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AN ESTIMATION OF THE NUMBER OF BIFURCATION VALUES FOR REAL POLYNOMIALS

HÀ HUY VUI AND PHẠM TIẾN SƠN

Abstract. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d with only isolated complex critical points. It is shown that the set of bifurcation values of f is contained in a set which has at most $(d-1)^n$ points. The proof of this result is done in such a way that all points of the last set can be explicitly calculated. As a consequence, we obtain a finite set containing the global infimum value of a bounded below polynomial.

1. Introduction

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d. By a result of R. Thom [27], there exists a finite minimal *set of bifurcation values* $A(f)$ of points of $\mathbb{R}$ such that the restriction

$$f\colon \mathbb{R}^n \setminus f^{-1}(A(f)) \to \mathbb{R} \setminus A(f)$$

is a C^∞-trivial fibration. The set of bifurcation values $A(f)$ consists of (i) the *set of affine critical values* $A_f(f)$ of f, and (ii) the set of so called *critical values, corresponding to the singularities at infinity* of f. The exact definition of the latter is the following.

Definition 1.1. (see [18]) A value $t_0 \in \mathbb{R}$ is called *regular at infinity* of f if there exist a compact set $K \subset \mathbb{R}^n$ and a real number $\delta > 0$ such that the restriction

$$f\colon f^{-1}(D_\delta(t_0)) \setminus K \to D_\delta(t_0) := \{t \in \mathbb{R}|\ |t - t_0| < \delta\}$$

is a C^∞-trivial fibration. If t_0 is not a regular value at infinity of f, then it is called a *critical value, corresponding to the singularities at infinity* of f.

Thus, denoting by $A_\infty(f)$ the set of all critical values corresponding to the singularities at infinity of f, we have (see, for example, [4], [19])

$$A(f) = A_f(f) \cup A_\infty(f).$$

In the natural way two fundamental questions appear: *how to determine the set* $A(f)$ and *how to estimate the number of points of this set.*

In general, it is difficult to compute the set of bifurcation values of f unless $n = 2$ (see [28], [2]). Moreover, for $n > 2$, there is no characterization of the

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critical values, corresponding to the singularities at infinity. In order to examine the set $A_\infty(f)$ one often constructs larger sets in which it is easier to study. There is a relation between such sets and the asymptotic growth at infinity of the gradient of f. For instance, let

$$\begin{aligned}\widetilde{K}_\infty(f) := \{t \in \mathbb{R} \quad | \quad & \text{there exists a sequence } x^k \to \infty \text{ such that} \\ & f(x^k) \to t \text{ and } \|\text{grad} f(x^k)\| \to 0\}.\end{aligned}$$

If $t \notin \widetilde{K}_\infty(f)$, then we say that f satisfies *Fedoryuk's condition* at t (see [3]). If one looks for a weaker condition then it is natural to consider the set

$$\begin{aligned}K_\infty(f) := \{t \in \mathbb{R} \quad | \quad & \text{there exists a sequence } x^k \to \infty \text{ such that} \\ & f(x^k) \to t \text{ and } \|x^k\|\|\text{grad} f(x^k)\| \to 0\}.\end{aligned}$$

Clearly, $K_\infty(f) \subset \widetilde{K}_\infty(f)$. (It is worth noting that the sets $\widetilde{K}_\infty(f)$ and $K_\infty(f)$ may also be defined for complex polynomials [8].) If $t \notin K_\infty(f)$ then it is usual to say that f satisfies *Malgrange's condition* at t (see [15], [23]). Let $\widetilde{K}(f) := A_f(f) \cup \widetilde{K}_\infty(f)$ and $K(f) := A_f(f) \cup K_\infty(f)$. Then it was proved (see, for example, [5], [6], [19]) that $A_\infty(f) \subset K_\infty(f) \subset \widetilde{K}_\infty(f)$. In particular, $A(f) \subset K(f) \subset \widetilde{K}(f)$. Moreover, according to the results of H. V. Hà (see [4], [5], [6]), for polynomials of two complex variables we have the equations $A_\infty(f) = K_\infty(f) = \widetilde{K}_\infty(f)$ and $A(f) = A_f(f) \cup \widetilde{K}_\infty(f)$. In [9] (see also [4], [14]) the author gave a sharp estimation of the number $\#A(f)$, provided that $\#\widetilde{K}_\infty(f) < \infty$, in terms of the degree $d := \deg f$ of f. That is if $\#\widetilde{K}_\infty(f) < \infty$, then $\#A(f) \leqslant (d-1)^n$. Unfortunately, it may happen that $\#\widetilde{K}_\infty(f) = \infty$ (see [20, Example 1.11], [8, Example 2.1]). Shortly thereafter, Z. Jelonek and K. Kurdyka (see [10]) improved results of Z. Jelonek and obtained the following estimation $\#A(f) < (d-1)^n + nd^{n-2}$. The authors also show that the sets $\widetilde{K}(f)$ and $K(f)$ can be computed effectively. Here "effectively" means that there is an algorithm (based on Gröbner basis) which works actually on a computer.

