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ON ZALCMAN COMPLEX SPACES AND NOGUCHI-TYPE CONVERGENCE-EXTENSION THEOREMS FOR HOLOMORPHIC MAPPINGS INTO WEAKLY ZALCMAN COMPLEX SPACES

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Abstract. In this article, some Noguchi-type convergence-extension theorems for holomorphic mappings into weakly Zalcman complex subspaces for a complex space are given. Moreover, the tautness of unbounded domains of a complex space with non-compact automorphism groups is investigated from the viewpoint of the theory of Zalcman complex spaces.

1. Introduction

The convergence-extension theorems of Noguchi-type have received much attention in the last few decades, and they are related to many problems in hyperbolic complex analysis and pluripotential theory (see the reference in [2], [13], [14] for the development in related subjects). More precisely, a "Noguchi-type convergence-extension theorem" means a theorem on mappings analogous to the theorem of Noguchi of extending holomorphic mappings [7, Thm.1.6.24], which would keep the local uniform convergence. In this paper we initiate the study of these problems from the viewpoint of the theory of Zalcman complex spaces

The notion of Zalcman complex spaces is introduced in [15]. At the same time, some important classes of Zalcman complex spaces are also given there. We think that Zalcman complex spaces have nice properties and are an useful subject to find new Noguchi-type convergence-extension theorems.

Modifying the above-mentioned notion, in this article, we introduce the notion of weakly Zalcman complex subspaces for a complex space and show some Noguchi-type convergence-extension theorems for holomorphic mappings into weakly Zalcman complex subspaces for a complex space. More precisely, we determine when the restricted mapping $R : Hol(M, Y) \to Hol(M \setminus A, X)$ is homeomorphic in the compact-open topology, where X is a weakly Zalcman complex subspace for a complex space Y and A is a complex hypersurface of a complex manifold M. Different from the approachs in [8], [4], [5], we use the weak-disc convexity of complex spaces. This tool was used in [13], [14].

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2. On Zalcmanness of complex spaces

First of all, we recall some definitions (see [15])

Definition 2.1. A family $\mathcal{F}$ of holomorphic maps from a complex space X to a complex space Y is said to be normal if $\mathcal{F}$ is relatively compact in $Hol(X, Y)$ in the compact-open topology.

Definition 2.2. Let X, Y be complex spaces and $\mathcal{F} \subset Hol(X, Y)$.

(i) A sequence $\{f_j\} \subset \mathcal{F}$ is compactly divergent if for every compact set $K \subset X$ and for every compact set $L \subset Y$ there is a number $j_0 = j(K, L)$ such that $f_j(K) \cap L = \emptyset$ for all $j \geqslant j_0$.

(ii) The family $\mathcal{F}$ is said to be not compactly divergent if $\mathcal{F}$ contains no compactly divergent subsequences.

Definition 2.3. Let X be a complex space. Let Δ be the open unit disc in $\mathbf{C}$.

The complex space X is said to be *a Zalcman space* if X satisfies the following:

For each non-normal family $\mathcal{F} \subset Hol(\Delta, X)$ such that $\mathcal{F}$ is not compactly divergent, then there exist sequences $\{p_j\} \subset \Delta$ with $\{p_j\} \to p_0 \in \Delta$, $\{f_j\} \subset \mathcal{F}$, $\{\rho_j\} \subset \mathbf{R}$ with $\rho_j > 0$ and $\{\rho_j\} \to 0^+$ such that

$$g_j(\xi) = f_j(p_j + \rho_j \xi), \quad \xi \in \mathbf{C},$$

converges uniformly on compact subsets of $\mathbf{C}$ to a non-constant holomorphic mapping $g : \mathbf{C} \to X$.

It is conventional that the taut space is Zalcman.

Examples 2.1. 1. By a theorem of [14,Thm 2.8], it follows that every compact complex space is a Zalcman space.

2. Let X be a compact complex space. Let H be a hyperbolic complex hypersurface of X. Then $X \setminus H$ is a Zalcman space. In particular, $\mathbf{C} = \mathbf{CP}^1 \setminus \{\text{one point}\}$ is a Zalcman space. For more details, see [15].

3. If X_1 is a taut space and X_2 is a Zalcman space, then $X_1 \times X_2$ is also Zalcman. Indeed, $\{f_j = (f_j^1, f_j^2)\} \subset Hol(\Delta, X_1 \times X_2)$ such that $\{f_j\}$ is not normal on Δ and is not compactly divergent on Δ. Then it is easy to see that $\{f_j^k\}$ is also not compactly divergent on Δ $(k = 1, 2)$. Since X_1 is taut, $\{f_j^1\}$ is normal on Δ. Thus $\{f_j^2\}$ is not normal on Δ. Without loss of generality we may assume that $\{f_j^1\} \to f^1$ in $Hol(\Delta, X_1)$. Since X_2 is Zalcman, without loss of generality we may assume that there exist sequences $\{p_j\} \subset \Delta$ with $\{p_j\} \to p_0 \in \Delta$, $\{\rho_j\} \subset \mathbf{R}$ with $\rho_j > 0$ and $\{\rho_j\} \to 0^+$ such that

$$g_j^2(\xi) = f_j^2(p_j + \rho_j \xi), \quad \xi \in \mathbf{C},$$

converges uniformly on compact subsets of $\mathbf{C}$ to a non-constant entire function $g^2 : \mathbf{C} \to X_2$. Then $\{g_j^1(\xi) = f_j^1(p_j + \rho_j \xi)\}, \xi \in \mathbf{C}$, converges also uniformly on compact subsets of $\mathbf{C}$ to a constant function $g^1 = f^1(p_0)$. The claim is proved.

