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$\mathbb{Q}$-FACTORIAL SUBALGEBRAS OF A POLYNOMIAL RING

MASAYOSHI MIYANISHI

Abstract. We give an algebraic characterization of a log affine pseudo-plane and the quotient surface $\mathbb{A}^2/G$ of $\mathbb{A}^2$ with a small finite subgroup G of $\mathrm{GL}(2, \mathbb{C})$ in terms of $\mathbb{Q}$-factorial normal subalgebras of a polynomial ring $\mathbb{C}[x, y]$. Then we consider the cancellation problem for these surfaces.

1. Subalgebras of a polynomial ring

Let X be a normal affine surface defined over the complex field $\mathbb{C}$ and let A be the coordinate ring of X. We then say that X (or A) is $\mathbb{Q}$-factorial if the divisor class group $\mathrm{C}\ell\,(X)$ (or $\mathrm{C}\ell\,(A)$) consists of elements of finite order. Further, X is said to be a *log affine pseudo-plane of type* d if there exists an $\mathbb{A}^1$-fibration $\rho : X \to C$ such that C is isomorphic to the affine line $\mathbb{A}^1$, every fiber is irreducible and only one fiber dF_0 is a multiple fiber with multiplicity $d > 1$. It is known by [7] that the singularity of X is at most cyclic quotient singularity, and that if P is a singular point then P lies on a multiple fiber and there are no other singular points on the fiber. Hence X has at most one singular point. If X is smooth, we simply say that X is an affine pseudo-plane of type d.

On the other hand, let $\varphi : A \hookrightarrow B$ be an injective homomorphism of $\mathbb{C}$-algebras by which we view A as the subalgebra $\varphi(A)$ of B. We call φ a *pure* embedding if the natural homomorphism $\varphi_M : M \to M \otimes_A B$ is injective for every A-module M. We call A also a *pure* subalgebra of B. For this definition and relevant results, the readers are referred to Hochster-Roberts [5]. Let us begin with the following result. For an integral domain A, we denote by $Q(A)$ the field of fractions.

Lemma 1.1. *Let $\varphi : A \to B$ be a pure embedding of $\mathbb{C}$-algebras. Then the following assertions hold.*

(1) *For any ideal I of A, we have $IB \cap A = I$. Hence if B is noetherian, so is A.*
(2) *Suppose that B is a noetherian domain. Let $X = \mathrm{Spec}\, A, Y = \mathrm{Spec}\, B$ and $p = {}^a\varphi$. Then $p : Y \to X$ is a surjective morphism.*

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(3) *Suppose that B is an integral domain and that A and B are birational, i.e., $Q(A) = Q(B)$. Then $\varphi(A) = B$.*
(4) *Suppose that B is normal. Then so is A.*

Proof. The assertions follow from the definition. $\square$

We set $B = \mathbb{C}[x, y]$ a polynomial ring in two variables and specify further the properties of a (pure) subalgebra A of dimension two.

Lemma 1.2. *Let $\varphi : A \hookrightarrow B = \mathbb{C}[x, y]$ be a subalgebra with $\dim A = 2$. Let $p : \mathbb{A}^2 \to X = \operatorname{Spec} A$ be the associated morphism. Then we have the following assertions.*

(1) *A is a finitely generated, normal domain provided φ is a pure embedding.*
(2) *Suppose that p is a quasi-finite morphism. Let X° be the smooth part of X. Then X° has log Kodaira dimension $\overline{\kappa}(X^\circ) = -\infty$. Hence either X contains an open set isomorphic to $\mathbb{A}^2/G$ with a small finite subgroup G of $\mathrm{GL}\,(2, \mathbb{C})$ or X has an $\mathbb{A}^1$-fibration $\rho : X \to C$, where C is isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$.*
(3) *If X is smooth and $p : \mathbb{A}^2 \to X$ is a dominant morphism, the assertion (2) holds with X replacing X°.*
(4) *Suppose that p is quasi-finite and X is $\mathbb{Q}$-factorial. Then either X is isomorphic to $\mathbb{A}^2/G$ or X has an $\mathbb{A}^1$-fibration $\rho : X \to C \cong \mathbb{A}^1$ whose fibers are all irreducible.*

Proof. The assertion (1) is due to Hashimoto [4].

(2) If p is quasi-finite, the set $p^{-1}(X - X^\circ)$ is a finite subset of $\mathbb{A}^2$. Hence $p^{-1}(X^\circ)$ has log Kodaira dimension $-\infty$, and so does X° (cf. [9, Lemma 1.14.1 in Chap. 2]). There are two cases to consider (see [9, Theorem 5.1.2 in Chap. 2 and Lemma 1.6.2 in Chap. 3] and [8]).

