ACTA MATHEMATICA VIETNAMICA
Volume 32, Number 1, 2007, pp. 41-50

CONDITIONAL MARTINGALES

DARIUSZ MAJEREK AND WIESŁAW ZIĘBA

Abstract. In this paper we would like to present a concept of conditional martingales given σ-field $\mathcal{F}$, which is a natural generalization of martingales. Based on $\mathcal{F}$-independence, we are able to show that there exist conditional martingales, which are not martingales in pure sense. In the first part of the paper we will show different types of conditional convergence and relations between them. This article is intended to give a generalization of the theorem about almost sure convergence of martingales and some properties of conditional martingales.

1. Definitions and examples

Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space, $\{\mathcal{F}_n\}_{n=1}^{\infty}$ a nondecreasing family of σ-fields and $\mathcal{F} \subset \mathcal{F}_1$ a σ-field. Then the following definition of conditional supermartingale (martingale) can be introduced:

Definition 1.1. Let $\{X_n\}_{n=1}^{\infty}$ be an adapted sequence with respect to $\{\mathcal{F}_n\}_{n=1}^{\infty}$. Then we say that the sequence $\{X_n\}_{n=1}^{\infty}$ is a conditional supermartingale with respect to σ-field $\mathcal{F}$ if it fulfills the following conditions:

(1) $\mathrm{E}^{\mathcal{F}}|X_n| < \infty$ a.s.

(2) $\mathrm{E}^{\mathcal{F}_n} X_{n+1} \leqslant X_n$ a.s.

This sequence is a conditional martingale given σ-field $\mathcal{F}$ if we have equality in (2).

Now we introduce a concept of conditional independence [2].

Definition 1.2. We say that $A_1, A_2, \ldots, A_n$ are $\mathcal{F}$-independent if

$$\bigwedge_{1\leqslant k\leqslant n} \bigwedge_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} \mathrm{E}^{\mathcal{F}} \prod_{s=1}^{k} I_{A_{i_s}} = \prod_{s=1}^{k} \mathrm{E}^{\mathcal{F}} I_{A_{i_s}}.$$

In case $\mathcal{F} = (\emptyset, \Omega)$ we obtain independence. Note that if $\mathcal{F} = \mathcal{A}$, then any events are $\mathcal{F}$- independent.

A sequence of families $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ where $\mathcal{G}_k \subset \mathcal{A}$ is $\mathcal{F}$-independent if any sequence of events $A_1, A_2, \ldots, A_n$ where $A_i \in \mathcal{G}_i$, $i = 1, 2, \ldots, n$ is $\mathcal{F}$-independent.

Received March 27, 2006; in revised form July 13, 2006.

2000 Mathematics Subject Classification. 60F15.

Key words and phrases. Conditional independence, conditional martingale, conditional convergence in distribution.

Let $X : \Omega \to \mathbb{R}$ be a random variable. Then $\mathcal{A}_X$ denotes the σ-field generated by the random variable X. It is obvious that $\mathcal{A}_X = X^{-1}(\mathcal{B})$ where $\mathcal{B}$ denotes a Borel σ-field on real. We say that $\mathcal{A}_X$ is the σ-field generated by X.

The random variables $X_1, X_2, \ldots, X_n$ are $\mathcal{F}$-independent if σ-fields $\mathcal{A}_{X_1}, \mathcal{A}_{X_2}, \ldots, \mathcal{A}_{X_n}$ are $\mathcal{F}$-independent.

In the next part of this paper we shall need the following lemma.

Lemma 1.1. *Let $\mathcal{F}, \mathcal{G}$ be σ-fields fulfilling the following condition $\mathcal{F} \subset \mathcal{G}$ and let $\sigma(X)$ and $\mathcal{G}$ be $\mathcal{F}$-independent, then*

$$\mathrm{E}^{\mathcal{G}} X = \mathrm{E}^{\mathcal{F}} X \quad a.s.$$

Proof. Let us assume that the lemma is not true. Let

$$D = \{\omega : \mathrm{E}^{\mathcal{G}} X \neq \mathrm{E}^{\mathcal{F}} X\}$$

such that $\mathrm{P}(D) > 0$ and $D \in \mathcal{G}$. Let

$$D_1 = [\omega \in \Omega : \mathrm{E}^{\mathcal{F}} X > \mathrm{E}^{\mathcal{G}} X],$$

$$D_2 = [\omega \in \Omega : \mathrm{E}^{\mathcal{F}} X < \mathrm{E}^{\mathcal{G}} X].$$

