorphism $N \in G(\mathbb{C}[Z])$:

$$N := (X - 2Y(XZ + Y^2) - Z(XZ + Y^2)^2, Y + Z(XZ + Y^2))$$

This is the famous Nagata's automorphism. For all $z \in \mathbb{C}^*$, $N_{Z\to z}$ has multidegree $(2, 2)$ (this is the generic multidegree of N) but $N_{Z\to 0}$ is a triangular automorphism of degree 3. Hence Nagata's automorphism gives the following: $(X - ZY^3, Y) \in G_{(3)} \cap \overline{G_{(2,2)}} \neq \emptyset$ (cf.[2] §4.3). The main ingredient of the proof of Theorem 1.2, is the construction of automorphisms in $G(\mathbb{C}[Z])$ generalizing Nagata's one.

In this paper we improve Theorem 1.2 in the following two ways (Theorem 1.3 and Theorem 1.4):

Theorem 1.3. *Let $d, e \geqslant 1$ be integers. If $e \in d\,\mathbb{N}$ then*

$$G_{(d+e+1)} \subset \overline{G_{(e+1,d+1)}}.$$

Remark 1. If $e \in d\,\mathbb{N}$, for each triangular automorphism τ of degree $d + e + 1$, we construct effectively an automorphism $\sigma \in G(\mathbb{C}[Z])$ with generic multidegree $(e + 1, d + 1)$ and such that $\sigma_{Z\to 0} = \tau$. There are two steps. First, we construct an automorphism $\sigma' \in G(\mathbb{C}[Z])$ such that the d largest coefficients of τ and $\sigma'_{Z\to 0}$ are equal (see Lemma 2.3). Second, we adjust the $e + 1$ low coefficients by multiplication by a triangular automorphism of degree $e + 1$.

Remark 2. As a particular case (when $d = 1$), for all $e \geqslant 1$, Theorem 1.3 gives $G_{(e+2)} \subset \overline{G_{(e+1,2)}}$.

Remark 3. Theorem 1.1 gives $G_{(7)} \cap \overline{G_{(2,3,2)}} \neq \emptyset$, but $G_{(7)} \not\subset \overline{G_{(2,3,2)}}$. We have no example of such a phenomenon in the length 2 case. Theorem 1.3 is very important because it is a particular case of the following conjecture.

Conjecture 1.1. *For all integers $d, e \geqslant 1$, we have $G_{(d+e+1)} \subset \overline{G_{(e+1,d+1)}}$.*

The aim is to completely describe the closure of the set of automorphisms of fixed multidegree and, in the length 2 case, to prove

Conjecture 1.2. *For all integers $d, e \geqslant 1$, we have*

$$\overline{G_{(e+1,d+1)}} = \bigcup_{e' \leqslant e\,,\, d' \leqslant d} G_{(e'+1,d'+1)} \cup \bigcup_{c \leqslant d+e+1} G_{(c+1)}.$$

Remark 4. One can easily show that $\overline{G_{(e+1,d+1)}} \supset \bigcup_{e' \leqslant e\,,\, d' \leqslant d} G_{(e'+1,d'+1)}$. Conjecture 1.2 is very interesting because the two parts of the inclusions are difficult to prove. In this paper we only study the $\supset$ part. Theorem 1.3 implies the inclusion $\supset$ under the assumption $e \in d\,\mathbb{N}$. The other inclusion is already not easy for $d = e = 1$. For example we have to prove that $\overline{G_{(2,2)}} \cap G_{(4)} = \emptyset$.

In the following theorem we try to understand what happens when the assumption $e \in d\,\mathbb{N}$ is not satisfied.

Theorem 1.4. *Let $d, e \geqslant 1$ be integers, if d is even and $e \in \frac{d}{2}\mathbb{N}$ then*

$$G_{(d+e+1)} \cap \overline{G_{(e+1,d+1)}} \neq \emptyset.$$

Remark 5. In the proof of Theorem 1.4, an unexpected link appears between this problem and the theory of hypergeometric functions.

Remark 6. As a particular case (when $d = 2$), for all $e \geqslant 1$, Theorem 1.44 gives: $G_{(e+3)} \cap \overline{G_{(e+1,3)}} \neq \emptyset$.

Remark 7. Conjecture 1.1 seems to be difficult as soon as d and e are "big" with $\gcd(d, e) = 1$. The next step will be to prove that $G_{(8)} \cap \overline{G_{(5,4)}} \neq \emptyset$.

2. Proof of Theorem 1.3

We are beginning with the following lemma.