(i) X° contains an open set U which is isomorphic to $\mathbb{A}^2/G - \{\overline{O}\}$, where $\overline{O}$ is the unique singular point, where G is, as above, a small finite subgroup of $\mathrm{GL}\,(2, \mathbb{C})$. Furthermore, $X - U$ is a disjoint union of contractible curves which are isomorphic to $\mathbb{A}^1$ if X is smooth.
(ii) X° has an $\mathbb{A}^1$-fibration $\rho^\circ : X^\circ \to C^\circ$.

We consider the case (i) first. Since $\mathbb{A}^2/G$ is normal, the natural immersion $U \hookrightarrow X$ extends to a morphism $\mathbb{A}^2/G \to X$ which must be an open immersion by the Zariski Main Theorem. In the case (ii), since X is affine, the $\mathbb{A}^1$-fibration ρ° extends to an $\mathbb{A}^1$-fibration $\rho : X \to C$, where C contains C° as an open set and is isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$ because X is dominated by $\mathbb{A}^2$.

(3) If X is smooth, it follows that $\overline{\kappa}(X) = -\infty$ (cf. [9, Lemma 1.14.1 in Chap. 2]). Then we can argue in the same way as in the assertion (2) with X° replaced by X.

(4) Suppose that X contains an open set $\mathbb{A}^2/G$. If $X \neq \mathbb{A}^2/G$, let C be an irreducible component of $X - \mathbb{A}^2/G$. Since X is $\mathbb{Q}$-factorial, there exists an integer $N > 0$ such that NC is defined by an element f of A. Then f is invertible on

$\mathbb{A}^2/G$. Since there is a finite morphism $\pi : \mathbb{A}^2 \to \mathbb{A}^2/G$, the element $\pi^*(f)$ is an invertible element on $\mathbb{A}^2$, which is a constant. This is absurd. So, $X \cong \mathbb{A}^2/G$. Suppose next that X has an $\mathbb{A}^1$-fibration $\rho : X \to C$ which we may assume to be surjective. Then $C \cong \mathbb{A}^1$, for otherwise X would have positive Picard number and therefore X would not be $\mathbb{Q}$-factorial. If ρ has reducible fibers then X has again positive Picard number, which contradicts the $\mathbb{Q}$-fatoriality of X. ☐

We can strengthen the assertion (4) in Lemma 1.2 by the following result.

Lemma 1.3. *Let X be a normal affine surface with an $\mathbb{A}^1$-fibration $\rho : X \to C$, where $C \cong \mathbb{A}^1$. Suppose that there exists a dominant morphism $p : \mathbb{A}^2 \to X$ and that X is $\mathbb{Q}$-factorial. Then X is either the affine plane $\mathbb{A}^2$ or a log affine pseudo-plane of type $d > 1$.*

Proof. By the assertion (4) of Lemma 1.2, every fiber of ρ is irreducible. If there is no multiple fiber, then X is smooth and hence isomorphic to $\mathbb{A}^2$. Otherwise, let $d_1F_1, \ldots, d_sF_s$ be all multiple fibers of ρ. Since $p : \mathbb{A}^2 \to X$ is dominant, there exists a general line ℓ on $\mathbb{A}^2$ such that the image of ℓ by p lies horizontally along the fibration ρ. If $s \geqslant 2$ this is impossible by [10, Lemma 2.4]. So, $s = 1$ and we are done. ☐

Now we can state the following result.

Theorem 1.1. *Let X be a $\mathbb{Q}$-factorial affine surface and let A be the coordinate ring of X. Then the following conditions are equivalent.*

(1) *X is isomorphic to the affine plane, $\mathbb{A}^2/G$ with a small finite subgroup G of $\mathrm{GL}\,(2, \mathbb{C})$ or a log affine pseudo-plane of type $d > 1$.*
(2) *There exists a surjective quasi-finite morphism $p : \mathbb{A}^2 \to X$.*
(3) *The ring A is a pure subalgebra of a polynomial ring $\mathbb{C}[x, y]$ with a surjective quasi-finite morphism $p : \mathbb{A}^2 \to X$.*
(4) *There exists a quasi-finite morphism $p : \mathbb{A}^2 \to X$.*

Proof. (1) $\Longrightarrow$ (2). For the case $X \cong \mathbb{A}^2$, the assertion is obvious. For the case $X \cong \mathbb{A}^2/G$, the quotient morphism $q : \mathbb{A}^2 \to \mathbb{A}^2/G$ will do. For the case X is a log affine pseudo-plane, we refer to [10, Lemma 2.1]. In fact, we can take p to be a surjective étale morphism.