4. If X_1 is a compact space and X_2 is a Zalcman space, then $X_1 \times X_2$ is also Zalcman. Indeed, assume that $\{f_j = (f_j^1, f_j^2)\} \subset Hol(\Delta, X_1 \times X_2)$ such that $\{f_j\}$ is not normal on Δ and is not compactly divergent on Δ. Then it is easy to see that $\{f_j^2\}$ is also not compactly divergent on Δ. Consider two cases.
Case 1. $\{f_j^2\}$ is normal on Δ.
Then $\{f_j^1\}$ is not normal on Δ. Without loss of generality, we can assume that $\{f_j^2\} \to f^2$ in $Hol(\Delta, X_2)$. Since X_1 is compact, without loss of generality we may assume that there exist sequences $\{p_j\} \subset \Delta$ with $\{p_j\} \to p_0 \in \Delta$, $\{\rho_j\} \subset \mathbf{R}$ with $\rho_j > 0$ and $\{\rho_j\} \to 0^+$ such that
$$g_j^1(\xi) = f_j^1(p_j + \rho_j \xi), \quad \xi \in \mathbf{C},$$
converges uniformly on compact subsets of $\mathbf{C}$ to a non-constant entire function $g^1 : \mathbf{C} \to X_1$. Then $\{g_j^2(\xi) = f_j^2(p_j + \rho_j \xi)\}, \xi \in \mathbf{C}$, converges also uniformly on compact subsets of $\mathbf{C}$ to a constant function $g^2 = f^2(p_0)$.
Case 2. $\{f_j^2\}$ is not normal on Δ.
Since X_2 is Zalcman, without loss of generality we may assume that there exist sequences $\{p_j\} \subset \Delta$ with $\{p_j\} \to p_0 \in \Delta$, $\{\rho_j\} \subset \mathbf{R}$ with $\rho_j > 0$ and $\{\rho_j\} \to 0^+$ such that
$$g_j^2(\xi) = f_j^2(p_j + \rho_j \xi), \quad \xi \in \mathbf{C},$$
converges uniformly on compact subsets of $\mathbf{C}$ to a non-constant entire function $g^2 : \mathbf{C} \to X_2$.
Consider the sequence $g_j^1(\xi) := f_j^1(p_j + \rho_j \xi), \quad \xi \in \mathbf{C}$.
(a) If $\{g_j^1\}$ is normal, then without loss of generality we may assume that $\{g_j\} \to g \in Hol(\mathbf{C}, X_1 \times X_2)$, $g \neq constant$.
(b) If $\{g_j^1\}$ is not normal, then by the compactness of X_1, without loss of generality we may assume that there exist sequences $\{p'_j\} \subset \mathbf{C}$ with $\{p'_j\} \to p'_0 \in \mathbf{C}$, $\{\rho'_j\} \subset \mathbf{R}$ with $\rho'_j > 0$ and $\{\rho'_j\} \to 0^+$ such that
$$h_j^1(\xi) = g_j^1(p'_j + \rho'_j \xi), \quad \xi \in \mathbf{C},$$
converges uniformly on compact subsets of $\mathbf{C}$ to a non-constant entire function $h^1 : \mathbf{C} \to X_1$. Then $\{h_j^2(\xi) := g_j^2(p'_j + \rho'_j \xi)\}, \xi \in \mathbf{C}$, converges also uniformly on compact subsets of $\mathbf{C}$ to a constant function $h^2 = g^2(p'_0)$.
This proves that $X_1 \times X_2$ is Zalcman.

We now prove the first result of this section.

Theorem 2.1. *Let M_1, M_2 be two complex spaces. Let $\pi : M_1 \to M_2$ be a holomorphic covering. Then the complex space M_1 is Zalcman if and only if M_2 is also Zalcman.*

Proof. $(\Leftarrow)$ Assume that M_2 is a Zalcman space. Let $\mathcal{F} \subset Hol(\Delta, M_1)$ be such that $\mathcal{F}$ is not normal on Δ and $\mathcal{F}$ is not compactly divergent on Δ.

(i) We show that the familly $\pi \circ \mathcal{F}$ is also not normal on Ω. Indeed, suppose on the contrary that this family is normal on Δ. Let $\{f_n\} \subset \mathcal{F}$. Without loss of generality we may assume that $\{\pi \circ f_n\} \to g \in Hol(\Delta, M_2)$. For each $y \in M_2$, choose a taut neighbourhood U_y of y in M_2. Then $\pi^{-1}(U_y)$ is taut. Put $V_y = g^{-1}(U_y)$ for each $y \in M_2$. Take a countable covering $\{V_i\}_{i=1}^{\infty}$ of Δ such that $V_i \Subset V_y$ for some $y \in M_2$.

Consider the sequence $\{f_n \mid_{V_1}\}$. Without loss of generality we may assume that $f_n(V_1) \subset \pi^{-1}(U_{y_1})$ for each $n \geqslant 1$. Since $\{\pi \circ f_n \mid_{V_1}\} \to g \mid_{V_1}$, it follows that there exists a subsequence $\{f_n^{(1)}\} \subset \{f_n\}$ which is convergent in $Hol(V_1, M_1)$. Consider the sequence $\{f_n^{(1)} \mid_{V_2}\}$. As above, this sequence contains a subsequence $\{f_n^{(2)} \mid_{V_2}\}$ being convergent in $Hol(V_2, M_1)$. Continuing this process we can find sequences $\{f_n^{(k)}\}$ such that $\{f_n^{(k)}\} \subset \{f_n^{(k-1)}\}$ for all $k \geqslant 2$ and $\{f_n^{(k)}\}$ is convergent in $Hol(V_k, M_1)$. Then the sequence $\{f_n^{(n)}\}$ is convergent in $Hol(\Delta, M_1)$. Thus the family $\mathcal{F}$ is normal. This is a contradiction.

(ii) We now show that the family $\pi \circ \mathcal{F}$ is not compactly divergent on Δ. Indeed, suppose on the contrary that there exists a sequence $\{f_n\}$ such that $\{\pi \circ f_n\}$ is compactly divergent. Let K be any compact subset of Δ and L be any compact subset of M_1. Then there is n_0 such that

$$\pi \circ f_n(K) \cap \pi(L) = \emptyset \text{ for all } n \geqslant n_0.$$

Hence $f_n(K) \cap L = \emptyset$ for all $n \geqslant n_0$. This implies that the sequence $\{f_n\}$ is also compactly divergent. This is impossible.

(iii) Since M_2 is a Zalcman space, there exist sequences $\{p_j\} \subset \Delta$ with $\{p_j\} \to p_0 \in \Delta$, $\{f_j\} \subset \mathcal{F}$, $\{\rho_j\} \subset \mathbf{R}$ with $\rho_j > 0$ and $\{\rho_j\} \to 0^+$ such that

$$g_j(\xi) = \pi \circ f_j(p_j + \rho_j \xi), \quad \xi \in \mathbf{C},$$

converges uniformly on compact subsets of $\mathbf{C}$ to a non-constant entire function $g_0 : \mathbf{C} \to M_2$.

Put $\theta_j(\xi) = f_j(p_j + \rho_j \xi), \quad \xi \in \mathbf{C}$. Then

$$\{\pi \circ \theta_j\} \to g_0 \text{ in } Hol(\mathbf{C}, M_2).$$

Repeating the argument in (i), without loss of generality we can assume that $\{\theta_j\} \to \theta_0$ in $Hol(\mathbf{C}, M_1)$. Since $\pi \circ \theta_0 = g_0$, this implies that $\theta_0 \neq$ *constant*. This yields that M_1 is Zalcman.