Then D_1, $D_2 \in \mathcal{G}$, $D_1 \cup D_2 = D$ and

$$\left[\int_{D_1} \mathrm{E}^{\mathcal{G}} X d\mathrm{P} < \int_{D_1} \mathrm{E}^{\mathcal{F}} X d\mathrm{P}\right] \quad \vee \quad \left[\int_{D_2} \mathrm{E}^{\mathcal{G}} X d\mathrm{P} > \int_{D_2} \mathrm{E}^{\mathcal{F}} X d\mathrm{P}\right],$$

so

$$\int_{D_1} \mathrm{E}^{\mathcal{F}} X d\mathrm{P} = \mathrm{E}[I_{D_1} \mathrm{E}^{\mathcal{F}} X] = \mathrm{E}[\mathrm{E}^{\mathcal{F}} I_{D_1} \mathrm{E}^{\mathcal{F}} X]$$
$$= \mathrm{E}[I_{D_1} X] = \int_{D_1} X d\mathrm{P} = \int_{D_1} \mathrm{E}^{\mathcal{G}} X d\mathrm{P},$$

which contradicts the assumption. □

We present two types of conditional convergence in distribution.

Definition 1.3. The random variables X and Y have the same conditional distribution if

$$\bigwedge_{a \in \mathbb{R}} \mathrm{E}^{\mathcal{F}} I_{[X \leqslant a]} = \mathrm{E}^{\mathcal{F}} I_{[Y \leqslant a]} \quad a.s.$$

Let ζ_{F_X} be the set of continuity of a distribution function F_X.

Definition 1.4 ([5])**.** A sequence $\{X_n\}_{n=1}^{\infty}$ of random variables is conditionally convergent in distribution to some X if

$$\bigwedge_{x \in \zeta_{F_X}} \mathrm{E}^{\mathcal{F}} I_{[X_n < x]} \xrightarrow{a.s.} \mathrm{E}^{\mathcal{F}} I_{[X < x]}, \quad n \to \infty.$$

We denote this convergence by $X_n \xrightarrow{\mathrm{D}^{\mathcal{F}}} X,\ n \to \infty$, and call it the conditional strong convergence in distribution.

Now we introduce conditional weak convergence in distribution as follows.

Definition 1.5. A sequence $\{X_n\}_{n=1}^{\infty}$ of random variables is conditionally weak convergent in distribution to some X if

$$\bigwedge_{x \in \zeta_{F_X}} \mathrm{E}^{\mathcal{F}} I_{[X_n < x]} \xrightarrow{\mathrm{P}} \mathrm{E}^{\mathcal{F}} I_{[X < x]}.$$

This convergence is denoted by $X_n \xrightarrow{\mathrm{WD}^{\mathcal{F}}} X,\ n \to \infty$.

It is easy to see that if $X_n \xrightarrow{\mathrm{D}^{\mathcal{F}}} X$ then $X_n \xrightarrow{\mathrm{WD}^{\mathcal{F}}} X$. Moreover, if $\mathcal{F} = \{\emptyset, \Omega\}$ then conditional weak convergence in distribution means convergence in distribution because

$$\bigwedge_{x \in \zeta_{F_X}} \lim_{n \to \infty} F_{X_n}(x) = \mathrm{E} I_{[X_n < x]} \to \mathrm{E} I_{[X < x]} = F_X(x).$$

Theorem 1.1. [6]

$$X_n \xrightarrow{\mathrm{P}} X \Longleftrightarrow \bigwedge_{x \in \zeta_{F_X}} \mathrm{P}\left([X_n < x] \triangle [X < x]\right) \to 0,\ n \to \infty.$$

Therefore we can claim:

Theorem 1.2. *If* $\bigwedge_{x \in \zeta_{F_X}} I_{[X_n < x]} \xrightarrow{\mathrm{P}} I_{[X < x]},\ n \to \infty,$ *then* $X_n \xrightarrow{\mathrm{P}} X$.