The first aim of this paper is to give an *explicitly constructed* set in which its cardinality is less than or equal to $(d-1)^n$ and it contains the set of bifurcation values of f. More precisely, we denote by $f_{\mathbb{C}}$ the complexification of the polynomial f, and let $A_f(f_{\mathbb{C}})$ be the set of critical values of $f_{\mathbb{C}}$. We will prove the following.

Theorem 1.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d with only isolated complex critical points. Then there exists a finite set $T_\infty(f_{\mathbb{C}}) \subset \mathbb{C}$ such that the following statements hold*

(i) $A(f) \subset A_f(f) \cup (T_\infty(f_{\mathbb{C}}) \cap \mathbb{R})$;
(ii) $(T_\infty(f_{\mathbb{C}}) \cap \mathbb{R}) \subset K_\infty(f)$; *and*
(iii) $\#A_f(f_{\mathbb{C}}) + \#T_\infty(f_{\mathbb{C}}) \leqslant (d-1)^n$.

Observe that, in view of [10, Example 2.1]), our estimate is the best possible result: *For every $n > 0$ and $d > 0$ there is a polynomial $f\colon \mathbb{R}^n \to \mathbb{R}$ of degree d such that $\#A(f) = (d-1)^n$.*

Moreover, emphasis is made on being able to calculate all points of the set $A_f(f_{\mathbb{C}}) \cup T_\infty(f_{\mathbb{C}})$. Our method is actually different from the argument of the previous authors: the proof uses only the homotopy continuation methods as a tool (see, for example, [31], [30], [12], [13]). Namely, we can define a trivial system:

$$\mathrm{grad}\rho_{\mathbb{C}}(x) = 0$$

with known solution set, where $\rho\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial function and $\rho_{\mathbb{C}}$ is its complexification. We then follow the curves in the variable s which make up the solution set of

$$H(x,s) := s\mathrm{grad}f_{\mathbb{C}}(x) + \eta(1-s)\mathrm{grad}\rho_{\mathbb{C}}(x) = 0$$

for some nonzero parameter $\eta \in \mathbb{C}$. More precisely, if ρ is chosen correctly, the following three properties hold:

(T) (*Triviality*) The set of critical points of $\rho_{\mathbb{C}}$ is known.
(S) (*Smoothness*) The set of solutions of $H(x,s) = 0$ for $0 \leqslant s < 1$ consists of a finite number of smooth paths, each parametrized by s in $[0,1)$.
(A) (*Accessibility*) Every isolated critical point of $f_{\mathbb{C}}$ is reached by some path originating at $s = 0$; and consequently, the set of affine critical values of f is computed. It follows that this path starts at an isolated critical point of $\rho_{\mathbb{C}}$.

When these three properties hold, the solution paths can be followed from the initial points (known because of Property (T)) at $s = 0$ to all critical points of the original polynomial f at $s = 1$, using standard numerical techniques.

It is important to realize that even though properties (T), (S) and (A) imply that each critical point of $f_{\mathbb{C}}$ will lie at the end of a solution path, it is also consistent with these properties that some of the paths may diverge to infinity as the parameter s approaches 1. Then it is shown that for any $t \in A_\infty(f)$ there exists such a path on which f tends to t when s approaches 1. (The smoothness property rules this out for $s \to s_0 < 1$.)

Another reason, which motivates our research work, is the following basic problem: *Given a polynomial function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *which is bounded from below on* $\mathbb{R}^n$, *find the global infimum*

$$f^* := \inf\{f(x) \mid x \in \mathbb{R}^n\}.$$

If a polynomial f attains a minimum in $x^* \in \mathbb{R}^n$, i.e., $f(x^*) \leqslant f(x)$ for all $x \in \mathbb{R}^n$, then the gradient of f vanishes at x^*; in other words, $f^* = f(x^*)$ is a critical value of f. However, there are polynomials that are bounded from below on $\mathbb{R}^n$ and yet do not attain a minimum on $\mathbb{R}^n$. The simplest example is perhaps

$$f(x,y) := (1-xy)^2 + x^2 \in \mathbb{R}[x,y]$$

for which we have $f > 0$ on $\mathbb{R}^2$ but $f^* = 0$ since $\lim_{t\to 0} f(t, 1/t) = 0$.