($\Rightarrow$) Assume that M_1 is a Zalcman space. Let $\{f_j\} \subset Hol(\Delta, M_2)$ be such that $\{f_j\}$ is not normal on Δ and $\{f_j\}$ is not compactly divergent on Δ. Then there exists a sequence $\{z_j\} \subset \Delta$ with $\{z_j\} \to z_0 \in \Delta$ and $\{f_j(z_j)\} \to p \in M_2$.

Let $y_j := f_j(z_j)$ and take $\widetilde{y_j} \in \pi^{-1}(y_j)$. Then there is a holomorphic map $\widetilde{f_j} : \Delta \to M_2$ satisfying

$$\pi \circ \widetilde{f_j} = f_j \quad \text{and} \quad \widetilde{f_j}(z_j) = \widetilde{y_j}.$$

(i) We now show that the sequence $\{\widetilde{f_j}\}$ is not normal on Δ and is not compactly divergent on Δ. Indeed, if the sequence $\{\widetilde{f_j}\}$ is normal on Δ then $\{f_j = \pi \circ \widetilde{f_j}\}$ is also normal on Δ. This is impossible.

Suppose that the sequence $\{\widetilde{f_j}\}$ is compactly divergent on Δ. Let K be any compact subset of Δ and L be any compact subset of M_2. It is easy to see that there exists a compact subset $\widetilde{L}$ of M_1 such that $\pi \circ \widetilde{L} \supset L$. Since $\{\widetilde{f_j}\}$ is compactly divergent on Δ, there is j_0 such that

$$\widetilde{f_j}(K) \cap \widetilde{L} = \emptyset \text{ for all } j \geqslant j_0.$$

Hence $f_j(K) \cap L = \emptyset$ for all $j \geqslant j_0$. This implies that $\{f_j\}$ is also compactly divergent. This is a contradiction.

(ii) Since M_1 is a Zalcman space, without loss of generality we may assume that there exist sequences $\{p_j\} \subset \Delta$ with $\{p_j\} \to p_0 \in \Delta$, $\{\rho_j\} \subset \mathbf{R}$ with $\rho_j > 0$ and $\{\rho_j\} \to 0^+$ such that

$$\widetilde{g}_j(\xi) = \widetilde{f_j}(p_j + \rho_j \xi), \quad \xi \in \mathbf{C},$$

converges uniformly on compact subsets of $\mathbf{C}$ to a non-constant entire function $\widetilde{g}_0 : \mathbf{C} \to M_1$. Hence

$$g_j(\xi) = \pi \circ \widetilde{f_j}(p_j + \rho_j \xi), \quad \xi \in \mathbf{C},$$

converges uniformly on compact subsets of $\mathbf{C}$ to an entire function $g_0 := \pi \circ \widetilde{g}_0$. Since $\widetilde{g}_0 \neq$ *constant*, $g_0 \neq$ *constant*. This implies that M_2 is Zalcman. □

We prove the second result of this section.

Theorem 2.2. *Let X be a complex space. Then X is Zalcman if and only if $\widetilde{S^iX}$ is Zalcman for all $i \geqslant 0$, where $S^0X = X$, $S^1X = S(X)$ is the singular locus of X, and $S^iX = S(S^{i-1}X)$ for all $i \geqslant 2$.*

Proof. ($\Rightarrow$) In order to prove this assertion, we need the following lemma

Lemma 2.1 ([14], Prop. 2.15)**.** *Let M_1, M_2 be two complex spaces. Let $\pi : M_1 \to M_2$ be a proper holomorphic mapping such that $\pi^{-1}(y)$ is hyperbolic for every $y \in M_2$. Then the complex space M_1 is a Zalcman space if so is M_2.*

Let X be a Zalcman space. Then S^iX is Zalcman for every $i \geqslant 0$. By the above mentioned lemma, it follows that $\widetilde{S^iX}$, the normalization of S^iX, is also Zalcman for every $i \geqslant 0$.

($\Leftarrow$) Now assume that $\widetilde{S^iX}$ is Zalcman for every $i \geqslant 0$.

Let $\mathcal{F} \subset Hol(\Delta, X)$ be given such that $\mathcal{F}$ is not normal and $\mathcal{F}$ is not compactly divergent on Δ.

Then there exists a sequence $\{f_n\}$ in $\mathcal{F}$ such that $\{f_n\}$ contains no uniformly convergent subsequences and contains no compactly divergent subsequences (*).

It is easy to see that we can find $i \geqslant 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}(\Delta) \subset S^iX$ but $f_{n_k}(\Delta) \not\subset S^{i+1}X$ for all $k \geqslant 1$.

Consider the commutative diagram:

$$\begin{array}{ccc}
\Delta \times_{S^iX} \widetilde{S^iX} & \xrightarrow{\tilde{f}_{n_k}} & \widetilde{S^iX} \\
\theta_i \big\downarrow & & \big\downarrow \pi_i \\
\Delta & \xrightarrow{f_{n_k}} & S^iX,
\end{array}$$

where $\theta_i : \Delta \times_{S^iX} \widetilde{S^iX} \to \Delta$ is a pull-back bundle of the bundle $\pi_i : \widetilde{S^iX} \to S^iX$ by the holomorphic map f_{n_k}.

Since π_i is finite and proper, so is θ_i. It is easy to see that $\theta_i : \Delta \times_{S^iX} S^iX \to \Delta$ is an analytic covering map. This yields that card $\theta_i^{-1}(z) = 1$ for every $z \in \Delta$. Hence, we deduce that $g_k = \tilde{f}_{n_k} \circ \theta_i^{-1} : \Delta \to \widetilde{S^iX}$ is holomorphic for every $k \geqslant 1$.

Put $\mathcal{G} :=\{g_k\}$

We now prove the following two assertions

(i) $\mathcal{G}$ *is not normal.*

Indeed, assume on the contrary that $\mathcal{G}$ is normal. Then $\mathcal{G}$ contains $\{g_{k_l}\}$ which converges uniformly to a map $G \in Hol(\Delta, \widetilde{S^iX})$ in $Hol(\Delta, \widetilde{S^iX})$. Hence $\{f_{n_{k_l}}\}$ locally uniformly converges to $\pi_i \circ G = F$ in $Hol(\Delta, X)$. This contradicts to (*).

(ii) $\mathcal{G}$ *is not compactly divergent.*

Indeed, assume on the contrary that there exists a compactly divergent subsequence of $\{g_k\}$. Without loss of generality we may assume that $\{g_k\}$ itself is compactly divergent. Let K and L be two compact subsets in Δ and S^iX respectively. Since $\pi_i^{-1}(L)$ is compact, there is k_0 such that

$$g_k(K) \cap \pi_i^{-1}(L) = \emptyset \text{ for all } k > k_0.$$

This implies that $f_{n_k}(K) \cap L = \emptyset$ for all $k > k_0$, and hence the sequence $\{f_{n_k}\}$ is compactly divergent. This is a contradiction.