Proof. It is obvious that

$$I_{([X_n < x] \triangle [X < x])} = |I_{[X_n < x]} - I_{[X < x]}|.$$

By the convergence

$$\bigwedge_{x \in \zeta_{F_X}} I_{[X_n < x]} \xrightarrow{\mathrm{P}} I_{[X < x]},\ n \to \infty$$

we have

$$I_{([X_n < x] \triangle [X < x])} \xrightarrow{\mathrm{P}} 0,$$

which means that

$$\mathrm{P}\left([X_n < x] \triangle [X < x]\right) \to 0,\ n \to \infty.$$

By Theorem 1.1 the assertion is proved. □

From the above theorem we can say that if $\mathcal{F} = \mathcal{A}$ then

$$X_n \xrightarrow{\mathrm{WD}^{\mathcal{F}}} X \Longleftrightarrow X_n \xrightarrow{\mathrm{P}} X.$$

Theorem 1.3. *If* $\mathcal{F} \subset \mathcal{G}$ *and* $X_n \xrightarrow{\mathrm{WD}^{\mathcal{G}}} X,\ n \to \infty,$ *then* $X_n \xrightarrow{\mathrm{WD}^{\mathcal{F}}} X,\ n \to \infty.$

Proof. Let us assume that $X_n \stackrel{\mathrm{WD}^{\mathcal{G}}}{\longrightarrow} X,\ n \to \infty$. Then

$$|\mathrm{E}^{\mathcal{F}} I_{[X_n<x]} - \mathrm{E}^{\mathcal{F}} I_{[X<x]}| = |\mathrm{E}^{\mathcal{F}}(I_{[X_n<x]} - I_{[X<x]})| =$$
$$|\mathrm{E}^{\mathcal{F}}\mathrm{E}^{\mathcal{G}}(I_{[X_n<x]} - I_{[X<x]})| \stackrel{\mathrm{P}}{\longrightarrow} 0,\ n \to \infty.$$

□

This relation points out that there exists a continuous link between weak convergence and convergence in probability. If we consider conditional weak convergence in distribution, then a similar statement holds.

Theorem 1.4 ([5])**.** *If $\mathcal{F} \subset \mathcal{G}$ and $X_n \stackrel{\mathrm{D}^{\mathcal{G}}}{\longrightarrow} X,\ n \to \infty$, then $X_n \stackrel{\mathrm{D}^{\mathcal{F}}}{\longrightarrow} X,\ n \to \infty$.*

Hence, if $\mathcal{F} = \{\emptyset, \Omega\}$ then the following conditions are equivalent:

(i) $X_n \stackrel{\mathrm{D}^{\mathcal{F}}}{\longrightarrow} X$,
(ii) $X_n \stackrel{\mathrm{WD}^{\mathcal{F}}}{\longrightarrow} X$,
(iii) $X_n \stackrel{\mathrm{D}}{\longrightarrow} X$.

To show the reason of introducing conditional martingales, we present some examples.

Example 1. Let us consider an experiment relying on tossing coins n-times. We have at our disposal two coins of types A and B. Let p_a and p_b are probabilities of heads for the coin of type A and B respectively.

It is easy to see that a sequence $\{A_n\}_{n=1}^{\infty}$, where A_i denote "head in i-th toss", is not independent, but it is conditionally independent with respect to events C_B and C'_B, where C_B denotes that "coin A is selected". Therefore events $\{A_n\}_{n=1}^{\infty}$ are conditionally independent given σ-field $\mathcal{F} = \sigma(C_B)$.

Let us assume that

$$X_n = I_{A_n} - \mathrm{E}^{\mathcal{F}} I_{A_n} = I_{A_n} - \mathrm{P}_{\mathcal{F}}(A_n),$$

and

$$S_n = X_1 + X_2 + \cdots + X_n.$$

Then obviously

$$\mathrm{E}^{\mathcal{F}}|S_n| < \infty \quad , n \in \mathbb{N}.$$

Choose $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ and note that $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ Then by conditional independence we have

$$\mathrm{E}^{\mathcal{F}_n} X_{n+1} = \mathrm{E}^{\mathcal{F}_n}[I_{A_{n+1}} - \mathrm{E}^{\mathcal{F}_n} I_{A_{n+1}}] = \mathrm{E}^{\mathcal{F}} I_{A_{n+1}} - \mathrm{E}^{\mathcal{F}_n} I_{A_{n+1}} = 0 \quad \text{a.s.}$$

So we have

$$\mathrm{E}^{\mathcal{F}_n} S_{n+1} = \mathrm{E}^{\mathcal{F}_n}[S_n + X_{n+1}] = \mathrm{E}^{\mathcal{F}_n} S_n + \mathrm{E}^{\mathcal{F}_n} X_{n+1} = S_n \quad \text{a.s.}$$

Hence $\{S_n\}_{n=1}^{\infty}$ is conditional martingale given σ-field $\mathcal{F}$. It is clear that

$$\mathrm{E}|S_n| < \infty \quad , n \in \mathbb{N}.$$

So $\{S_n\}_{n=1}^{\infty}$ is also martingale in pure sense.