There are at least three techniques to compute the value f^* : Gröbner bases and eigenvalues, Resultants and discriminants, and Homotopy methods. Exact methods can be found in [7], [29] and [22]. These algorithms work when the given polynomial has a minimum, without considering an approach for finding the infimum.

Different approaches, based on solving a certain convex relaxation of the problem, can be found in [25], [11], [26], [21], [22]. Such methods seem to have better computational properties. However, in general, they only guarantee finding a lower bound of the infimum.

The second aim of this note is to investigate the general case: *the polynomial f may not attain its infimum.* We will prove that if a polynomial f is bounded from below, then $f^* \in A_f(f) \cup (T_\infty(f_{\mathbb{C}}) \cap \mathbb{R})$. Moreover, since the degree of f is even, it is very easy to choose ρ that satisfies the three properties (T), (S) and (A). Consequently, as mentioned above, we can effectively compute a finite set containing all critical values and the infimum value of f.

The paper is organized as follows. We give a proof of the main theorem in Section 2. The problem of computing the global infimum of a real polynomial f on $\mathbb{R}^n$ is considered in Section 3. Finally, computations are given in Section 4.

2. Proof of Theorem 1.1

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $d := \deg f$.

Theorem 1.1 is clear in the case $d = 1$. Hence, one has only to prove the theorem when $d > 1$.

We first need some definitions. Let m be the greatest integer $\leqslant d/2$. Then we may introduce on $\mathbb{R}^n$ the "control function"

$$\rho(x) := x_1^{2m} + x_2^{2m} + \cdots + x_n^{2m} + g(x_1, x_2, \ldots, x_n),$$

where g is a polynomial of degree $< 2m$. We have

Lemma 2.1. (i) *The polynomial function $\rho\colon \mathbb{R}^n \to \mathbb{R}$ is proper and bounded from below.*

(ii) *The polynomial mapping* $\mathrm{grad}\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ *is proper.*

(iii) *There exists a positive constant c such that the following inequality holds*

$$\|\mathrm{grad}\rho(x)\|\|x\| \leqslant c\rho(x) \quad \text{for} \quad \|x\| \gg 1.$$

Proof. The statements (i) and (ii) are clear from the definition.

Assume to the contrary that (iii) does not hold. Then, by the Curve Selection Lemma at infinity (see [16], [17]), there exists a real analytic curve $\varphi(\tau)$, $\tau \in (0, \epsilon]$, such that

(a1) $\lim_{\tau \to 0} \|\varphi(\tau)\| = \infty$; and
(a2) $\|\mathrm{grad}\rho[\varphi(\tau)]\|\|\varphi(\tau)\| \gg \rho[\varphi(\tau)]$ as $\tau \to 0$.

We can write

$$\|\varphi(\tau)\| = a\tau^\alpha + \text{ higher order terms in } \tau \qquad (a \neq 0).$$

It follows from (a1) that $\alpha < 0$. Moreover, it is not difficult to verify, asymptotically as $\tau \to 0$, that

$$\begin{aligned} \rho[\varphi(\tau)] &\simeq \tau^{2m\alpha}, \\ \|\mathrm{grad}\rho[\varphi(\tau)]\| &\simeq \tau^{(2m-1)\alpha}. \end{aligned}$$

Therefore

$$\|\mathrm{grad}\rho[\varphi(\tau)]\|\|\varphi(\tau)\| \simeq \tau^{2m\alpha} \simeq \rho[\varphi(\tau)],$$

which contradicts (a2). □

Let $f_{\mathbb{C}}$ (resp., $\rho_{\mathbb{C}}$) be the complexification of f (resp., ρ). Consider the homotopy mapping $\mathcal{H}\colon \mathbb{C}^n \times \mathbb{P} \to \mathbb{C}$ defined as follows

$$\mathcal{H}(x, \lambda) := \lambda_1 \mathrm{grad} f_{\mathbb{C}}(x) + \lambda_2 \mathrm{grad}\rho_{\mathbb{C}}(x) \quad \text{with } \lambda := (\lambda_1 : \lambda_2) \in \mathbb{P}.$$

Lemma 2.2. *There exists a finite set $E \subset \mathbb{P}$ such that the natural projection*

$$\pi\colon \{(x, \lambda) \in \mathbb{C}^n \times \mathbb{P} \mid \mathcal{H}(x, \lambda) = 0\} - \pi^{-1}(E) \to \mathbb{P} - E, \quad (x, \lambda) \mapsto \lambda,$$

is a finite sheeted covering mapping.