By (i), (ii) and the Zalcmanness of $\widetilde{S^iX}$, it follows that there exist sequences $\{p_h\} \subset \Delta$ with $\{p_h\} \to p_0 \in \Delta$, $\{g_{k_h}\} \subset \mathcal{G}$, $\{\rho_h\} \subset \mathbf{R}$ with $\rho_h > 0$ and $\{\rho_h\} \to 0^+$ such that

$$\varphi_h(\xi) = g_{k_h}(p_h + \rho_h \xi), \quad \xi \in \mathbf{C},$$

converges uniformly on compact subsets of $\mathbf{C}$ to a non-constant mapping $\varphi : \mathbf{C} \to \widetilde{S^iX}$.

Then $\gamma_h(\xi) = f_{n_{k_h}}(p_h + \rho_h \xi), \quad \xi \in \mathbf{C}$, converges uniformly on compact subsets of $\mathbf{C}$ to a mapping $\gamma : \mathbf{C} \to X$, where $\gamma = \pi_i \circ \varphi$.

Assume that γ is constant, i.e. $\gamma \equiv a$ on $\mathbf{C}$. Then $\varphi(\mathbf{C}) \subset \pi_i^{-1}(a)$ which is a finite set. This is impossible. Thus γ is a non constant mapping.

It follows that X is Zalcman. $\square$

We now give a sufficient condition for tautness of (not necessary bounded) domains in a complex space through the geometrical conditions near boundary points. First of all, we recall the following definitions.

Definition 2.4. [6], [7]

(i) A complex space X is said to be *weakly Brody hyperbolic* if each holomorphic mapping $f : \mathbf{C} \to X$ with $f(\mathbf{C}) \Subset X$ is constant.
(ii) A complex space X is said to be *Brody hyperbolic* if each holomorphic mapping $f : \mathbf{C} \to X$ is constant.

By Liouville theorem, it is easy to see that $\mathbf{C}^n$ is weakly Brody hyperbolic, but is not Brody hyperbolic.

Definition 2.5. Let M be a domain in a complex space X. Let $X^+ = X \cup \{\infty\}$ be the 1-point Alexandrov compactification of X. Denote by $\overline{M}$ the closure of M in X^+. We say that M is unbounded if $\infty \in \overline{M}$. If M is unbounded and ϕ is a function defined on M, we set $\phi(\infty) = c \in \overline{\mathbf{R}}$ if $\lim_{z\to\infty} \phi(z) = c$.

Let M be an unbounded domain in a complex space X.

(i) A function φ is called a local peak plurisubharmonic function at p in $\partial M \cup \{\infty\}$ if there exists a neighbourhood U of p in X^+ such that φ is plurisubharmonic on $U \cap M$ continuous up to $U \cap \overline{M}$ and satisfies

$$\begin{cases} \varphi(p) = 0, \\ \varphi(z) < 0 \quad \text{for all } z \in (U \cap \overline{M}) \setminus \{p\}. \end{cases}$$

(ii) A function ψ is called a local antipeak plurisubharmonic function at p in $\partial M \cup \{\infty\}$ if there exists a neighbourhood U of p in X^+ such that ψ is plurisubharmonic on $U \cap M$ continuous up to $U \cap \overline{M}$ and satisfies

$$\begin{cases} \psi(p) = -\infty, \\ \psi(z) > -\infty \quad \text{for all } z \in (U \cap \overline{M}) \setminus \{p\}. \end{cases}$$

Remark that the existence of an antipeak plurisubharmonic function at a finite point p can always be ensured by setting $\psi(z) = \ln |z - p|$.

Theorem 2.3. *Let M be a domain in a complex space X and $\xi_0 \in \partial M \cup \{\infty\}$. Assume that there are local peak and antipeak plurisubharmonic functions φ and ψ at ξ_0. Moreover, assume that $W \cap M$ is Zalcman for some weakly Brody hyperbolic neighbourhood W of ξ_0 and that there exists a sequence $\{\sigma_p\} \subset Aut(M)$ such that $\lim \sigma_p(x_0) = \xi_0$ for some $x_0 \in M$. Then M is taut.*

In order to prove this theorem, we need the following lemma which is the multi-dimensional version of Gaussier's lemma (see [3], Lemma 2.1.1) .

Lemma 2.2. *Let M be an unbounded domain in a complex space X. Assume that there are local peak and antipeak plurisubharmonic functions φ and ψ at ξ_0 in $\partial M \cup \{\infty\}$. Then for every neighbourhood $\widetilde{U}$ of ξ_0 in X^+ there exists a* *neighbourhood $\widetilde{U}'$ of ξ_0 in X^+ such that every holomorphic map $f : \Delta^N \to M$ satisfies*

$$f(0) \in \widetilde{U}' \Rightarrow f(\Delta^N_{\frac{1}{2}}) \subset \widetilde{U}.$$

Proof. Since φ is a local peak plurisubharmonic function at ξ_0, there exist two neighbourhoods U, V of ξ_0 ($\overline{U} \subset V$) and two positive constants c, c' ($c > c'$) such that :

$$\begin{cases} \inf_{z\in\overline{M}\cap\partial U} & \varphi(z) & = & -c', \\ \\ \sup_{z\in\overline{M}\cap\partial V} & \varphi(z) & = & -c. \end{cases}$$

Then the function $\tilde{\varphi}$ defined on $\overline{M}$ by:

$$\begin{cases} \tilde{\varphi}(z) & = & \varphi(z) & \text{if } z \in \overline{M} \cap U, \\ \tilde{\varphi}(z) & = & \sup(\varphi(z), -(c+c')/2) & \text{if } z \in \overline{M} \cap (V \backslash \overline{U}), \\ \tilde{\varphi}(z) & = & -(c+c')/2 & \text{if } z \in \overline{M} \backslash V, \end{cases}$$

is a global peak plurisubharmonic function at ξ_0.