Lemma 2.1 (Computing coefficients)**.** *Let R be a $\mathbb{Q}$-algebra, let $m \geqslant 0$ be an integer and let $P(Y) \in R[Y]$ be a polynomial of degree m ($\deg(P) = m$). Let $S(T) = \sum_{n=0}^{\infty} s_n(Y)\,T^n \in R[Y][[T]]$ be a formal series with polynomial coefficients. We set:*

$$\begin{cases} Q_1(Y,Z) & := & Y + ZY^2P(Y) & \in R[Y,Z] \\ Q_2(Y,Z) & := & Y + ZY^2S(ZY) & \in R[Y][[Z]]. \end{cases}$$

The following assumptions are equivalent:

i) $Q_2(Q_1(Y,Z),Z) = Y$ *in* $R[Y][[Z]]$, i)' $Q_1(Q_2(Y,Z),Z)=Y$ *in* $R[Y][[Z]]$,

ii) $P(Y)+(1+TP(Y))^2S(T(1+TP(Y)))=0$ *in* $R[Y][[T]]$,

ii)' $S(T)+(1+TS(T))^2P(Y(1+TS(T)))=0$ *in* $R[Y][[T]]$,

iii) *we have* $s_0(Y)=-P(Y)$ *and for all* $n\geqslant 1$

$$s_n(Y)=\frac{(-1)^{n+1}}{n}\sum_{j=0}^{m(n+1)}\binom{j+2n+1}{n-1}p_{n,j}\,Y^j$$

where for all $j\in\{0,\dots,m(n+1)\}$ *the coefficients* $p_{n,j}$ *are defined by*

$$(*)\ (2P(Y)+YP'(Y))\,P(Y)^n=\sum_{j=0}^{m(n+1)}p_{n,j}\,Y^j.$$

Remark 8. With the notations of iii) in Lemma 2.1, we have

$$s_1(Y)=(2P(Y)+YP'(Y))\,P(Y).$$

Proof of Lemma 2.1. The equivalence i) $\iff$ i)' is clear. We set $T=ZY$. Easy computations give

$$Q_2(Q_1(Y,Z),Z)=Y+ZY^2\{P(Y)+(1+TP(Y))^2S(T(1+TP(Y)))\}$$

and

$$Q_1(Q_2(Y,Z),Z)=Y+ZY^2\{S(T)+(1+TS(T))^2P(Y(1+TS(T)))\}.$$

We deduce i) $\iff$ ii) and i)' $\iff$ ii)'.

Let us assume ii)'. Setting $T=0$ in ii)', we obtain: $s_0(Y)=-P(Y)$. Let $n\geqslant 1$ be an integer. We set:

$$P_1(Y):=(2P(Y)+YP'(Y))\,P(Y)^n=\sum_{j=0}^{m(n+1)}p_{n,j}\,Y^j.$$

According to Cauchy's formula, we have (the path of integration is any little circle around 0)

$$s_n(Y)=\frac{1}{2\pi i}\oint\frac{S(T)}{T^{n+1}}dT=-\frac{1}{2\pi i}\oint\frac{(1+TS(T))^2P(Y(1+TS(T)))}{T^{n+1}}dT$$

We change from the variable T to $W=TS(T)$ with

$$T=v(W)=-\frac{W}{(1+W)^2P(Y(1+W))}.$$

We obtain

$$\begin{aligned}
s_n(Y) =& -\frac{1}{2\pi i}\oint (1+W)^2 P(Y(1+W))\,\frac{v'(W)}{v(W)^{n+1}}\,dW \\
=&\frac{(-1)^{n+1}}{2\pi i n}\oint (1+W)^{2n+1}\{2P(Y(1+W)) \\
&+ Y(1+W)P'(Y(1+W))\}P(Y(1+W))^n\frac{dW}{W^n} \\
=&\frac{(-1)^{n+1}}{2\pi i n}\oint (1+W)^{2n+1}P_1(Y(1+W))\frac{dW}{W^n} \\
=&\frac{(-1)^{n+1}}{2\pi i n}\oint \sum_{j=0}^{m(n+1)} p_{n,j}\,Y^j(1+W)^{j+2n+1}\frac{dW}{W^n} \\
=&\frac{(-1)^{n+1}}{2\pi i n}\sum_{j=0}^{m(n+1)} p_{n,j}\,Y^j\oint (1+W)^{j+2n+1}\frac{dW}{W^n} \\
=&\frac{(-1)^{n+1}}{n}\sum_{j=0}^{m(n+1)} \binom{j+2n+1}{n-1} p_{n,j}\,Y^j.
\end{aligned}$$

By uniqueness of the coefficients $s_n(Y)$ this computation proves ii)' $\Longleftrightarrow$ iii). □

Remark 9. One can use a purely algebraic version of Cauchy's formula (Lagrange's formula: see Corollary 5.4.3. p. 42 in [12]) in the proof of Lemma 2.1. We use complex analysis only for the comfort of the reader.