Proof. We introduce the set

$$X := \{(x, \lambda) \in \mathbb{C}^n \times \mathbb{P} \mid \mathcal{H}(x, \lambda) = 0\}.$$

Then $\dim X = 1$. Let X_0 be the set of nonsingular points of X. By an algebraic version of Sard's theorem (see [1]), the set Y of critical values of the restriction of π on X_0 is finite. Then it is not hard to verify that the set $E := Y \cup \pi(X - X_0)$ is finite and that π is a fibration outside E. This completes the proof of the lemma. □

By Lemma 2.2, for each $\lambda \notin E$ the system of equations

$$\mathcal{H}(x, \lambda) = 0$$

consists of a fixed number of solutions, say μ. By Bezout's theorem,

$$\mu \leqslant (d-1)^n. \tag{2.1}$$

We now consider the homotopy equation

$$H(x, s) := s\mathrm{grad} f_{\mathbb{C}}(x) + \eta(1-s)\mathrm{grad}\rho_{\mathbb{C}}(x) \quad \text{for} \quad x \in \mathbb{C}^n, \ s \in \mathbb{R}$$

for some nonzero parameter $\eta \in \mathbb{C}$. Obviously,

$$H(x, s) = \mathcal{H}(x, (s\colon \eta(1-s))), \quad s \in \mathbb{R}.$$

An immediate consequence of this representation is

Lemma 2.3. *For almost each parameter $\eta \in \mathbb{C}$ on the unit circle, there are μ analytic functions*

$$\varphi_k\colon \{s \in \mathbb{R} \mid s \neq 1\} \to \mathbb{C}^n, \quad s \to \varphi_k(s), \quad k = 1, 2, \ldots, \mu,$$

such that

(i) $H(\varphi_k(s), s) = 0, k = 1, 2, \ldots, \mu,$ *for* $s \neq 1$; *and*

(ii) *for any critical point x of $f_{\mathbb{C}}$ there is k such that $x = \lim_{s\to 1}\varphi_k(s)$.*

Proof. The claim follows very closely along the lines of the proof of [30, Theorem 1] (see also [31]). We will leave to the reader to verify these facts. □

We denote by $A_f(f_{\mathbb{C}})$ the set of critical values of $f_{\mathbb{C}}$ and let $T_\infty(f_{\mathbb{C}})$ be the set of all finite limit values $\lim_{s\to 1} f_{\mathbb{C}}[\varphi_k(s)]$ with $k \in \{1, 2, \ldots, \mu\}$ as $\lim_{s\to 1}\|\varphi_k(s)\| = \infty$. Clearly,

$$\#A_f(\mathbb{C}) + \#T_\infty(\mathbb{C}) \leqslant \mu. \tag{2.2}$$

We need the following lemma.

Lemma 2.4. *If $t_0 \notin T_\infty(f_{\mathbb{C}}) \cap \mathbb{R}$, then the vectors* $\mathrm{grad}f(x)$ *and* $\mathrm{grad}\rho(x)$ *are linearly independent for all x sufficiently large and $f(x)$ sufficiently close to t_0.*

Proof. Indeed, suppose that this is not the case. Then, by the Curve Selection Lemma at infinity (see [16], [17]), there exist a real analytic curve $\psi(\tau)$ and a real analytic function $\lambda(\tau), \tau \in (0, \delta]$, such that

(b1) $\lim_{\tau\to 0}\|\psi(\tau)\| = \infty$;
(b2) $\lim_{\tau\to 0} f[\psi(\tau)] = t_0$; and
(b3) $\mathrm{grad}f[\psi(\tau)] = \lambda(\tau)\mathrm{grad}\rho[\psi(\tau)]$.