Let $f : \Delta^N \to M$ be a holomorphic mapping. Assume that α is an arbitrary negative number such that $(\tilde{\varphi} \circ f)(0) > \alpha$. Denote by $mes(E_\alpha)$ the measure of the set

$$E_\alpha = \{\theta = (\theta_1, \theta_2, \cdots, \theta_N) \in [0, 2\pi]^N / (\tilde{\varphi} \circ f)(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_N}) \geqslant 2\alpha\}.$$

Since the function $\tilde{\varphi} \circ f$ is subharmonic, the mean value inequality implies that

$$\begin{aligned} \alpha < (\tilde{\rho} \circ f)(0) &\leqslant \frac{1}{(2\pi)^N} \int_{[0,2\pi]^N} (\tilde{\rho} \circ f)(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_N}) d\theta_1 d\theta_2 \cdots d\theta_N \\ &\leqslant \frac{2\alpha}{(2\pi)^N} mes([0, 2\pi]^N \setminus E_\alpha) = \frac{2\alpha}{(2\pi)^N}((2\pi)^N - mes(E_\alpha)). \end{aligned} \tag{2.1}$$

Thus $mes(E_\alpha) > \frac{(2\pi)^N}{2}$. Take ε small enough such that

$$\begin{cases} \inf_{\overline{M}\cap\partial U} (\varphi + \varepsilon\psi) & = & -c_1 < 0, \\ \\ \sup_{\overline{M}\cap\partial V} (\varphi + \varepsilon\psi) & = & -c_2 < -c_1. \end{cases}$$

The function ρ defined on $\overline{M}$ by

$$\begin{cases} \rho(z) & = & (\varphi + \varepsilon\psi)(z) & \text{if } z \in \overline{M} \cap U, \\ \rho(z) & = & \sup((\varphi + \varepsilon\psi)(z), -(c_1 + c_2)/2) & \text{if } z \in \overline{M} \cap (V \backslash \overline{U}), \\ \rho(z) & = & -(c_1 + c_2)/2 & \text{if } z \in \overline{M} \backslash V, \end{cases}$$

is a continuous negative plurisubharmonic function on M and satisfies $\rho^{-1}(-\infty) = \{\xi_0\}$.

Let $g : \Delta \to M$ be a holomorphic mapping. Using the Poisson integral, for any point λ on $\Delta_{1/2}$ we get

$$(\rho \circ f)(\lambda) \leqslant \frac{1}{2\pi} \int_0^{2\pi} Re(\frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda})(\rho \circ f)(e^{i\theta}) d\theta. \tag{2.2}$$

By a computation, we have

$$\min_{\lambda\in\overline{\Delta}_{1/2}} Re(\frac{e^{i\theta}+\lambda}{e^{i\theta}-\lambda}) = \frac{1}{3}.$$

Thus,

$$(\rho\circ f)(\lambda) \leqslant \frac{1}{6\pi}\int_0^{2\pi}(\rho\circ f)(e^{i\theta})d\theta \text{ for all } \lambda\in\Delta_{1/2}.$$

Let $f:\Delta^N\to M$ be a holomorphic mapping. Then, for all $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_N)\in\Delta^N_{1/2}$, we get

$$\begin{aligned}(\rho\circ f)(\lambda_1,\lambda_2,\cdots,\lambda_N) &\leqslant \frac{1}{6\pi}\int_0^{2\pi}(\rho\circ f)(e^{i\theta_1},\lambda_2,\cdots,\lambda_N)d\theta_1\\ &\leqslant \frac{1}{(6\pi)^N}\int_{[0,2\pi]^N}(\rho\circ f)(e^{i\theta_1},e^{i\theta_2},\cdots,e^{i\theta_N})d\theta_1 d\theta_2\cdots d\theta_N.\end{aligned} \tag{2.3}$$

Since $\tilde{\varphi}$ is a peak function at ξ_0 and ρ satisfies $\rho(p)=-\infty$, there exists for each $n\geqslant 1$ a negative constant α_n such that for any z in $\overline{M}$, the inequality $\tilde{\varphi}(z)\geqslant 2\alpha_n$ implies $\rho(z)<-n$.

Since $\rho^{-1}(-\infty)=\{p\}$, the family $(U_n=\{z\in\overline{M}:\rho(z)<-\frac{1}{2.3^N}n\})_{n=1}^{\infty}$ is a neighbourhood basis of ξ_0 in $\overline{M}$. Let U'_n be a neighbourhood ξ_0 in $\overline{M}$ defined by $U'_n=\{z\in\overline{M}:\tilde{\varphi}(z)>\alpha_n\}$.

Let $f:\Delta^N\to M$ be a holomorphic mapping such that $f(0)\in U'_n$. Then $\tilde{\varphi}(f(0))>\alpha_n$ and hence, by (2.1) we have $mes(E_{\alpha_n})>\frac{(2\pi)^N}{2}$. Using (2.2) and the fact that ρ is a negative function, for every $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_N)\in\Delta^N_{1/2}$, we have

$$\begin{aligned}(\rho\circ f)(\lambda_1,\cdots,\lambda_N) \leqslant&\frac{1}{(6\pi)^N}\Big(\int_{E_{\alpha_n}}(\rho\circ f)(e^{i\theta_1},\cdots,e^{i\theta_N})d\theta_1\cdots d\theta_N\\ &+\int_{[0,2\pi]^N\setminus E_{\alpha_n}}(\rho\circ f)(e^{i\theta_1},\cdots,e^{i\theta_N})d\theta_1\cdots d\theta_N\Big)\\ \leqslant&\frac{1}{(6\pi)^N}\int_{E_{\alpha_n}}(-n)d\theta_1\cdots d\theta_N=-\frac{1}{(6\pi)^N}n\cdot mes(E_{\alpha_n})\\ <&-\frac{1}{(6\pi)^N}n\cdot\frac{(2\pi)^N}{2}=-\frac{1}{2\cdot 3^N}\cdot n.\end{aligned} \tag{2.4}$$

Thus $f(\Delta^N_{1/2})\subset U_n$. This proves the Lemma. □

Proof of Theorem 2.3. (i) By Lemma 2.2, we see that the sequence $\{f_\nu\}$ converges uniformly on compact subsets of M to ξ_0, when M is the unit polydisc Δ^N in $\mathbf{C}^N$. But one easily generalizes this to an arbitrary domain by some cover argument.