Example 2.1 (Lemma 2.1)**.** We set $R = \mathbb{Q}$, $m = 2$ and $P(Y) = 25Y^2 + 10Y - 4$. We have

$$s_0(Y) = -P(Y) = -25Y^2 - 10Y + 4,$$

$$s_1(Y) = (2P(Y) + YP'(Y))P(Y) = 2500Y^4 + 1750Y^3 - 300Y^2 - 200Y + 32,$$

$$(2P(Y) + YP'(Y))P(Y)^2 = 2(31250Y^6 + 34375Y^5 - 300Y^2 - 80Y + 16),$$

hence

$$-s_2(Y) = 31250 \times 11Y^6 + 34375 \times 10Y^5 - 300 \times 7Y^2 - 80 \times 6Y + 16 \times 5).$$

Lemma 2.2 (A triangular automorphism)**.** *Let $m \geqslant 0$ be an integer and let $a \in \mathbb{C}^*$ be a nonzero complex number. We consider the $\mathbb{Q}$-algebra of polynomials in m variables: $R = \mathbb{C}[X_1, \ldots, X_m]$. We set*

$$P(Y) = aY^m + \sum_{i=1}^{m} X_i\,Y^{i-1} \in R[Y] \quad (\deg_Y(P) = m).$$

Let $n \geqslant 1$ be an integer. For all $j \in \{0, \ldots, m(n+1)\}$, we denote by $p_{n,j}$ the coefficients associated to the polynomial P by the formula $(*)$ in Lemma 2.1.

Then $\tau = (p_{n,mn}, p_{n,mn+1}, \ldots, p_{n,mn+m-1}) \in R^m$ *is a* $\mathbb{C}$*-automorphism of* R *and* $p_{n,m(n+1)} = (m+2)\, a^{n+1}$.

Proof of Lemma 2.2. We show that τ is a triangular $\mathbb{C}$-automorphism. Let $k \in \{1, \ldots, m\}$. Modulo $\mathbb{C}[X_{k+1}, \ldots, X_m]$ we have

$$\begin{aligned}&(2P(Y) + YP'(Y))P(Y)^n\\ &= \left((m+2)aY^m + \sum_{i=1}^{k}(i+1)X_i\, Y^{i-1}\right)\left(aY^m + \sum_{i=1}^{k} X_i\, Y^{i-1}\right)^n\\ &=(m+2)\, a^{n+1}\, Y^{m(n+1)} + ((m+2)n + k + 1)a^n X_k Y^{mn+k-1} + L_n(Y).\end{aligned}$$

where $L_n(Y) \in R[Y]$ and $\deg_Y(L_n) < mn + k - 1$.
Hence $p_{n,mn+k-1} = ((m+2)n + k + 1)\, a^n\, X_k$ modulo $\mathbb{C}[X_{k+1}, \ldots, X_m]$ and the coefficient $((m+2)n + k + 1)\, a^n$ is nonzero. This shows that τ is a triangular $\mathbb{C}$-automorphism of R and (when $k = m$) $p_{n,m(n+1)} = (m+2)\, a^{n+1}$. □

Example 2.2 (Lemma 2.2)**.** Let $a \in \mathbb{C}^*$ be a nonzero complex number. We set $m = 2$ and $n = 2$. Let $\tau = (p_{2,4}, p_{2,5})$ be the triangular automorphism defined in Lemma 2.2. We have $R = \mathbb{C}[X_1, X_2]$ and $P(Y) = aY^2 + X_2Y + X_1 \in R[Y]$. From (Formula $(*)$ in Lemma 2.1)

$$(2P(Y) + YP'(Y))P(Y)^2 = \sum_{j=0}^{6} p_{2,j} Y^j$$

we get

$$\tau = (10a^2X_1 + 10aX_2^2\,,\, 11a^2X_2)$$

and

$$\tau^{-1} = ((10a^2)^{-1}X_1 - (121a^5)^{-1}X_2^2\,,\, (11a^2)^{-1}X_2).$$

Lemma 2.3 (The incomplete triangular automorphism)**.** *Let* $m \geqslant 0$ *and let* $n \geqslant 1$ *be integers. Let* $x_0 \in \mathbb{C}^*$ *be a nonzero complex number and let* $x_1, \ldots, x_m \in \mathbb{C}$ *be* m *complex numbers. There exist* $(m+1)n+2$ *complex numbers* $x_{m+1}, \ldots,$ $x_{(m+1)(n+1)+1} \in \mathbb{C}$ *such that*

$$(X + \sum_{k=0}^{(m+1)(n+1)+1} x_k\, Y^{(m+1)(n+1)+1-k}, Y) \in \overline{G_{((m+1)n+1,m+2)}}.$$

Proof of Lemma 2.3. We set

$$c_0 := \left\{\frac{(-1)^n}{n}\begin{pmatrix} m(n+1) + 2n + 1\\ n-1\end{pmatrix}\right\}^{-1}.$$

Let $a \in \mathbb{C}^*$ be a $(n+1)$st root of $\frac{c_0}{m+2}$ (*i.e.* a is such that $(m+2)a^{n+1} = c_0$). For all $k \in \{1, \ldots, m\}$, we set

$$c_k := \left\{\frac{(-1)^n}{n}\begin{pmatrix} m(n+1) + 2n + 1 - k\\ n-1\end{pmatrix}\right\}^{-1} x_k.$$