Let

$$\rho[\psi(\tau)] \;:=\; a\tau^\alpha + \text{ higher order terms in } \tau, \quad a \neq 0.$$

Since $\|\psi(\tau)\| \to +\infty$ as $\tau \to 0$, $\rho[\psi(\tau)] \to +\infty$ as $\tau \to 0$, and hence $\alpha < 0$. We expand also

$$f[\psi(\tau)] \;:=\; t_0 + b\tau^\beta + \text{ higher order terms in } \tau, \quad b \neq 0.$$

Since $f[\psi(\tau)] \to t_0$ as $\tau \to 0$, we have $\beta > 0$. These imply that

$$\beta - \alpha > 0.$$

On the other hand, differentiating $f[\psi(\tau)]$ with respect to τ yields

$$\begin{aligned}
\frac{d}{d\tau} f[\psi(\tau)] &= \left\langle \mathrm{grad}f[\psi(\tau)], \frac{d\psi(\tau)}{d\tau} \right\rangle \\
&= \lambda(\tau)\left\langle \mathrm{grad}\rho[\psi(\tau)], \frac{d\psi(\tau)}{d\tau} \right\rangle \\
&= \lambda(\tau)\frac{d}{d\tau}\rho[\psi(\tau)].
\end{aligned}$$

Consequently,

$$\left|\frac{d}{d\tau} f[\psi(\tau)]\right| = |\lambda(\tau)| \left|\frac{d}{d\tau}\rho[\psi(\tau)]\right|.$$

It follows that

$$|b\beta\tau^{\beta-1} + \text{ higher order terms in } \tau| = |\lambda(\tau)||a\alpha\tau^{\alpha-1} + \text{ higher order terms in } \tau|.$$

Thus,

$$|\lambda(\tau)| \simeq \tau^{\beta-\alpha} \;\to 0 \;\text{ as }\; \tau \to 0.$$

Let $s := \frac{\eta}{\eta-\lambda(\tau)}$. Then we can write

$$\tau = \theta(s) := c_1 s^{\frac{n_1}{N}} + c_2 s^{\frac{n_2}{N}} + \cdots, \quad c_i \in \mathbb{C},$$

where N and $n_1 < n_2 < \cdots$ are positive integers, having no common divisor, such that $\theta((s-1)^N)$ has positive radius of convergence.

Let us define now a mapping

$$\varphi\colon \{s \in \mathbb{R} \mid 0 < |s-1| \ll 1\} \to \mathbb{C}^n, \quad s \mapsto \psi[\theta(s)].$$

Then it is clear that

(c1) $\varphi(s)$ is a continuous mapping;
(c2) $\lim_{s\to 1} \|\varphi(s)\| = \infty$;
(c3) $\lim_{s\to 1} f_{\mathbb{C}}[\varphi(s)] = t_0$; and
(c4) $H(\psi(s), s) = s\mathrm{grad} f_{\mathbb{C}}[\varphi(s)] + \eta(1-s)\mathrm{grad}\rho_{\mathbb{C}}[\varphi(s)] = 0$ for all $0 < |s-1| \ll 1$.

But, according to Lemma 2.3, the system $H(x, s) = 0$ has exactly μ continuous solutions $x = \varphi_k(s), k = 1, 2, \ldots, \mu$. Therefore, there is an integer number $k \in \{1, 2, \ldots, \mu\}$ such that $\varphi \equiv \varphi_k$. This implies that $t_0 \in T_\infty(f_{\mathbb{C}})$, which is a contradiction. $\square$

Now we can pass to the proof of Theorem 1.1.

Proof. (i). Let $t_0 \in \mathbb{R} - T_\infty(f_{\mathbb{C}})$. By Lemma 2.4, there exist $r \gg 1$ and $0 < \delta \ll 1$ such that the vectors $\mathrm{grad} f(x)$ and $\mathrm{grad}\rho(x)$ are linearly independent for all $x \in f^{-1}(D_\delta(t_0)) \cap \{\rho(x) \geqslant r\}$. And therefore we can define the smooth vector fields

$$\begin{aligned} v(x) &:= \mathrm{grad} f(x) - \frac{\langle \mathrm{grad}\rho(x), \mathrm{grad} f(x)\rangle}{\|\mathrm{grad}\rho(x)\|^2}\mathrm{grad}\rho(x), \\ w(x) &:= \frac{v(x)}{\langle v(x), \mathrm{grad} f(x)\rangle}. \end{aligned}$$

Clearly, $w(x)$ is the well-defined vector field and $w(x) \neq 0$ provided that $\|x\|$ is sufficiently large and $f(x)$ is sufficiently close to t_0. Thus, integrating w we get the desired trivialization of f outside a compact set $\{x \in \mathbb{R}^n \mid \rho(x) \leqslant r\}$, which implies that $t_0 \notin A_\infty(f)$.