(ii) Let $\widetilde{W}$ be a relatively compact neighbourhood of ξ_0 in W. By Lemma 2.2, there exists a neighbourhood W' of ξ_0 in X^+ such that every holomorphic mapping $f : \Delta \to M$ satisfies

$$f(0) \in W' \Rightarrow f(\Delta_{\frac{1}{2}}) \subset \widetilde{W} \cap M.$$

Shrinking W' if necessary, we may assume that $W' \Subset \widetilde{W}$. Let $\{f_k\}$ be any sequence in $Hol(\Delta, M)$ such that $\{f_k(0), k \geqslant 0\}$ is relatively compact in M. Therefore, $\sigma[\{f_k(0), k \geqslant 0\}] \subset W' \cap M$ for some $\sigma := \sigma_\nu \in Aut(M)$. Let us denote $\sigma \circ f_k$ by $\tilde{f}_k$. Then $\tilde{f}_k(\Delta_{\frac{1}{2}}) \subset \widetilde{W} \cap M$ for any $k \geqslant 0$. Define a holomorphic mapping $g_k : \Delta \to \widetilde{W} \cap M$ by putting $g_k(z) = \tilde{f}_k(\frac{z}{2})$, $\forall z \in \Delta$. Since $\widetilde{W} \cap M \Subset W \cap M$, the sequence $\{g_k\}$ is not compactly divergent. Assume that $\{g_k\}$ is not normal. Then, since $W \cap M$ is a Zalcman space, there exist sequences $\{z_j\} \subset \Delta$ with $z_j \to z_0$, $\{\tilde{f}_j\} \subset \{\tilde{f}_k\}$, $\{\rho_j\} \subset \mathbf{R}^+$ with $\rho_j \to 0$ such that

$$h_j(z) = g_j(z_j + \rho_j z),\ z \in \mathbf{C}$$

converges uniformly on compact subsets of $\mathbf{C}$ to a nonconstant holomorphic mapping $h : \mathbf{C} \to W \cap M$. Since $h(\mathbf{C}) \Subset W$ and by the weak Brody hyperbolicity of W, the mapping h must be constant. This is a contradiction. Therefore, there exists a subsequence $\{g_{k_l}\} \subset \{g_k\}$ which converges uniformly on compact subsets of Δ to some element of $Hol(\Delta, W \cap M)$. This implies that the sequence $\{\tilde{f}_{k_l}|_{\Delta_{1/2}}\} \subset Hol(\Delta_{1/2}, M \cap W)$ converges uniformly on compact subsets of $\Delta_{1/2}$ to some element of $Hol(\Delta_{1/2}, W \cap M)$. The corresponding subsequence $\{f_{k_l}|_{\Delta_{1/2}} = \sigma^{-1} \circ (\tilde{f}_{k_l}|_{\Delta_{1/2}})\}$ converges uniformly on compact subsets of $\Delta_{1/2}$ to some element of $Hol(\Delta_{1/2}, M)$. Finally, a diagonal process shows that $\{f_k\}$ admits a subsequence which converges uniformly on compact subsets of Δ to some element of $Hol(\Delta, M)$. $\square$

3. On weak disc-convexity of complex subspaces

Modifying the definition of the weak disc-convexity (see [10]), we now give the following

Definition 3.1. Let X be a complex subspace of a complex space Y.

X is said to be *weakly disc-convex for* Y if every sequence $\{f_n\} \subset Hol(\Delta, Y)$ converges in $Hol(\Delta, Y)$ whenever the sequence $\{f_n|_{\Delta^*}\} \subset Hol(\Delta^*, X)$ converges in $Hol(\Delta^*, X)$. Here, $Hol(X, Y)$ denotes the space of holomorphic mappings from a complex space X into a complex space Y equipped with the compact-open topology and $\Delta^* = \Delta \setminus \{0\}$.

When $X = Y$ we get the weakly disc-convex concept of the complex space X.

Example 3.1. If X is relatively compact and hyperbolically imbedded in Y then X is weakly disc-convex for Y.

Indeed, this is deduced from Noguchi's theorem on Δ (see [6, Thm 4.1, p.56]).

Definition 3.2. Let X be a complex subspace of a complex space Y.

X is said to be *A-disc-convex for* Y if for every compact subset $K \subset \overline{X}$, there exists a compact subset $L \subset Y$ which satisfies the following condition:

For all $f \in Hol(\Delta, Y) \cap \mathcal{C}(\overline{\Delta}, Y)$, if $f(\partial\Delta) \subset K$ then $f(\overline{\Delta}) \subset L$.

If $X = Y$ then X is said to be *A-disc-convex* .

Concerning the A-disc-convexity of complex spaces we refer to [1].

Theorem 3.1. *Let X be a complex subspace of a weakly Brody hyperbolic complex space Y such that X is A-disc-convex for Y. Then X is weakly disc-convex for Y.*

Proof. Assume that $\{f_n\} \subset Hol(\Delta, Y)$ such that the sequence $\{f_n|_{\Delta^*}\}$ converges, uniformly on compact subsets, to a mapping $f \in Hol(\Delta^*, X)$. Let $\{f_{n_k}\}$ be any subsequence of the sequence $\{f_n\}$.

Put $K = \overline{\bigcup\limits_{k\geqslant 1} f_{n_k}(\partial\Delta_s)}$, where $0 < s < 1$. It is easy to see that K is compact in $\overline{X}$.

By the hypothesis, there exists a compact subset L such that $\overline{\bigcup\limits_{k\geqslant 1} f_{n_k}(\Delta_s)} \subset L \subset Y$. Since Y is weakly Brody hyperbolic then the compact subset L contains no complex line. Hence, by the theorem of Brody-Urata-Zaidenberg (see [8]), there exists a hyperbolic neighbourhood W of L in Y. This implies that the family $\{f_{n_k}|_{\Delta_s}\}$ is equicontinuous.

On the other hand, since $\{f_{n_k}(\lambda)\}$ is relatively compact for each $\lambda \in \Delta_s$, by the Ascoli theorem the family $\{f_{n_k} : k \geqslant 1\}$ is relatively compact in $Hol(\Delta_s, Y)$. Thus there exists a subsequence $\{f_{n_{k_l}}\}$ of $\{f_{n_k}\}$ which converges, uniformly on compact subsets, to the mapping F in $Hol(\Delta, Y)$. The equality $F|_{\Delta^*} = f$ determines F uniquely, hence independently of the choices of subsequences $\{f_{n_k}\}$ of the sequence $\{f_n\}$. It follows that the sequence $\{f_n\}$ converges, uniformly on compact sets, to the mapping F in $Hol(\Delta, Y)$. Hence X is weakly disc-convex. □

We now give another character of the weak disc-convexity from the view point of the Zalcmanness of complex spaces. First, we formulate some definitions.

Definition 3.3. Let X be a complex subspace of a complex space Y.

X is said to be *weakly Zalcman for Y* if for every compact subset $K \subset \overline{X}$, there exists an open neighbourhood U of K in Y which satisfies the two following condition:

(i) U is Zalcman,
(ii) for each $f \in Hol(\Delta, Y) \cap \mathcal{C}(\overline{\Delta}, Y)$, if $f(\partial\Delta) \subset K$ then $f(\overline{\Delta}) \subset U$.

If $X = Y$ then X is said to be *weakly Zalcman.*

Example 3.2. Every complex subspace X of a Zalcman complex space Y is weakly Zalcman for Y.