(2) $\Longrightarrow$ (3). Let $\mathbb{C}[x, y]$ be the coordinate ring of $\mathbb{A}^2$ and let $\varphi : A \hookrightarrow \mathbb{C}[x, y]$ be the homomorphism associated to p. Then φ is a pure embedding by [3, Lemma 2.2].

(3) $\Longrightarrow$ (4). This is obvious.

(4) $\Longrightarrow$ (1). This follows from Lemmas 1.2 and 1.3. ☐

The following is a fundamental question concerning pure subalgebras.

Problem 1.1. *Let A be a pure subalgebra of an affine normal domain with $Q(B)$ algebraic over $Q(A)$. Is the associated morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ a quasi-finite morphism ?*

Hereafter we consider the affine plane as a log affine pseudo plane of type $d = 1$. In view of the problem, we can pose the following.

Conjecture 1.1. *Let $\varphi : A \hookrightarrow B = \mathbb{C}[x, y]$ be a pure embedding with $\dim A = 2$. If A is $\mathbb{Q}$-factorial then either $X \cong \mathbb{A}^2/G$ or X is a log affine pseudo-plane of type d.*

In the smooth case, we have the following algebraic characterization of an affine pseudo-plane.

Theorem 1.2. *Let X be a $\mathbb{Q}$-factorial, smooth affine surface. Then X is an affine pseudo-plane of type d if and only if there exists a dominant morphism $p : \mathbb{A}^2 \to X$.*

Proof. If X is an affine pseudo-plane of type d, it follows from Theorem 1.1 that there exists a dominant morphism $p : \mathbb{A}^2 \to X$. Suppose that there exists a dominant morphism p. By Lemma 1.2, X is isomorphic to $\mathbb{A}^2/G$ or X has an $\mathbb{A}^1$-fibration $\rho : X \to C$ with $C \cong \mathbb{A}^1$ or $\mathbb{P}^1$. Since X is smooth, the case $X \cong \mathbb{A}^2/G$ does not take place. Since X is $\mathbb{Q}$-factorial, C is isomorphic to $\mathbb{A}^1$ and ρ has only irreducible fibers. By Lemma 1.3, X is an affine pseudo-plane of type d, where we understand that $X \cong \mathbb{A}^2$ if $d = 1$. □

2. Cancellation problem for affine pseudo-planes

Affine pseudo-planes have geometric structures which are quite close to the affine plane. Since the affine plane has the cancellation property, it is interesting to ask whether the affine pseudo-planes have the same property. We begin with the following result.

Lemma 2.1. *Let A be a noetherian normal domain and let $A[x_1, \ldots, x_n]$ be a polynomial ring over A. Then we have:*

(1) *The natural injection $A \hookrightarrow A[x_1, \ldots, x_n]$ induces an isomorphism between the divisor class groups $\mathrm{C}\ell\,(A)$ and $\mathrm{C}\ell\,(A[x_1, \ldots, x_n])$.*
(2) *A is $\mathbb{Q}$-factorial if and only if so is $A[x_1, \ldots, x_n]$.*

Proof. (1) By induction on n, it suffices to verify the assertions in the case $n = 1$. Let $\mathfrak{p}$ be a prime ideal of A of height 1. Then $\mathfrak{p}A[x]$ is a prime ideal of $A[x]$ of height 1. Suppose that $\mathfrak{p}A[x]$ is principal. Then $\mathfrak{p}A[x] = f(x)A[x]$ for $f(x) \in A[x]$. For a nonzero element $a \in \mathfrak{p}$, we have $a = f(x)g(x)$ for some $g(x) \in A[x]$. This implies that $f(x) \in A$. Set $f(x) = f$. It is now clear that $\mathfrak{p} = fA$. So, the natural homomorphism $\mathrm{C}\ell\,(A) \to \mathrm{C}\ell\,(A[x])$ is injective. On the other hand, let $S = A - 0$ and $K = Q(A)$. Then S is a multiplicatively closed subset of $A[x]$ and $S^{-1}A[x] = K[x]$. Note that $\mathrm{C}\ell\,(S^{-1}A[x])$ is generated by prime ideals $\mathfrak{P}$ of $A[x]$ of height one such that $\mathfrak{P} \cap S = \emptyset$. Consider the natural homomorphism $\pi : \mathrm{C}\ell\,(A[x]) \to \mathrm{C}\ell\,(K[x])$. Since $\mathrm{C}\ell\,(K[x]) = (0)$, $\mathrm{C}\ell\,(A[x]) = \mathrm{Ker}\,\pi$. Hence, for any prime ideal $\mathfrak{P}$ of $A[x]$ of height 1 which represents a non-zero class of $\mathrm{C}\ell\,(A[x])$, we have $S \cap \mathfrak{P} \neq \emptyset$. Let $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\mathfrak{p}$ is a non-zero prime ideal of A and $\mathfrak{p}A[x] \subseteq \mathfrak{P}$. Since $\text{ht}(\mathfrak{P}) = 1$, we have $\mathfrak{P} = \mathfrak{p}A[x]$. This implies that $\text{C}\ell(A) \cong \text{C}\ell(A[x])$.