(ii). Let $t_0 \in T_\infty(f_{\mathbb{C}}) \cap \mathbb{R}$. Then, by definition, there exist a real analytic curve $\psi(\tau)$ and a real analytic function $\lambda(\tau), \tau \in (0, \delta]$, satisfying the properties (b1)-(b3); in other words, we are in the situation of the proof of Lemma 2.4. Hence, asymptotically as $\tau \to 0$, we have

$$\begin{aligned} |f[\psi(\tau)] - t_0| \simeq \tau^\beta &\simeq |\lambda(\tau)||\rho[\psi(\tau)]| \\ &\simeq \frac{\|\mathrm{grad} f[\psi(\tau)]\|}{\|\mathrm{grad}\rho[\psi(\tau)]\|}|\rho[\psi(\tau)]|. \end{aligned}$$

(The last relation follows from (b3).) Hence, by Lemma 2.1(iii), there exists a positive constant c such that

$$|f[\psi(\tau)] - t_0| \quad \geqslant \quad c\|\mathrm{grad} f[\psi(\tau)]\|\|\psi(\tau)\| \quad \text{for} \quad 0 < \tau \ll 1.$$

Consequently,

$$\|\mathrm{grad} f[\psi(\tau)]\| \|\psi(\tau)\| \to 0 \quad \text{as} \quad \tau \to 0,$$

which shows that $t_0 \in K_\infty(f)$.

(iii). The claim is a direct consequence of the inequalities (2.1) and (2.2).

All is now proven. □

Remark 2.1. It is possible that in Theorem 1.1 the upper bound $(d-1)^n$ can be replaced by $\prod_{i=1}^n d_i$, where $d_i, i = 1, 2, \ldots, n$, is the highest degree of a monomial occurring in the derivative $\frac{\partial f}{\partial x_i}$.

3. Minimizing Polynomial Functions

This section is concerned with the following basic problem: *Given a bounded below polynomial $f : \mathbb{R}^n \to \mathbb{R}$, find the global infimum*

$$f^* := \inf\{f(x) \mid x \in \mathbb{R}^n\}.$$

We first recall that

$$\rho(x) := x_1^{2m} + x_2^{2m} + \cdots + x_n^{2m} + g(x_1, x_2, \ldots, x_n),$$

where g is a polynomial of degree $< 2m$. Put

$$\Gamma(f, \rho) := \{x \in \mathbb{R}^n \mid \mathrm{grad} f(x) \neq 0 \text{ and rank} \begin{pmatrix} \mathrm{grad} f(x) \\ \mathrm{grad} \rho(x) \end{pmatrix} \leqslant 1\}.$$

Then, by an easy consequence of Sard's theorem, $\Gamma(f, \rho)$ is an algebraic curve for almost every $g \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. We fix such a polynomial g. It is not hard to see that for large $r > 0$, the set $\{x \in \Gamma(f, \rho) \mid \rho(x) \geqslant r\}$ consists of a fixed number of one dimensional connected components, say $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$. Taking r large enough, we have, for $i = 1, 2, \ldots, s$, that there exist $\delta > 0$ and a Nash (i.e., analytic algebraic) function $\psi_i : (0, \delta] \to \mathbb{R}^n, \tau \mapsto \psi_i(\tau)$, such that Γ_i is the germ of the curve $x = \psi_i(\tau)$ as $\tau \to 0$. Note that ψ_i (or rather its germ at 0) is given by a real algebraic Puiseux series in τ. The function $f \circ \psi_i : (0, \delta] \to \mathbb{R}, \tau \mapsto f[\psi_i(\tau)]$, is strictly increasing, or strictly decreasing or constant for small δ. Hence, it has a limit $t_i := \lim_{\Gamma_i} f$ in $\mathbb{R} \cup \{+\infty, -\infty\}$. Let

$$\widetilde{T}_\infty(f) := \{t_1, t_2, \ldots, t_s\} \subset \mathbb{R} \cup \{+\infty, -\infty\}.$$

We have

Proposition 3.1. *The following inclusions hold*

$$A_\infty(f) \subset \widetilde{T}_\infty(f) \cap \mathbb{R} \subset T_\infty(f_{\mathbb{C}}) \cap \mathbb{R}.$$

Proof. The claim follows very closely along the lines of the proof of Theorem 1.1. We will leave to the reader to verify these facts. □

We now assume that the components $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ are numbered in a way such that $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_s$.