Example 2. Similarly to the previous example we choose a sequence of events $\{A_n\}_{n=1}^{\infty}$ and σ-field $\mathcal{F}$. Let

$$X_n = I_{A_n} + \mathrm{E}^{\mathcal{F}} I_{A'_n} = I_{A_n} + \mathrm{P}_{\mathcal{F}}(A'_n)$$

and

$$S_n = X_1 \cdot X_2 \dots X_n.$$

Obviously $\mathrm{E}^{\mathcal{F}}|S_n| < \infty$ for all $n \in \mathbb{N}$. Similarly to the previous example we choose a sequence of σ-fields $\mathcal{F}_n$.

Then by conditional independence, we have

$$\begin{aligned} \mathrm{E}^{\mathcal{F}_n}[S_{n+1}] &= \mathrm{E}^{\mathcal{F}_n}[S_n \cdot X_{n+1}] = S_n[\mathrm{E}^{\mathcal{F}_n} I_{A_n} + \mathrm{E}^{\mathcal{F}} I_{A'_n}] \\ &= S_n \mathrm{E}^{\mathcal{F}}[I_{A_n} + \mathrm{E}^{\mathcal{F}} I_{A'_n}] = S_n. \end{aligned}$$

Therefore, this sequence is conditional martingale and martingale, like in the previous example, because

$$\mathrm{E}|S_n| < \infty, \quad n \in \mathbb{N}.$$

Example 3. Let us consider an experiment relying on tossing coins n-times. We have at our disposal countably many coins of types $\{B_n\}_{n=1}^{\infty}$. Let p_h^i be probabilities of heads for the coin of type B_i. Moreover, p_i is the probability of choosing coin of type B_i. Let us assume that $p_i = \frac{1}{2_i}$ and $p_h^i = \frac{1}{3^i}$, $i = 1, 2, \dots$.

It is easy to see [2] that the sequence $\{A_n\}_{n=1}^{\infty}$, where A_i denote "head in i-th toss", is not independent, but it is conditionally independent with respect to events $\{B_n\}_{n=1}^{\infty}$. Putting $\mathcal{F} = \sigma(B_1, B_2, \dots)$, we have

$$\begin{aligned} \mathrm{P}_{\mathcal{F}}(A_{i_1}, \dots, A_{i_k}) &= \sum_{n=1}^{\infty} I_{B_n} \mathrm{P}(A_{i_1}, \dots, A_{i_k} | B_n) \\ &= \sum_{n=1}^{\infty} \left(I_{B_n} \prod_{i=i_1}^{i_k} \mathrm{P}(A_i | B_n) \right) \\ &= \sum_{n=1}^{\infty} \prod_{i=i_1}^{i_k} \mathrm{P}(A_i | B_n) I_{B_n} = \prod_{i=i_1}^{i_k} \sum_{n=1}^{\infty} I_{B_n} \mathrm{P}(A_i | B_n) \\ &= \prod_{i=i_1}^{i_k} \mathrm{P}_{\mathcal{F}}(A_i). \end{aligned}$$

Hence $\{A_n\}_{n=1}^{\infty}$ is $\mathcal{F}$-independent but is not independent.

Let us assume that

$$X_n = I_{A_n} - \mathrm{E}^{\mathcal{F}} I_{A_n} = I_{A_n} - \mathrm{P}_{\mathcal{F}}(A_n),$$

$$Y(\omega) = \sum_{n=1}^{\infty} 2^n I_{B_n}$$

and

$$S_n = X_1 + X_2 + \cdots + X_n.$$

It is clear that $\mathrm{E}^{\mathcal{F}}|S_n| < \infty$ a.s. for all $n \in \mathbb{N}$, Y is $\mathcal{F}$-measurable and $\mathrm{E}^{\mathcal{F}}|Y| < \infty$ a.s., but

$$\begin{aligned} \mathrm{E}|Y| &= \mathrm{E}Y = \sum_{n=1}^{\infty} \mathrm{E}(Y|B_n)\mathrm{P}(B_n) = \sum_{n=1}^{\infty} 2^n \mathrm{E}(|I_{B_n}||B_n)\mathrm{P}(B_n) \\ &= \sum_{n=1}^{\infty} 2^n \mathrm{P}(B_n) = \infty. \end{aligned}$$