Definition 3.4. Let X be a complex subspace of a complex space Y. We say that X has the Δ^*-EP for Y if every holomorphic mapping f from Δ^* into X extends to a holomorphic mapping F from Δ into Y. If X has the Δ^*-EP for itself then X is said to have the Δ^*-EP (shortly X has Δ^*-EP).

Example 3.3. (i) By a theorem of Kobayashi [6, Thm 6.3.7, p.284], if X is relatively compact and hyperbolically imbedded into Y then X has the Δ^*-EP for Y.

(ii) It is easy to see from the Riemann extension theorem that if D is a bounded domain in $\mathbf{C}^n$ and Ω is an open neighbourhood of $\overline{D}$ in $\mathbf{C}^n$ then D has the Δ^*-EP for Ω.

For details concerning the Δ^*-EP we refer the readers to [9], [11], [12].

We now prove the following.

Theorem 3.2. *Let X be a complex subspace of a Brody hyperbolic complex space Y. If X is weakly Zalcman for Y and has the $\Delta^* - EP$ for Y then X is weakly disc-convex for Y.*

Proof. Assume that $\{f_n\}_{n\geqslant 1} \subset Hol(\Delta, Y)$ such that the sequence

$$\{f_n|_{\Delta^*}\} \subset Hol(\Delta^*, X)$$

converges, uniformly on compact subsets, to a mapping $f \in Hol(\Delta^*, X)$. Since X has the $\Delta^* - EP$ for Y, there exists $g \in Hol(\Delta, Y)$ such that $g = f$ on Δ^*.

Put $x_0 := g(0) \in \overline{X}$. Take $0 < s < 1$ and take a relatively compact neighbourhood $\tilde{K}$ of $f(\partial\Delta_s)$ in X . Then there exists $n_0 \geqslant 1$ such that $f_n(\partial\Delta_s) \subset \tilde{K}$ for all $n \geqslant n_0$. Put $K = \{x_0\} \cup \overline{\tilde{K}}$. Then K is a compact subset of $\overline{X}$. Since X is weakly Zalcman for Y, there is an open Zalcman neighbourhood U of K in Y which satisfies (ii) in Definition 3.3. It follows that

$$f_n(\overline{\Delta}_s) \subset U \text{ for all } n \geqslant n_0.$$

For each $n \leqslant n_0$, define a holomorphic mapping $\tilde{f}_n : \Delta \to U$ by setting $\tilde{f}_n(z) = f_n(sz)$ for each $z \in \Delta$. Then $\mathcal{F} = \{\tilde{f}_n\}_{n\geqslant n_0} \subset Hol(\Delta, U)$.

It is easy to see that the family $\mathcal{F}$ is not compactly divergent.

Assume that the family $\mathcal{F}$ is not normal. Then by the Zalcmanness of U, there exist sequences $\{p_j\} \subset \Delta$ with $\{p_j\} \to p_0 \in \Delta$, $\{f_j\} \subset \mathcal{F}$, $\{\rho_j\} \subset \mathbf{R}$ with $\rho_j > 0$ and $\{\rho_j\} \to 0$ such that

$$g_j(\xi) = f_j(p_j + \rho_j\xi), \quad \xi \in \mathbf{C}$$

converges uniformly on compact subsets of $\mathbf{C}$ to a nonconstant holomorphic map $g : \mathbf{C} \to U$. This is a contradiction, because Y contains no complex lines.

Thus $\mathcal{F}$ is normal. Hence $\{f_n\}$ converges uniformly on compact subsets of Δ to g in $Hol(\Delta, Y)$. ∎

4. Noguchi-type convergence-extension theorems for holomorphic mappings into weakly Zalcman complex spaces

In this section we show some convergence-extension theorems for holomorphic mappings into weakly Zalcman complex spaces.

Theorem 4.1. *Let X be a complex subspace of a Brody hyperbolic complex space Y such that X is weakly Zalcman for Y and has the $\Delta^* - EP$ for Y. Let A be any non-singular analytic hypersurface of a complex manifold M. Let $\{f_j : M\backslash A \to X\}_{j=1}^{\infty}$ be a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \setminus A$ to a holomorphic mapping $f : M \setminus A \to X$. Then there are unique holomorphic extensions $\overline{f}_j : M \to Y$ and $\overline{f} : M \to Y$ of f_j and f over M, and $\{\overline{f}_j\}_{j=1}^{\infty}$ converges uniformly on compact subsets of M to $\overline{f}$.*

Proof. By Theorem 3.2, X is weakly disc-convex for Y.

(i) We now prove that every holomorphic mapping $f : M \setminus A \to X$ extends to a holomorphic mapping $\overline{f} : M \to Y$.

By localizing the mapping f, we may assume that $M = \Delta^m = \Delta^{m-1} \times \Delta$ and $A = \Delta^{m-1} \times \{0\}$.

For each $z' \in \Delta^{m-1}$, consider the holomorphic mapping $f_{z'} : \Delta^* \to X$ given by $f_{z'}(z) = f(z', z)$ for each $z \in \Delta^*$. By the hypothesis, there exists a holomorphic extension $\overline{f}_{z'} : \Delta \to Y$ of $f_{z'}$ for each $z' \in \Delta^{m-1}$. Define the mapping $\overline{f} : \Delta^{m-1} \times \Delta \to Y$ by $\overline{f}(z', z) = \overline{f}_{z'}(z)$ for all $(z', z) \in \Delta^{m-1} \times \Delta$. It suffices to prove that $\overline{f}$ is continuous at $(z'_0, 0) \in \Delta^{m-1} \times \Delta$.

Indeed, assume that $\{(z'_k, z_k)\} \in \Delta^{m-1} \times \Delta$ such that $\{(z'_k, z_k)\} \to (z'_0, 0)$.

Put $\sigma_k = \overline{f}_{z'_k}$ for each $k \geqslant 1$ and $\sigma_0 = \overline{f}_{z'_0}$. Then the sequence $\{\sigma_k|_{\Delta^*}\}$ converges uniformly to the mapping $\{\sigma_0|_{\Delta^*}\}$ in $Hol(\Delta^*, X)$. Since X is weakly disc-convex for Y, the sequence $\{\sigma_k\}$ converges uniformly to the mapping σ_0 in $Hol(\Delta, Y)$. Therefore, $\{\sigma_k(z_k) = \overline{f}(z'_k, z_k)\} \to \sigma_0(0) = \overline{f}(z'_0, 0)$ and hence, $\overline{f}$ is continuous at $(z'_0, 0)$.