Proposition 3.2. *A polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is lower bounded if and only if $t_1 > -\infty$.*

Proof. Let

$$C := \{x \in \mathbb{R}^n \mid f(x) = \min\{f(y) \mid \rho(y) = \rho(x),\ y \in \mathbb{R}^n\}\}.$$

Clearly, the set C is nonempty and unbounded. Moreover, we have $C \subset \Gamma(f, \rho)$. Thus, C must contain the component Γ_1 and, possibly, some other components. This implies that

$$t_1 = \lim_{x \in C,\ \rho(x) \to \infty} f(x).$$

The proof of the proposition follows now from the definition of the set C. □

Remark 3.1. Checking that a given polynomial function is lower (or upper) bounded function is far from trivial (see [24]).

With the notations in Section 2, the main result of this section is then the following.

Proposition 3.3. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a bounded below polynomial. Then*

$$f^* \in A_f(f) \cup (\widetilde{T}_\infty(f) \cap \mathbb{R}).$$

Proof. Indeed, if the polynomial f attains its infimum f^*, then it is well known that f^* is a critical value of f; that is $f^* \in A_f(f)$.

We now suppose that f^* is not attained by f. Put

$$C := \{x \in \mathbb{R}^n \mid f(x) = \min\{f(y) \mid \rho(y) = \rho(x),\ y \in \mathbb{R}^n\}\}.$$

Then it is not difficult to verify that

(d1) C is an unbounded semi-algebraic set (this follows from Tarski's theorem);
(d2) For all $x \in C$ there is $\lambda \in \mathbb{R}$ such that $\mathrm{grad} f(x) = \lambda \mathrm{grad} \rho(x)$; and
(d3) For every sequence $x_k \in C, x_k \to \infty$, we have $f(x_k) \to f^*$ (since f does not attain its infimum).

Hence, by using a version at infinity of the Curve Selection Lemma (see [16], [17]), there exist a real analytic curve $\psi(\tau)$ and a real analytic function $\lambda(\tau), \tau \in (0, \delta]$, satisfying the properties (b1)-(b3). In other words, we are in the situation of the proof of Lemma 2.4. Then it is not difficult to verify that $f^* \in \widetilde{T}_\infty(f) \cap \mathbb{R}$, which completes the proof. □

Remark 3.2. If a polynomial f is bounded from below, then the degree $d := \deg f$ of f is even. In this case, we have $m = d/2 \in \mathbb{N}$. Moreover, it is easy to choose ρ such that the set of critical points of ρ is known and has exactly $(d-1)^n$ points. Consequently, ρ satisfies the three properties (T), (S) and (A).

4. Algorithmic aspect and illustrative examples

4.1. **Algorithmic aspect.** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d with only isolated complex critical points. In this section we recall the homotopy methods to effectively compute the set $A_f(f) \cap (T_\infty(f_{\mathbb{C}}) \cap \mathbb{R})$.

We consider the problem of finding the solutions of the equation

$$0 = \mathrm{grad} f_{\mathbb{C}}(x).$$

We define

$$H(x,s) := s\mathrm{grad} f_{\mathbb{C}}(x) + \eta(1-s)\mathrm{grad}\rho_{\mathbb{C}}(x).$$

The equation $0 = \mathrm{grad}\rho_{\mathbb{C}}(x)$ should have known solutions.

Let $\varphi(s)$ be a smooth solution of the equation

$$0 = H(x,s).$$

Since H is differentiable, the Implicit Function Theorem enables us to compute $\frac{d}{ds}\varphi(s)$. By pursuing this idea, we can describe the curve $\varphi(s)$ by a differential equation. We have

$$0 = H(\varphi(s), s).$$

On differentiating with respect to s, we obtain

$$0 = H_s(\varphi(s), s) + H_x(\varphi(s), s)\frac{d}{ds}\varphi(s)$$

in which subscripts denote partial derivatives. Thus,

$$\frac{d}{ds}\varphi(s) = -\left[H_x(\varphi(s), s)\right]^{-1} H_s(\varphi(s), s). \tag{4.1}$$

This is a differential equation for $\varphi(s)$. It has a known initial value because $\varphi(0)$ is supposedly known. On integrating this differential equation (usually by numerical procedures), we shall have the value $\varphi(1)$, which is the solution.