We denote by $\tau = (p_{n,mn}, p_{n,mn+1}, \dots, p_{n,mn+m-1})$ the triangular automorphism of $\mathbb{C}[X_1, \dots, X_m]$ defined in Lemma 2.2 and we set

$$(a_1, \dots, a_m) := \tau^{-1}(c_m, \dots, c_1) \in \mathbb{C}^m,$$

(i.e we have $c_k = p_{n,m(n+1)-k}(a_1, \dots, a_m)$ for all $k \in \{1, \dots, m\}$).
Consider

$$P(Y) = aY^m + \sum_{i=1}^{m} a_i\, Y^{i-1} \in \mathbb{C}[Y] \quad (\deg(P) = m).$$

We define $S(T) := \sum_{j=0}^{\infty} s_j(Y)T^j \in \mathbb{C}[Y][[T]]$ by assumptions of Lemma 2.1 ($R = \mathbb{C}$). We set $Q_1(Y,Z) = Y + ZY^2P(Y) \in \mathbb{C}[Z][Y]$, $S_{\leqslant n-1}(T) = \sum_{j=0}^{n-1} s_j(Y)T^j$ and $Q_2(Y,Z) := Y + ZY^2 S_{\leqslant n-1}(YZ) \in \mathbb{C}[Z][Y]$. We consider the following two polynomials:

$$\begin{cases} g(X,Y) & := & Z^{n+1}X + Q_1(Y,Z) & \in \mathbb{C}[Z][X,Y] \\ f(X,Y) & := & Z^{-n-1}(Q_2(g(X,Y),Z) - Y) & \in \mathbb{C}(Z)[X,Y] \end{cases}$$

By i) of Lemma 2.1, we have $Q_2(g(X,Y),Z) = Q_2(Q_1(Y,Z),Z) = Y$ modulo Z^{n+1}, hence $f(X,Y) \in \mathbb{C}[Z][X,Y]$. Since $\mathrm{Jac}(f,g) = 1$ (the determinant of the jacobian matrix), we have $\sigma := (f,g) \in G(\mathbb{C}[Z])$ (cf. Lemma 1.1.8, p. 5 in [1] or Proposition 4.4 in [2]). In $G(\mathbb{C}(Z))$, the automorphism σ splits into $\sigma = b_2\pi b_1$ with $b_1 = (Z^{n+1}X + Q_1(Y,Z)\,,\, Y)$, $b_2 = (Z^{-n-1}(Q_2(Y,Z) - X)\,,\, Y)$ and $\pi = (Y,X)$. The generic multidegree of σ is $(\deg(b_2), \deg(b_1))$ equal to

$$(\deg_Y(Q_2)\,,\, \deg_Y(Q_1)) = (2{+}n{-}1{+}\deg(s_{n-1})\,,\, 2{+}\deg(P)) = ((m{+}1)n{+}1, m{+}2).$$

We have $\sigma_{Z\to 0} \in \overline{G_{((m+1)n+1,m+2)}}$ and the second component of $\sigma_{Z\to 0}$ is $Q_1(Y,0) = Y$. We prove that for all $k \in \{0, \dots, m\}$ the coefficient of $Y^{(m+1)(n+1)+1-k}$ in the first component of $\sigma_{Z\to 0}$ is x_k.

Modulo T^{n+1}, we have $S(T) = S_{\leqslant n-1}(T) + s_n(Y)T^n$. Then, by ii) of Lemma 2.1, we deduce

$$0 = P(Y) + (1+TP(Y))^2\{S_{\leqslant n-1}(T(1+TP(Y)) + s_n(Y)T^n(1+TP(Y))^n\}.$$

Hence (again modulo T^{n+1}), we have

$$P(Y) + (1+TP(Y))^2 S_{\leqslant n-1}(T(1+TP(Y)) = -s_n(Y)T^n$$

On the other hand,

$$Q_2(Q_1(Y,Z),Z) = Y + ZY^2\{P(Y) + (1+TP(Y))^2 S_{\leqslant n-1}(T(1+TP(Y)))\}.$$

Hence, modulo Z^{n+2}, we have (with $T = ZY$)

$$\begin{aligned} Q_2(Q_1(Y,Z),Z) &= Y - Z^{n+1}Y^{n+2}s_n(Y) \\ &= Y + \frac{(-1)^n}{n} Z^{n+1} \sum_{j=0}^{m(n+1)} \binom{j+2n+1}{n-1} p_{n,j} Y^{n+2+j}. \end{aligned}$$

For all $k \in \{1, \ldots, m\}$, the coefficient of $Y^{(m+1)(n+1)+1-k}$ of the first component of $\sigma_{Z\to 0}$ is, (where $j = m(n+1) - k$),

$$\begin{aligned}
&\frac{(-1)^n}{n}\begin{pmatrix} m(n+1)+2n+1-k \\ n-1 \end{pmatrix} p_{n,m(n+1)-k}(a_1, \ldots, a_m) \\
&= \frac{(-1)^n}{n}\begin{pmatrix} m(n+1)+2n+1-k \\ n-1 \end{pmatrix} c_k \\
&= x_k.
\end{aligned}$$

The coefficient of $Y^{(m+1)(n+1)+1}$ of the first component of $\sigma_{Z\to 0}$ is (cf. Lemma 2.2)

$$\frac{(-1)^n}{n}\begin{pmatrix} m(n+1)+2n+1 \\ n-1 \end{pmatrix}(m+2)a^{n+1} = \frac{(-1)^n}{n}\begin{pmatrix} m(n+1)+2n+1 \\ n-1 \end{pmatrix} c_0 = x_0.$$