(2) This is straightforward from the assertion (1). □

We need the following result to proceed further.

Lemma 2.2. *Let X be a normal affine surface with one singular point $\overline{O}$. Suppose that there exists a surjective quasi-finite morphism $p : \mathbb{A}^2 \to X$ and that there is given an isomorphism $\theta : X \times \mathbb{A}^n \xrightarrow{\sim} Y \times \mathbb{A}^n$ for an algebraic variety Y. Then the following assertions hold.*

(1) *Y is a normal affine surface with one singular point.*
(2) *If $n = 1$, there exists a quasi-finite morphism $q : \mathbb{A}^2 \to Y$.*

Proof. (1) It is clear that Y is a normal affine surface. Hence Y has finitely many isolated singular points, say $Q_1, \ldots, Q_s$. Since X has a unique singular point $\overline{O}$, the singular locus of $X \times \mathbb{A}^n$ is $\{\overline{O}\} \times \mathbb{A}^n$. Since the singular locus of $Y \times \mathbb{A}^n$ is the disjoint union $\coprod_{i=1}^s \{Q_i\} \times \mathbb{A}^n$, it follows that Y has a unique singular point Q and $\{\overline{O}\} \times \mathbb{A}^n$ is mapped isomorphically onto $\{Q\} \times \mathbb{A}^n$ under the isomorphism θ.

(2) Consider the given morphism $p : \mathbb{A}^2 \to X$. Let O be a point of $\mathbb{A}^2$ such that $p(O) = \overline{O}$. We consider the point O as the origin of a certain coordinate system $\{x_1, x_2\}$ on $\mathbb{A}^2$. Let L be the linear subspace $L := \{O\} \times \mathbb{A}^n$ in the affine space $\mathbb{A}^2 \times \mathbb{A}^n \cong \mathbb{A}^{n+2}$ which surjects to the space $\overline{L} := \{\overline{O}\} \times \mathbb{A}^n$ in $X \times \mathbb{A}^n$ via $\widetilde{p} := p \times 1_{\mathbb{A}^n}$. Let $W(2, n+2)$ be the set of all linear planes in $\mathbb{A}^{n+2}$. Let $x_1, x_2, \ldots, x_{n+2}$ be coordinates of $\mathbb{A}^{n+2}$ and let $X_0, \ldots, X_{n+2}$ be homogeneous coordinates of $\mathbb{P}^{n+2}$ when $\mathbb{A}^{n+2}$ is embedded into $\mathbb{P}^{n+2}$ in such a way that $x_i = X_i/X_0$ for $1 \leqslant i \leqslant n+2$. Let P be a linear plane of $\mathbb{A}^{n+2}$. Then P is defined by n equations

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n+2}x_{n+2} + a_{10} & = & 0 \\ \cdots \quad \cdots \quad \cdots & & \\ a_{n1}x_1 + \cdots + a_{nn+2}x_{n+2} + a_{n0} & = & 0 \end{cases},$$

or equivalently

$$\begin{cases} a_{10}X_0 + a_{11}X_1 + \cdots + a_{1n+2}X_{n+2} & = & 0 \\ \cdots \quad \cdots \quad \cdots & & \\ a_{n0}X_0 + a_{n1}X_1 + \cdots + a_{nn+2}X_{n+2} & = & 0 \end{cases}.$$

Since $P \subset \mathbb{A}^{n+2}$, we have $\text{rank}\, A = \text{rank}\, \widetilde{A} = n$, where

$$A = (a_{ij})_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n+2}} \quad \text{and} \quad \widetilde{A} = (a_{ij})_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant n+2}}.$$