If we choose $\mathcal{F}_n$ similarly as in Example 1, then

$$\mathrm{E}^{\mathcal{F}_n}[S_{n+1} + Y] = \mathrm{E}^{\mathcal{F}_n}[S_n + X_{n+1} + Y] = S_n + Y \quad \text{a.s.}$$

As $\mathrm{E}^{\mathcal{F}}|S_n+Y| < \infty$ a.s. for all $n \in \mathbb{N}$, the sequence $\{S_n + Y\}_{n=1}^{\infty}$ fulfills conditions (1.1),(**??**) in Definition 1.1 and it is conditional martingale.

On the other hand, $\mathrm{E}|S_n + Y| \geqslant \mathrm{E}S_n + \mathrm{E}Y = \infty$, hence $\{S_n + Y\}_{n=1}^{\infty}$ is not martingale in pure sense.

Example 4. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{F}$-independent random variables with the same conditional distribution such that $\mathrm{E}^{\mathcal{F}}X = \eta < \infty$ a.s. Then the sequence $\{S_n - n\eta; \mathcal{F}_n\}_{n=1}^{\infty}$ is conditional martingale because

$$\mathrm{E}^{\mathcal{F}}|S_n - n\eta| \leqslant \mathrm{E}^{\mathcal{F}}|S_n| + \mathrm{E}^{\mathcal{F}}|n\eta| \leqslant 2n\eta < \infty \quad \text{a.s.}$$

and

$$\begin{aligned} \mathrm{E}^{\mathcal{F}_n}[S_{n+1} - (n+1)\eta] &= S_n + \mathrm{E}^{\mathcal{F}_n}X_{n+1} - \mathrm{E}^{\mathcal{F}_n}[(n+1)\eta] = \\ & S_n + \mathrm{E}^{\mathcal{F}}X_{n+1} - (n+1)\eta = S_n - n\eta. \end{aligned}$$

2. Almost sure convergence of conditional martingales

We are going to prove a generalization of Doob's theorem. But first let us give the following theorem.

Theorem 2.1. *Let $\{X_n\}_{n=1}^{\infty}$ be a conditional supermartingale given $\mathcal{F}$ and let $\tau_1 \leqslant \tau_2$ are bounded stopping times, then $(X_{\tau_n}, \mathcal{F}_{\tau_n})_{n=1}^{2}$ is conditional supermartingale given $\mathcal{F}$.*

To prove this theorem we use a similar method as in the classical case.

Theorem 2.2 (On almost sure convergence of conditional supermartingale)**.** *Let $\{X_n\}_{n=1}^{\infty}$ be a conditional supermartingale given σ-field $\mathcal{F}$ and let $\sup_n \mathrm{E}^{\mathcal{F}}|X_n| < \infty$ a.s., then the sequence $\{X_n\}_{n=1}^{\infty}$ is convergent to some random variable X such that $\mathrm{E}^{\mathcal{F}}X < \infty$ a.s.*

In the proof of this theorem we use a well-known fact concerning sequences:

Lemma 2.1. *The sequence* $\{x_n\}_{n=1}^{\infty}$ *is convergent (possibly to infinity limit), iff* $U_a^b < \infty$ *for all pairs of rational numbers* a, b, *where* U_a^b *is the number of upcrossing of* (a, b) *by the sequence* $\{x_n\}_{n=1}^{\infty}$ *defined by*

$$U_a^b = \begin{cases} \sup\{k \geqslant 1 : \tau_{2k-1} < \infty\}, & \tau_1 < \infty; \\ 0, & \tau_1 = \infty, \end{cases}$$

where

$$\begin{aligned} \tau_0 &= \inf\{n : x_n < a\} \\ \tau_1 &= \inf\{n : n > \tau_0, x_n > b\} \\ &\dots \\ \tau_{2k} &= \inf\{n : n > \tau_{2k-1}, x_n < a\} \\ \tau_{2k+1} &= \inf\{n : n > \tau_{2k}, x_n > b\}. \end{aligned}$$

To prove Theorem 2.2 we need also the following lemma.