□

Example 2.3 (Lemma 2.3)**.** We set $m = 2$, $n = 2$, $x_0 = 1$, $x_1 = 1$ and $x_2 = 0$. We compute $x_3, \ldots, x_{10}$ such that

$$(X + \sum_{k=0}^{10} x_k Y^{10-k}, Y) \in G_{(7,4)}.$$

We have $c_0 = \dfrac{2}{11}$, $c_1 = \dfrac{1}{5}$, $c_2 = 0$ and $a = \dfrac{1}{\sqrt[3]{22}}$.
We have $(a_1, a_2) = \tau^{-1}(c_2, c_1) = \tau^{-1}(0, \frac{1}{5})$, where (cf. Example of Lemma 2.2)

$$\tau^{-1} = ((10a^2)^{-1}X_1 - (121a^5)^{-1}X_2^2\,,\,(11a^2)^{-1}X_2).$$

We obtain $a_1 = -\dfrac{4}{25}\,a$ and $a_2 = \dfrac{2}{5}\,a$.
We have
$P(Y) = aY^2 + a_2Y + a_1 = \dfrac{a}{25}\,(25Y^2 + 10Y - 4)$,
$-s_2(Y) = (\frac{a}{25})^3(31250 \times 11Y^6 + 34375 \times 10Y^5 - 300 \times 7Y^2 - 80 \times 6Y + 16 \times 5)$
$\qquad = Y^6 + Y^5 - \dfrac{42}{6875}Y^2 - \dfrac{48}{34375}Y + \dfrac{8}{34375}$.
Finally,

$$(X + Y^4(Y^6 + Y^5 - \frac{42}{6875}Y^2 - 4834375Y + \frac{8}{34375})\,,\,Y) \in G_{(7,4)},$$

in other words, $x_3 = 0$, $x_4 = -\dfrac{42}{6875}$, $x_5 = -\dfrac{48}{34375}$, $x_6 = 834375$ and $x_7 = x_8 = x_9 = x_{10} = 0$.

Proof of Theorem 1.3. Let $d, e \geqslant 1$ be integers such that $e \in d\,\mathbb{N}$. We set $m = d - 1 \geqslant 0$ and $n = ed \geqslant 1$ (m and n are integers). Let $\gamma \in G_{(d+e+1)}$. There exist affine automorphisms $\alpha_1, \alpha_2 \in A$ and a polynomial $b \in \mathbb{C}[Y]$ such that $\gamma = \alpha_2\beta\alpha_1$, where $\beta = (X + b(Y), Y) \in B$ is a triangular automorphism of degree $\deg(b) = d{+}e{+}1 = (m{+}1)(n{+}1){+}1$. By Lemma 2.3, there exist a triangular automorphism $\beta_1 \in B$ of degree $(m+1)n+1$ and an automorphism $\sigma \in G(\mathbb{C}[Z])$ of generic multidegree $((m+1)n+1, m+2)$ such that $\beta = \beta_1 \sigma_{Z\to 0}$. More precisely (see the proof of Lemma 2.3), there exist b_1, b_2 two triangular automorphisms of $G(\mathbb{C}(Z))$ such that: $\sigma = b_2 \pi b_1$ and $(\deg(b_2), \deg(b_1)) = ((m+1)n+1, m+2)$. We consider the automorphism $\sigma' := \alpha_2 \beta_1 b_2 \pi b_1 \alpha_1 \in G(\mathbb{C}[Z])$. Since $\deg(\beta_1 b_2) = (m+1)n+1$ and $\deg(b_1) = m+2$, the generic multidegree of σ' is $(n(m+1)+1, m+2)$. Hence,

$$\gamma = \alpha_2 \beta \alpha_1 = \alpha_2 \beta_1 \sigma_{Z\to 0} \alpha_1 = \sigma'_{Z\to 0} \in \overline{G_{(n(m+1)+1,m+2)}} = \overline{G_{(e+1,d+1)}}.$$

□

3. Proof of Theorem 1.4

Theorem 1.4. *Let $d, e \geqslant 1$ be integers, if d is even and $e \in \frac{d}{2}\mathbb{N}$ then:*

$$G_{(d+e+1)} \cap \overline{G_{(e+1,d+1)}} \neq \emptyset.$$

Lemma 3.1 (Computing coefficients)**.** *Let $m \geqslant 1$ be an integer. Let*

$$S(T) = \sum_{n=0}^{\infty} P_{m,n}(u) T^n \in \mathbb{C}[u][[T]] \quad (*)$$

be a formal series with polynomial coefficients. We set

$$\begin{cases} Q_1(Y,Z) := Y(1 - ZY^m - u(ZY^m)^2) & \in \mathbb{C}[u][Y,Z] \\ Q_2(Y,Z) := YS(ZY^m) & \in \mathbb{C}[u][Y][[Z]]. \end{cases}$$