Note that $P \subset \mathbb{P}^{n+2} \backslash \mathbb{A}^{n+2}$ if and only if $\text{rank} A < \text{rank} \widetilde{A}$. Then the set $W(2, n+2)$ is bijectively coordinated by $\binom{n+3}{n}$ minors

$$\det \begin{vmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_n} \\ a_{2i_1} & a_{2i_2} & \cdots & a_{2i_n} \\ & \cdots & \cdots & \\ a_{ni_1} & a_{ni_2} & \cdots & a_{ni_n} \end{vmatrix}$$

where $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant n+2$. Thus, $W(2, n+2)$ identified with the projective space having the above coodinates has dimension

$$\dim W(2, n+2) = \binom{n+3}{n} - 1 = \frac{1}{6}(n+1)(n+2)(n+3) - 1.$$

On the other hand, for any point $y \in Y$, let $W'(y)$ be the subset of $W(2, n+2)$ consisting of linear planes P such that $\dim(P \cap (\theta \cdot \widetilde{p})^{-1}(\{y\} \times \mathbb{A}^n) > 0$. When $n = 1$, for the existence of a desired linear plane P in $\mathbb{A}^{n+2}$ with an induced quasi-finite morphism $P \to Y$, we need to prove that

$$\dim \bigcup_{y \in Y} W'(y) < \dim W(2, n+2)$$

which seems to be valid even if $n > 1$ though we could not prove it. So, assume that $n = 1$. Suppose that an irreducible component of $(\theta \cdot \widetilde{p})^{-1}(\{y\} \times \mathbb{A}^1)$ with a general point $y \in Y$ is contained in two distinct linear planes P, P'. Then the component is a linear line ℓ in $\mathbb{A}^3 = \mathbb{A}^2 \times \mathbb{A}^n$ with $n = 1$. Hence one irreducible component of $(\theta \cdot \widetilde{p})^{-1}(\{y\} \times \mathbb{A}^1)$ for every $y \in Y$ is contained in a linear plane and parallel to the line ℓ. Hence those linear planes when y moves in Y form a two-dimensional family. Let $\mathcal{F}$ be the set of linear planes P satisfying one of the following conditions:

(i) P contains an irreducible component of $(\theta \cdot \widetilde{p})^{-1}(\{y\} \times \mathbb{A}^1)$ with a general point $y \in Y$ but does not share the component with other linear planes;
(ii) P contains an irreducible component of $(\theta \cdot \widetilde{p})^{-1}(\{y\} \times \mathbb{A}^1)$ for a special point $y \in Y$ which is a linear line.

Then every irreducible component of $\mathcal{F}$ has dimension at most two. Since $\dim W(2, 3) = 3$, we find a linear plane P which contains no irreducible components of $(\theta \cdot \widetilde{p})^{-1}(\{y\} \times \mathbb{A}^1)$ for all points $y \in Y$. Then the projection $p_Y : \mathbb{A}^3 \to Y$ restricts to a quasi-finite morphism $(p_Y) |_P : P \to Y$. $\square$

Given a normal algebraic variety X, we consider the *quasi-universal covering* of X when $\pi_1(X^\circ)$ is a finite group, where X° is the smooth part of X. Let $\widetilde{X^\circ}$ be the universal covering of X° which is a smooth algebraic variety since $\pi_1(X^\circ)$ is finite. Let $\widetilde{X}$ be the normalization of X in the function field of $\widetilde{X^\circ}$. We call $\widetilde{X}$ together with the normalization morphism $\pi : \widetilde{X} \to X$ the quasi-universal covering of X. The fundamental group $G := \pi_1(X^\circ)$ acts on $\widetilde{X}$ and X is the algebraic quotient $\widetilde{X}//G$.

The following result shows that the cancellation holds in the class of log affine pseudo-planes of type d but does not hold individually upto isomorphisms.

Theorem 2.1. *Let X be a log affine pseudo-plane of type d. Suppose we have an isomorphism $X \times \mathbb{A}^n \cong Y \times \mathbb{A}^n$ for an algebraic variety Y. Suppose further that either X is smooth and n arbitrary or X is singular and $n = 1$. Then Y is a log affine pseudo-plane of type d. But X is not necessarily isomorphic to Y.*

Proof. It is clear that Y is a normal affine surface and Y is smooth if so is X. By Lemma 2.1, Y is $\mathbb{Q}$-factorial since

$$\mathrm{C}\ell\,(A) \cong \mathrm{C}\ell\,(A[x_1,\ldots,x_n]) \cong \mathrm{C}\ell\,(B[y_1,\ldots,y_n]) \cong \mathrm{C}\ell\,(B),$$