(ii) Let $\{f_k\} \subset Hol(M \setminus A, X)$ be such that $\{f_k\} \to f_0$ in $Hol(M \setminus A, X)$.

We will show that $\{\overline{f}_k\} \to \overline{f}_0$ in $Hol(M, Y)$.

By localizing the mappings, we may assume that $M = \Delta^m = \Delta^{m-1} \times \Delta$ and $A = \Delta^{m-1} \times \{0\}$. Let $\{(z'_k, z_k)\} \subset \Delta^{m-1} \times \Delta$ be any sequence converging to $(z'_0, z_0) \in \Delta^{m-1} \times \Delta$. We now prove that the sequence $\{\overline{f}_k(z'_k, z_k)\}$ converges to $\overline{f}_0(z'_0, z_0)$.

Indeed, for each $k \geqslant 0$ consider the holomorphic mapping $\varphi_k : \Delta \to X$ given by $\varphi_k(z) = \overline{f}_k(z'_k, z)$ for all $z \in \Delta$. Then $\{\varphi_k|_{\Delta^*}\} \to \varphi_0|_{\Delta^*}$ in $Hol(\Delta^*, X)$. Since X is weakly disc-convex for Y, we have $\{\varphi_k\} \to \varphi_0$ in $Hol(\Delta, Y)$. Hence $\{\varphi_k(z_k) = \overline{f}_k(z'_k, z_k)\} \to \varphi_0(z_0) = \overline{f}_0(z'_0, z_0)$. □

Remark 1. Using the same above argument, we also get the following: Let X be a complex subspace of a weakly Brody hyperbolic complex space Y such that X is A-disc-convex for Y. Let A be any non-singular analytic hypersurface of a complex manifold M. Let $\{f_j : M \setminus A \to X\}_{j=1}^{\infty}$ be a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \setminus A$ to a holomorphic mapping $f : M \setminus A \to X$. Then there are unique holomorphic extensions $\overline{f}_j : M \to Y$ and $\overline{f} : M \to Y$ of f_j and f over M, and $\{\overline{f}_j\}_{j=1}^{\infty}$ converges uniformly on compact subsets of M to $\overline{f}$.

Theorem 4.2. *Let X be a complex subspace of a Brody hyperbolic complex space Y such that X is weakly Zalcman for Y and has the $\Delta^* - EP$ for Y. Let M be a complex manifold of dimension m, and let A be a subset which is nowhere dense in a non-singular complex submanifold $B \subset M$ of dimension $\leqslant m-1$. Let $\{f_j : M \setminus A \to X\}_{j=1}^{\infty}$ be a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \setminus A$ to a holomorphic mapping $f : M \setminus A \to X$. Then there are unique holomorphic extensions $\overline{f}_j : M \to Y$ and $\overline{f} : M \to Y$ of f_j and f over M, and $\{\overline{f}_j\}_{j=1}^{\infty}$ converges uniformly on compact subsets of M to $\overline{f}$.*

Proof. By Theorem 3.2, X is weakly disc-convex for Y.

(i) We now prove that every holomorphic mapping $f : M \setminus A \to X$ extends to a holomorphic mapping $\overline{f} : M \to Y$.

Take an arbitrary point $a \in A$. By localizing the mapping f, we may assume that $M = \Delta^m = \Delta^{m-1} \times \Delta,\ A = A' \times \{0\}$, where A' is a nowhere dense subset of Δ^{m-1}, and $a = (t_0, 0) \in A' \times \{0\}$. For every point $z \in \Delta^m$ denote $z = (t, u)$ with $t \in \Delta^{m-1}$ and $u \in \Delta$.

Assume that a sequence $\{a_j = (t_j, u_j)\} \subset (\Delta^{m-1} \setminus A') \times \Delta$ converges to a. Consider the holomorphic mappings $f_j : \Delta \to X, u \mapsto f_j(u) = f(t_j, u)$ for each $j \geqslant 1$, and $f_{t_0} : \Delta^* \to X, u \mapsto f_{t_0}(u) = f(t_0, u)$. It is easy to see that $\{f_j|_{\Delta^*}\} \to f_{t_0}$ in $Hol(\Delta^*, X)$. Since X is weakly disc-convex for Y, the sequence $\{f_j\}$ converges uniformly to the holomorphic mapping $g \in Hol(\Delta, Y)$, where $g|_{\Delta^*} = f_{t_0}$. Put $g(0) = p \in \overline{X}$. Then $\{f_j(u_j)\} \to g(0)$, i.e., $\{f(a_j)\} \to p$. Thus, the sequence $\{f(a_j)\}$ converges to p for any sequence $\{a_j\} \subset (\Delta^{m-1} \setminus A') \times \Delta$ converging to a (*). Choose a relatively compact neighbourhood V_p of p in Y such that $\overline{V}_p$ is contained in a holomorphic local coordinate neighbourhood of p in Y. By (*) there exists an open neighbourhood $T_0 \times U_0$ of $a = (t_0, 0)$ in $\Delta^{m-1} \times \Delta$ such that $f((T_0 \setminus A') \times U_0) \subset V_p$.

For every point $u \in U_0 \setminus \{0\}$, consider the holomorphic mapping $f_u : \Delta^{m-1} \to X, t \mapsto f_u(t) = f(t, u)$.

Since $f_u(T_0 \setminus A') \subset V_p$, it follows that $f_u(\overline{T_0 \setminus A'}) = f_u(T_0) \subset \overline{V}_p$. Thus $f(T_0 \times (U_0 \setminus \{0\})) \subset \overline{V}_p$. By the Riemann extension theorem, the mapping f extends holomorphically to $T_0 \times U_0$.

(ii) Repeating the argument in Theorem 3.1, we can show that if a sequence $\{f_k\} \subset Hol(M \setminus A, X)$ converges, locally uniformly, to a mapping f_0 in $Hol(M \setminus A, X)$, then the sequence $\{\overline{f}_k\}$ converges, locally uniformly, to $\overline{f}_0$ in $Hol(M, Y)$. □

Corollary 4.1. *Let X be a weakly Zalcman complex space such that X has the $\Delta^* - EP$. Let M be a complex manifold of dimension m, and let A be a subset which is nowhere dense in a complex subspace $B \subset M$ of dimension $\leqslant m-1$. Let $\{f_j : M \setminus A \to X\}_{j=1}^{\infty}$ be a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \setminus A$ to a holomorphic mapping $f : M \setminus A \to X$. Then there are unique holomorphic extensions $\overline{f}_j : M \to Y$ and $\overline{f} : M \to Y$ of f_j and f over M, and $\{\overline{f}_j\}_{j=1}^{\infty}$ converges uniformly on compact subsets of M to $\overline{f}$.*

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