The following assumptions are equivalent:

i) $Q_2(Q_1(Y,Z),Z) = Y$ *in* $\mathbb{C}[u][Y][[Z]]$,

i)' $Q_1(Q_2(Y,Z),Z) = Y$ *in* $\mathbb{C}[u][Y][[Z]]$,

ii) $(1 - T - uT^2)S(T(1 - T - uT^2)^m)) = 1$,

ii)' $S(T)(1 - TS(T)^m - u(TS(T)^m)^2) = 1$,

iii) *For all $n \in \mathbb{N}$, we have*

$$(*)\ P_{m,n}(u) = \frac{1}{(mn+1)!} \sum_{l=0}^{[\frac{n}{2}]} \frac{((m+1)\,n - l)!}{l!(n-2l)!}\, u^l$$

Proof. The equivalence i) $\iff$ i)' is clear. We set $T = ZY^m$. An easy computation gives

$$Q_2(Q_1(Y,Z),Z) = Y(1 - T - uT^2)S(T(1 - T - uT^2)^m)$$

and

$$Q_1(Q_2(Y,Z),Z) = YS(T)(1 - TS(T)^m - u(TS(T)^m)^2).$$

We deduce i) $\iff$ ii) and i)' $\iff$ ii)'.

Let us assume ii)'. Setting $T=0$ in ii)', we obtain $P_{m,0}(u)=1$. Let $n \geqslant 1$ be an integer. According to Cauchy's formula, we have (the path of integration is any little circle around 0)

$$P_{m,n}(u)=\frac{1}{2\pi i}\oint \frac{S(T)}{T^{n+1}}dT=\frac{1}{2\pi i}\oint \frac{1}{1-TS(T)^m-u(TS(T)^m)^2}\frac{dT}{T^{n+1}}$$

We change from the variable T to $W=TS(T)^m$ with

$$T=v(W)=W(1-W-uW^2)^m.$$

We obtain

$$P_{m,n}(u)=\frac{1}{2\pi i}\oint \frac{1}{1-W-uW^2}\frac{v'(W)}{v(W)^{n+1}}dW$$

Integrating by parts (4 minus signs):

$$nP_{m,n}(u)=\frac{1}{2\pi i}\oint \frac{1+2uW}{(1-W-uW^2)^2}\frac{dW}{v(W)^n}=\frac{1}{2\pi i}\oint \frac{W+2uW^2}{(1-W-uW^2)^{mn+2}}\frac{dW}{W^{n+1}}$$

Hence $nP_{m,n}(u)$ is the coefficient of W^n in $\dfrac{W+2uW^2}{(1-W-uW^2)^{mn+2}} \in \mathbb{C}[u][[W]]$.

We set $V=W+uW^2$. Then, we have $V^k=\sum_{l=0}^k \begin{pmatrix} k \\ l \end{pmatrix} u^l W^{l+k}$ and

$$\begin{aligned}
&\frac{W+2uW^2}{(1-W-uW^2)^{mn+2}}\\
=&\frac{2V-W}{(1-V)^{mn+2}}\\
=&\sum_{k=0}^{\infty}\begin{pmatrix} mn+k+1 \\ mn+1 \end{pmatrix}(2V-W)X^k\\
=&\sum_{k=1}^{\infty}\begin{pmatrix} mn+k \\ mn+1 \end{pmatrix}(2V^k-WV^{k-1}) \qquad (k\leftarrow k-1)\\
=&\sum_{k=1}^{\infty}\begin{pmatrix} mn+k \\ mn+1 \end{pmatrix}\left(\sum_{l=0}^{k-1}\left(2\begin{pmatrix} k \\ l \end{pmatrix}-\begin{pmatrix} k-1 \\ l \end{pmatrix}\right)u^lW^{l+k}+2u^kW^{2l}\right)\\
=&\sum_{k=1}^{\infty}\sum_{l=0}^{k}\frac{(mn+k)!}{(mn+1)!l!(k-l)!}(l+k)u^lW^{l+k} \qquad \left(2\begin{pmatrix} k \\ l \end{pmatrix}-\begin{pmatrix} k-1 \\ l \end{pmatrix}\right)\\
=&\frac{(k-1)!}{l!(k-l)!}(l+k).
\end{aligned}$$

When $l+k=n$, we have $k=n-l$ and

$$0\leqslant l\leqslant k \iff 0\leqslant 2l\leqslant n \iff 0\leqslant l\leqslant [n2].$$

We deduce the following formula:

$$nP_{m,n}(u) = \sum_{l=0}^{[\frac{n}{2}]} \frac{(mn+n-l)!}{(mn+1)!l!(n-2l)!} nu^l$$

and we obtain $(*)$ simplifying by n. □

Lemma 3.2 (Gauss hypergeometric polynomials)**.** *Let $m, n \geqslant 1$ be integers. Let $P_{m,n}$ be the polynomial defined by $(*)$ in Lemma 3.1. We have*

1) $\deg(P_{m,n}) = [\frac{n}{2}]$ *and* $P_{m,n}(t) \neq 0$ *for all* $t > 0$*,*