where A and B are respectively the coordinate rings of X and Y. On the other hand, by Theorem 1.1, there exists a surjective quasi-finite morphism $p : \mathbb{A}^2 \to X$. Hence $p \times 1_{\mathbb{A}^n} : \mathbb{A}^2 \times \mathbb{A}^n = \mathbb{A}^{n+2} \to X \times \mathbb{A}^n \cong Y \times \mathbb{A}^n$ composed with the projection onto Y induces a dominant morphism $q : \mathbb{A}^{n+2} \to Y$. If Y is smooth, Y is an affine pseudo-plane of type d by Theorem 1.2. If Y is singular and $n = 1$, we can take a linear plane P of $\mathbb{A}^3$ such that the restriction $q\,|_P \colon P \to Y$ is quasi-finite by Lemma 2.2. By Theorem 1.1, either Y is isomorphic to $\mathbb{A}^2/G$ or Y is a log affine pseudo-plane of type d. On the other hand, let X° and Y° be the smooth loci of X and Y. Then $X^\circ \times \mathbb{A}^1 \cong Y^\circ \times \mathbb{A}^1$, and hence $\pi_1(X^\circ) \cong \pi_1(Y^\circ)$, which is a cyclic group of order d by the hypothesis. Since $\pi_1(X^\circ) \cong G$, it follows that $\mathbb{A}^2/G$ has an $\mathbb{A}^1$-fibration (cf. [9, Theorem 2.5.1 of Chap. 3]). Then X is a log affine pseudo-plane of type d by Lemma 1.3. For the last assertion, we have an example of affine pseudo-planes X and Y which satisfy $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$ but $X \not\cong Y$ (see [6, Theorem 2.17]). ☐

By the same argument as in Theorem 2.1, we can prove the following result.

Theorem 2.2. *Let X be isomorphic to $\mathbb{A}^2/G$ with a small finite subgroup G of $\mathrm{GL}\,(2,\mathbb{C})$. Suppose that $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$ and that G is not a cyclic group. Then Y is isomorphic to X.*

Proof. By Lemma 2.2 and Theorem 1.1, either Y is isomorphic to $\mathbb{A}^2/G'$ for a small finite subgroup G' of $\mathrm{GL}\,(2,\mathbb{C})$ or Y is a log affine pseudo-plane of type $d > 1$. Since $\pi_1(X^\circ) \cong \pi_1(Y^\circ) \cong G$ as in the proof of Theorem 2.1 and since G is not cyclic by the hypothesis, Y is not a log affine pseudo-plane and Y is isomorphic to $\mathbb{A}^2/G$ as we have $G \cong G'$. In order to show that Y is isomorphic to X, we have to show that the linear representation ρ_G of G on $\mathbb{A}^2$ and that $\rho_{G'}$ of G' on $\mathbb{A}^2$ is the same upto an automorphism of $\mathbb{A}^2$. For this purpose, let X° and Y° be respectively the smooth parts of X and Y. Here $X \setminus X^\circ$ and $Y \setminus Y^\circ$ consist of single points $\overline{O}_X$ and $\overline{O}_Y$. Then $\{\overline{O}_X\} \times \mathbb{A}^1 \cong \{\overline{O}_Y\} \times \mathbb{A}^1$ is the singular locus of $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$. In particular, we have

$$G \cong \pi_1(X^\circ \times \mathbb{A}^1) \cong \pi_1(Y^\circ \times \mathbb{A}^1) \cong G'.$$

Let $\widetilde{X}$ and $\widetilde{Y}$ be the quasi-universal coverings of X and Y. Then $\widetilde{X} \cong \widetilde{Y} \cong \mathbb{A}^2$ and $\widetilde{X} \times \mathbb{A}^1 \cong \widetilde{Y} \times \mathbb{A}^1 \cong \mathbb{A}^3$. Since the induced actions of G and G' on $\mathbb{A}^3$ are

$\rho_G \oplus 1$ and $\rho_{G'} \oplus 1$ with the trivial representation 1 of G and G' on $\mathbb{A}^1$, a theorem of Krull-Schmidt in the representation theory implies that ρ_G and $\rho_{G'}$ are the same upto an automorphism of $\mathbb{A}^2$. Thence it follows that $X \cong Y$. □

The following result gives a criterion in terms of the Makar-Limanov invariant for a log affine pseudo-plane to be isomorphic to $\mathbb{A}^2/G$.