4.2. **Illustrative examples.** The computations can be performed with the software Mathematica. In order to test our method, we considered

Example 4.1. Let us consider the following polynomial of two real variables:

$$f(x,y) \quad := \quad (xy-1)^2 + x.$$

We have $\deg f = 4$. So, we can define $\rho \in \mathbb{R}[x,y]$ by

$$\rho(x,y) := \frac{x^4}{4} + \frac{y^4}{4} - 8x - y.$$

Then the polynomial $\rho_{\mathbb{C}}$ has nine non-degenerate critical points. So the above computation applies. It provides a homotopy and nine paths, beginning from the critical points of ρ, which lead to all values of $A_f(f_{\mathbb{C}}) \cup T_\infty(f_{\mathbb{C}})$. Table 1 shows the computed results (here, $i := \sqrt{-1}$). It is easy to see that $A_f(f) = \{1\}$ and $T_\infty(f_{\mathbb{C}}) \cap \mathbb{R} = \{0\}$ ($\eta \in \mathbb{C}$ randomly chosen on the unit circle). By the results of M. Coste and M. J. de la Puente [2] (see also [28]), we find that $A_\infty(f) = \{0\}$.

Example 4.2. Let us consider another example:

$$f(x,y) \quad := \quad (xy-1)^2 + (x-1)^2.$$

It is clear that the polynomial f is nonnegative on $\mathbb{R}^2$. Since $\deg f = 4$, our starting polynomial ρ can be chosen as in the previous example. Table 2 shows our computed results. It is clear that $A_f(f) = \{0, 2\}$ and $T_\infty(f_{\mathbb{C}}) \cap \mathbb{R} = \{1\}$. It is not hard to see that the polynomial f attains its infimum global value $f^* = f(1,1) = 0 \in A_f(f)$. Moreover, it follows from the results in [2] (see also [28]) that $A_\infty(f) = \{1\}$.

	x	y	f
1	$2.3\times10^{16}-4.0\times10^{16}i$	$0.$	$2.3\times10^{16}-4.0\times10^{16}i$
2	$2.3\times10^{16}+4.0\times10^{16}i$	$0.$	$2.3\times10^{16}+4.0\times10^{16}i$
3	-4.6×10^{16}	$0.$	-4.6×10^{16}
4	$-8.0\times10^{-9}-5.8\times10^{-9}i$	$-8.0\times10^{9}+5.8\times10^{9}i$	$-8.0\times10^{-9}-5.8\times10^{-9}i$
5	$-8.0\times10^{-9}+5.8\times10^{-9}i$	$-8.0\times10^{9}-5.8\times10^{9}i$	$-8.0\times10^{-9}+5.8\times10^{-9}i$
6	$3.0\times10^{-9}-9.5\times10^{-9}i$	$3.0\times10^{9}+9.5\times10^{9}i$	$3.0\times10^{-9}-9.5\times10^{-9}i$
7	$3.0\times10^{-9}+9.5\times10^{-9}i$	$3.0\times10^{9}-9.5\times10^{9}i$	$3.0\times10^{-9}+9.5\times10^{-9}i$
8	$1.\times10^{-10}$	$1.\times10^{10}$	$1.\times10^{-10}$
9	$0.$	0.5	$1.$

Table 1. Computations to the polynomial $(xy-1)^2+x$.

	x	y	f
1	$-0.5+1.4\times10^{25}i$	$0.$	$-2.\times10^{50}-4.2\times10^{25}i$
2	$-0.5-1.4\times10^{25}i$	$0.$	$-2.\times10^{50}+4.2\times10^{25}i$
3	$-2.6\times10^{-11}+8.2\times10^{-11}i$	$-3.5\times10^{9}-1.0\times10^{10}i$	$1.-1.6\times10^{-10}i$
4	$-2.6\times10^{-11}-8.2\times10^{-11}i$	$-3.5\times10^{9}+1.0\times10^{10}i$	$1.+1.6\times10^{-10}i$
5	$7.0\times10^{-11}+5.1\times10^{-11}i$	$9.2\times10^{9}-6.7\times10^{9}i$	$1.-1.0\times10^{-10}i$
6	$7.0\times10^{-11}-5.1\times10^{-11}i$	$9.2\times10^{9}+6.7\times10^{9}i$	$1.+1.0\times10^{-10}i$
7	-8.7×10^{-11}	-1.1×10^{10}	$1.$
8	$0.$	$-1.$	$2.$
9	$1.$	$1.$	$0.$

Table 2. Computations to the polynomial $(xy-1)^2+(x-1)^2$.

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Institute of Mathematics
18 Hoang Quoc Viet Road
10307 Hanoi, Vietnam.

E-mail address: hhvui@math.ac.vn

Department of Mathematics
University of Dalat
Dalat, Vietnam.

E-mail address: pham_ts@yahoo.co.uk