2) $P_{m,n}$ *has only simple roots.*

3) *There exists a complex number $u_{m,n} \in \mathbb{C}$ such that $P_{m,n+1}(u_{m,n}) = 0$ and $P_{m,n+2}(u_{m,n}) \neq 0$.*

Proof of Lemma 3.2. Part 1) of Lemma 3.2 is clear from the definition of $P_{m,n}$. To prove part 2), let us recall some basic definitions and formulas (see [4] or [13]). Let $a, b, c, z \in \mathbb{C}$ be complex numbers, the classical Gauss hypergeometric series is defined by

$${}_2F_1(a,b,c|z) = \sum_{k \in \mathbb{N}} \frac{(a)_l(b)_l}{(c)_l} \frac{z^l}{l!}$$

where, for a complex number $a \in \mathbb{C}$ and an integer $l \in \mathbb{N}$, the *Pochhammer symbol* $(a)_l$ is defined by $(a)_0 = 1$ and $(a)_l = a(a+1)\dots(a+l-1)$. □

The *Gauss hypergeometric function* ${}_2F_1(a,b,c|z)$ is well defined as soon as $c \in -\mathbb{N}$ implies $((a \in -\mathbb{N}$ and $c < a)$ or $(b \in -\mathbb{N}$ and $c < b))$.

When $a \in -\mathbb{N}$ or $b \in -\mathbb{N}$, ${}_2F_1(a,b,c|z)$ is a polynomial (in z), in other cases the radius of convergence is 1.

When the Gauss hypergeometric function is defined, it satisfies the following general linear differential equation:

$$\text{(GDE)} \quad z(z-1)F''(z) + ((a+b+1)z - c)F'(z) + abF(z) = 0.$$

When ${}_2F_1(a,b,c|z)$ is a polynomial (i.e. when $a \in -\mathbb{N}$ or $b \in -\mathbb{N}$), the value for $z = 1$ can be computed explicitly:

$$\text{(GSV)} \qquad {}_2F_1(a,b,c|1) = \begin{cases} \frac{(c-b)_{-a}}{(c)_{-a}} \text{ if } a \in -\mathbb{N} \\ \frac{(c-a)_{-b}}{(c)_{-b}} \text{ if } b \in -\mathbb{N}. \end{cases}$$

Let $m, n \geqslant 0$ be integers. The polynomial $P_{m,n}$ defined by $(*)$ in Lemma 4.1 is a Gauss hypergeometric polynomial:

$$P_{m,n}(u) = \frac{((m+1)n)!}{n!(n+1)!} {}_2F_1(-\frac{n}{2}, -\frac{n-1}{2}, -(m+1)n| - 4u).$$

In fact

$$
\begin{aligned}
&{}_2F_1(-\frac{n}{2},-\frac{n-1}{2},-(m+1)n|-4u)\\
=&\sum_{l=0}^{[\frac{n}{2}]}\frac{(-\frac{n}{2})_l(-\frac{n-1}{2})_l}{(-(m+1)n)_l}\frac{(-4u)^l}{l!}\\
=&\sum_{l=0}^{[\frac{n}{2}]}\frac{(-2)^l(-\frac{n}{2})\dots(-\frac{n}{2}+l-1)(-2)^l(-\frac{n-1}{2})\dots(-\frac{n-1}{2}+l-1)}{(-1)^l(-(m+1)n)\dots(-(m+1)n+l-1)}\frac{u^l}{l!}\\
=&\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\frac{((m+1)n-l)!}{((m+1)n)!}\frac{u^l}{l!}\\
=&\frac{n!(n+1)!}{((m+1)n)!}\,P_{m,n}(u).
\end{aligned}
$$

From the general formulas we obtain the following differential equation ($a=-\frac{n}{2}$, $b=-\frac{n-1}{2}$, $c=-(m+1)n$ and $z=-4u$):

(DE) $u(4u+1)\,P''_{m,n}(u)-((4n-6)u+(m+1)n)\,P'_{m,n}(u)+n(n-1)\,P_{m,n}(u)=0.$

and the following special value:

$$
\text{(SV)}\quad P_{m,n}(-\frac{1}{4})=\frac{((m+1)n)!}{n!(n+1)!}\frac{(-mn+[\frac{n-1}{2}]+\frac{1}{2})_{[\frac{n}{2}]}}{(-mn)_{[\frac{n}{2}]}}\neq 0.
$$

Formulas (DE) and (SV) imply that if $P_{m,n}$ and $P'_{m,n}$ have a common root then it is also a root of $P''_{m,n}$ and by induction of all derivatives of $P_{m,n}$, which is impossible. Hence $P_{m,n}$ has only simple roots in $\mathbb{C}$.