Theorem 2.3. *We have the following assertions.*

(1) *Let X be isomorphic to the quotient surface $\mathbb{A}^2/G$, where G is a cyclic group of order d. Then the Makar-Limanov invariant* ML (X) *is trivial.*
(2) *Let X be a singular, affine pseudo-plane of type $d > 1$. Then X is isomorphic to $\mathbb{A}^2/G$ if and only if X has the trivial Makar-Limanov invariant.*

Proof. (1) Suppose that $X \cong \mathbb{A}^2/G$ with a cyclic group G of order $d > 1$. Identify G with the group of all d-th roots of unity in $\mathbb{C}$. Then the G-action on $\mathbb{A}^2$ is given by ${}^{\zeta}(x, y) = (\zeta x, \zeta^q y)$ for $\zeta \in G$, where $q < d$ and $\gcd(d, q) = 1$. Then the coordinate ring A of X is given as

$$A = \mathbb{C}[x, y] \cap \mathbb{C}(x^d, y^d, y/x^q).$$

Let $u = x^d, v = y^d$ and $w = x^{d-q}y = u \cdot (y/x^q)$. Then $A[u^{-1}] = \mathbb{C}[u, u^{-1}, u^{-1}w]$. Hence $\delta = u^a \frac{\partial}{\partial w}$ with $a \gg 0$ defines a locally nilpotent derivation on A (cf. [9, p.219]). On the other hand, since $\gcd(d, q) = 1$, we find a positive integer q' so that $qq' \equiv 1 \pmod d$. Let ζ be a primitive d-th root of unity. Then $\zeta' := \zeta^{q'}$ is also primitive, and the ${}^{\zeta'}(x, y) = (\zeta^{q'} x, \zeta y)$. Hence $w' = xy^{d-q'} = (x/y^{q'})v \in A$, and $A[v^{-1}] = \mathbb{C}[v, v^{-1}, v^{-1}w']$. So, $\delta' = v^b \frac{\partial}{\partial w'}$ with $b \gg 0$ is a locally nilpotent derivation on A which is algebraically independent of δ. Hence ML $(X) = \mathbb{C}$.

(2) It suffices to show the "if "part. Suppose that ML$(X) = \mathbb{C}$. Let $\rho : X \to C$ be the $\mathbb{A}^1$-fibration with which X has a structure of singular affine pseudo-plane of type $d > 1$. Then there exists a normal projective surface V and a $\mathbb{P}^1$-fibration $\overline{\rho} : V \to \overline{C}$ such that the following conditions are satisfied.

(i) X is an open set of V and $D := V - X$ is a divisor with simple normal crossings. We may assume that the embedding $X \hookrightarrow V$ is minimal, i.e., D contains no (-1) curves which contract to smooth points without breaking the property of D being a divisor with simple normal crossings.
(ii) The restriction of $\overline{\rho}$ onto X is the given $\mathbb{A}^1$-fibration ρ.
(iii) The curve $\overline{C}$ is isomorphic to $\mathbb{P}^1$.

By [2, Theorem 2.9], we can assume that D is a linear chain. Let dF_0 be the unique multiple fiber of ρ and let Φ_0 be the fiber of $\overline{\rho}$ containing dF_0. Write $\Phi_{0,\mathrm{red}} = \Gamma + \overline{F}_0$, where $\overline{F}_0$ is the closure of F_0 in V. Let P be the unique singular point of X which lies on F_0. Let $\sigma : \widetilde{V} \to V$ be the minimal resolution of singularity at P and let $\Delta = \sigma^{-1}(P)$ be the exceptional divisor. The composite $\widetilde{\rho} = \overline{\rho} \cdot \sigma$ is a $\mathbb{P}^1$-fibration on $\widetilde{V}$ and $\sigma'(\Gamma + \overline{F}_0) + \Delta$ supports a degenerate fiber $\widetilde{\Phi}_0$ of $\widetilde{\rho}$, where $\sigma'(\cdot)$ signifies the proper transform. Let $\widetilde{F}_0 = \sigma'(\overline{F}_0)$ and $\widetilde{\Gamma} = \sigma'(\Gamma)$. Note that Δ is a linear chain with one end component meeting $\widetilde{F}_0$ and that $\widetilde{\Gamma}$

is a linear chain. Furthermore, $\widetilde{F_0}$ is the unique (-1) component of $\widetilde{\Phi}_0$ and its multiplicity is d.