Let us prove Part 3) of Lemma 3.2 by contradiction. Since $\deg(P_{m,n+1})=[\frac{n+1}{2}]\geqslant 1$, the polynomial $P_{m,n+1}$ has at least one root ($\mathbb{C}$ is an algebraically closed field). Let us assume that all roots of $P_{m,n+1}$ are also roots of $P_{m,n+2}$. Since $P_{m,n+1}$ has only simple roots and $\deg(P_{m,n+2})=\deg(P_{m,n+1})+\epsilon$ with $\epsilon=[\frac{n+2}{2}]-[\frac{n+1}{2}]\in\{0,1\}$, there exist complex numbers $a,b\in\mathbb{C}$ such that $P_{m,n+2}(u)=(au+b)P_{m,n+1}(u)$. Comparing coefficients of u^0 in this equation we can explicitly compute b:

$$
b=\frac{1}{n+2}\ \frac{(m(n+1)+1)!}{(m(n+2)+1)!}\frac{((m+1)(n+2))!}{((m+1)(n+1))!}\ .
$$

Comparing coefficients of u^1 we can explicitly compute a:

$$
a=\frac{(m(n+1)+1)!}{(m(n+2)+1)!}\frac{((m+1)(n+2)-1)!}{((m+1)(n+1))!}.
$$

Since $a\neq 0$, we have $\epsilon=1$ and n is even.

Now, we compare coefficients of u^2. If $n=2$, we obtain

$$
3(m+1)(3(m+1)-1)=2(5(m+1)-1)(4(m+1)-1)
$$

which is impossible. If $n\geqslant 4$, we obtain

$$(m+1)(n-1)(n+1)((m+1)(n+1)-1)$$
$$=((m+1)(n+2)-1)\{2((m+1)(n+1)-1)+(m+1)(n-2)(n-1)^2\},$$

which is impossible (since $n+1<(n-2)(n-1)$).

Finally, there exists a root $u_{m,n}$ of $P_{m,n+1}$ which is not a root of $P_{m,n+2}$.

Proof. (Proof of Theorem 1.4.) Let $d, e \geqslant 1$ be integers with d even and $e \in \frac{d}{2}\mathbb{N}$. We write $d=2m$ and $e=ml$ where $m, l \geqslant 1$ are two integers. By 3) of Lemma 4.4, there exists $u_l \in \mathbb{C}^*$ such that $P_{m,l+1}(u_l)=0$ and $P_{m,l+2}(u_l) \neq 0$.
We set $Q_1(Y,Z) := Y(1-ZY^m-u_l(ZY^m)^2) \in \mathbb{C}[Y,Z]$ and

$$Q_2(Y,Z) := Y\sum_{n=0}^{l} P_{m,n}(u_l)(ZY^m)^n = \sum_{n=0}^{l+1} P_{m,n}(u_l)Z^nY^{mn+1} \in \mathbb{C}[Y,Z].$$

We consider the following two polynomials:

$$\begin{cases} g(X,Y) & := & Z^{l+2}X+Q_1(Y,Z) & \in \mathbb{C}[Z][X,Y] \\ f(X,Y) & := & Z^{-l-2}(Q_2(g(X,Y),Z)-Y) & \in \mathbb{C}(Z)[X,Y] \end{cases}$$

By i) of Lemma 4.1, modulo Z^{l+2}, we have $Q_2(g(X,Y),Z)=Q_2(Q_1(Y,Z),Z)=Y$, hence $f(X,Y)\in\mathbb{C}[Z][X,Y]$. Since $\mathrm{Jac}(f,g)=1$ (the determinant of the jacobian matrix), we have $\sigma := (f,g) \in G(\mathbb{C}[Z])$ (cf. Lemma 1.1.8 p. 5 in [1] or Proposition 4.4 in [2]). In $G(\mathbb{C}(Z))$, the automorphism σ splits into $\sigma = b_2\pi b_1$ with $b_2 = (Z^{-l-2}(Q_2(Y,Z)-X)\,,\,Y)$, $b_1 = (Z^{l+2}X+Q_1(Y,Z)\,,\,Y)$ and $\pi=(Y,X)$. The generic multidegree of σ is equal to

$$(\deg_Y(Q_2), \deg_Y(Q_1)) = (ml+1, 2m+1) = (e+1, d+1).$$

We have, modulo Z^{l+3},

$$Q_2(Q_1(Y,Z),Z)$$
$$=\sum_{n=0}^{l+1} P_{m,n}(u_l)Z^nQ_1(Y,Z)^{mn+1}$$
$$=Y-P_{m,l+2}(u_l)Z^{l+2}Q_1(Y,Z)^{m(l+2)+1}$$
$$=Y-P_{m,l+2}(u_l)Z^{l+2}Y^{m(l+2)+1}.$$

The automorphism $\sigma_{Z\to 0}$ is triangular of degree $m(l+2)+1=e+d+1$. Finally, $\sigma_{Z\to 0} \in G_{(d+e+1)} \cap \overline{G_{(e+1,d+1)}}$. □

Remark. If $P_{m,l}(u_l)=0$ then we change b_2 to τb_2 where τ is the triangular automorphism $\tau=(X+ZY^{ml+1},Y)$. In this way, we always have $\deg(b_2) = ml+1=e+1$.

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Nouville Banian
University of New Caledonia
BPR4 - 98851 Noumea Cedex
New Caledonia

E-mail address: edo@univ-nc.nc