Now we consider the quasi-universal covering of X. Namely, we consider the universal covering $\widetilde{X^\circ}$ of $X^\circ := X \setminus \{P\}$ which is in fact a finite covering. The quasi-universal covering $\widetilde{X}$ of X is the normalization of X in the function field of $\widetilde{X^\circ}$. The surface $\widetilde{X^\circ}$ is obtained as the normalization of $X^\circ \times_C C'$, where $\nu : C' \to C$ is a d-th cyclic covering of C ramifying totally over the point $\rho(F_0)$. This process of producing $\widetilde{X}$ corresponds to the process of taking the fiber product $(\widetilde{V}, \widetilde{\rho}) \times_{\overline{C}} (\overline{C'}, \overline{\nu})$, where $\overline{\nu} : \overline{C'} \to \overline{C}$ is a d-th cyclic covering ramifying totally over the point $\rho(F_0)$ and the point at infinity $\overline{C} \setminus C$, taking the normalization of the fiber product and finally resolving minimally the singularities of the obtained normal surface. Let W be a smooth projective surface obtained in this manner and let $\mu : W \to \widetilde{V}$ be the natural morphism. By a general theory of d-th cyclic coverings of the above type, the component $\widetilde{F_0}$ does not ramify and the restriction $\mu : \mu'(\widetilde{F_0}) \to \widetilde{F_0}$ induced by the morphism μ ramifies totally over the points $\widetilde{F_0} \cap \Delta$ and $\widetilde{F_0} \cap \Gamma$, where $\mu'(\widetilde{F})_0$ is the induced proper transform of $\widetilde{F_0}$. Furthermore, $\mu'(\widetilde{F_0})$ has multiplicity one in the degenerate fiber $\mu^*(\widetilde{\Phi}_0)$ of the induced $\mathbb{P}^1$-fibration $\widetilde{\rho} \cdot \mu : W \to \overline{C}$. Since the degenerate fiber $\mu^*(\widetilde{\Phi}_0)$ can be contracted to a smooth fiber which is the image of the component $\mu'(\widetilde{F_0})$, it follows that $\mu^{-1}(\Delta)$ contracts to a smooth point on $\mu'(\widetilde{F_0})$. Hence $\widetilde{X}$ is isomorphic to the affine plane $\mathbb{A}^2$. Since $X \cong \widetilde{X}/G$ with $G = \pi_1(X^\circ) \cong \mathbb{Z}/d\mathbb{Z}$, it follows that X is isomorphic to $\mathbb{A}^2/G$. □

We shall now prove Theorem 2.2 in the case where G is a finite cyclic group.

Theorem 2.4. *Let X be isomorphic to $\mathbb{A}^2/G$ with a small finite subgroup G of* $\mathrm{GL}\,(2, \mathbb{C})$. *Suppose that $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$ and that G is a cyclic group. Then $X \cong Y$.*

Proof. As in the proof of Theorem 2.2, either Y is isomorphic to $\mathbb{A}^2/G'$ or Y is a singular affine pseudo-plane of type $d > 1$. With the same notations there, we have $\widetilde{X} \times \mathbb{A}^1 \cong \widetilde{Y} \times \mathbb{A}^1$, where $\widetilde{X} \cong \mathbb{A}^2$. By the cancellation theorem for $\mathbb{A}^2$, we have $\widetilde{Y} \cong \mathbb{A}^2$. Since $\pi_1(Y^\circ) \cong \pi_1(Y^\circ \times \mathbb{A}^1) \cong \pi_1(X^\circ \times \mathbb{A}^1) \cong G$, it follows that G' is isomorphic to G and that the group G acts on $\widetilde{Y} \cong \mathbb{A}^2$ as the same linear representation ρ_G upto an automorphism of $\mathbb{A}^2$. Since $Y \cong \widetilde{Y}/\!/G$, we have $Y \cong X$. □

Remark 1. In Theorem 2.1, when X is a log affine pseudo-plane, the condition $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$ does not necessarily imply $X \cong Y$. One of the reasons for this phenomenon is the following. With the above notations, we have $\widetilde{X} \times \mathbb{A}^1 \cong \widetilde{Y} \times \mathbb{A}^1$, while $\widetilde{X}$ is isomorphic to either $\mathbb{A}^2$ or a Danielewski surface. In the latter case, $\widetilde{X}$ is not necessarily isomorphic to $\widetilde{Y}$.

Remark 2. In [1, Theorems 4.2 and 4.3], given a finite morphism $\varphi : X \to Y$ of smooth affine surfaces, it is proved that Y has the trivial Makar-Limanov invariant provided so does X if φ satisfies one of the following conditions.

(1) φ is étale.
(2) φ is a Galois (possibly ramified) covering.
(3) X has the Picard number $\rho(X) = 0$.

It is most plausible that the same result holds when we replace X and Y by normal affine surfaces with quotient singularities, which we call *log affine surfaces* and an étale covering by a finite covering such that $X^\circ = \varphi^{-1}(Y^\circ)$ and $\varphi|_{X^\circ}: X^\circ \to Y^\circ$ is étale.

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School of Science and Technology
Kwansei Gakuin University
2-1 Gakuen, Sanda, Hyogo 669–1337, Japan
E-mail address: miyanisi@ksc.kwansei.ac.jp