Lemma 2.2. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a conditional supermartingale given* σ*-field* $\mathcal{F}$, *then for* $a < b$ *we have*

$$\mathrm{E}^{\mathcal{F}} U_a^b[m] \leqslant \frac{1}{b-a} \mathrm{E}^{\mathcal{F}} (X_m - a)^- \quad a.s.$$

where $U_a^b[m]$ *denotes the number of upcrossing of* (a, b) *by the sequence* X_1, X_2, ..., X_m *and it is random variable.*

The proof of this lemma is similar to the classical case.

Now, let us go back to the proof of Theorem 2.2.

Proof. Note that $U_a^b[m] \nearrow U_a^b$ while $m \to \infty$. By Lemma 2.2 we obtain

$$\begin{aligned} \mathrm{E}^{\mathcal{F}} U_a^b[m] &\leqslant \frac{1}{b-a} \mathrm{E}^{\mathcal{F}} (X_m - a)^- \\ &\leqslant \frac{1}{b-a} \mathrm{E}^{\mathcal{F}} \left(\frac{|X_m| - X_m}{2} + \frac{|a| - a}{2} \right) \\ &\leqslant \frac{1}{b-a} (\sup_n \mathrm{E}^{\mathcal{F}} |X_n| + a^+) < \infty \quad \text{a.s.} \end{aligned}$$

Then for each $a < b$

$$\mathrm{E}^{\mathcal{F}} U_a^b < \infty \quad \text{a.s.},$$

which means that U_a^b is finite almost sure for all a, b. Thus, by Lemma 2.1 the sequence $\{X_n\}_{n=1}^{\infty}$ is almost sure convergent.

It is sufficient to prove that the limit is finite. By the conditional version of Fatou lemma [3] we have

$$\mathrm{E}^{\mathcal{F}} (\liminf_{n \to \infty} |X_n|) \leqslant \liminf_{n \to \infty} \mathrm{E}^{\mathcal{F}} |X_n| < \infty \quad \text{a.s.}$$

Therefore, $\lim_{n \to \infty} X_n$ is almost sure finite and it has almost sure finite conditional expectation given σ-field $\mathcal{F}$, which completes the proof. ☐

3. Properties of conditional martingales

Theorem 3.1. (a) *If a sequence* $\{X_n\}_{n=1}^{\infty}$ *is a conditional martingale, and a function* $\phi : \mathbb{R} \to \mathbb{R}$ *is convex and such that* $\phi(X_n) \in L^1_{\mathcal{F}}$, $n \in \mathbb{N}$, *then the sequence* $\{\phi(X_n)\}_{n=1}^{\infty}$ *is a conditional submartingale.*
(b) *If a sequence* $\{X_n\}_{n=1}^{\infty}$ *is conditional submartingale, a function* $\phi : \mathbb{R} \to \mathbb{R}$ *is convex, nondecreasing and* $\phi(X_n) \in L^1_{\mathcal{F}}$, $n \in \mathbb{N}$, *then the sequence* $\{\phi(X_n)\}_{n=1}^{\infty}$ *is still a conditional submartingale.*

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function fulfilling assumptions of Theorem 3.1(a). Additionally, let us assume that the sequence $\{X_n\}_{n=1}^{\infty}$ is conditional martingale. Then, by the Jensen's inequality,

$$\mathrm{E}^{\mathcal{F}_n}[\phi(X_{n+1})] \geqslant \phi[\mathrm{E}^{\mathcal{F}_n} X_{n+1}] = \phi(X_n) \tag{3.1}$$

holds, which completes the proof of a).

Similarly, as before, we assume that the function $\phi : \mathbb{R} \to \mathbb{R}$ fulfills the assumption of Theorem 3.1(b). Additionally, it is assumed that the sequence $\{X_n\}_{n=1}^{\infty}$ is a conditional submartingale. Then we change equality by inequality in (3.1), which completes the proof of b). □

Theorem 3.2 (Maximal inequalities)**.** *Let* $\{X_n\}_{n=1}^{\infty}$ *and* $\{Y_n\}_{n=1}^{\infty}$ *be a conditional submartingale and a conditional supermartingale respectively. Fix* $\lambda > 0$ *as* $\mathcal{F}$*-measurable random variable, then we get the following inequalities*

$$\lambda \mathrm{P}_{\mathcal{F}} \left\{ \max_{k \leqslant n} X_k \geqslant \lambda \right\} \leqslant \mathrm{E}^{\mathcal{F}}[X_n^+ I_{[\max_{k\leqslant n} X_k \geqslant \lambda]}] \leqslant \mathrm{E}^{\mathcal{F}} X_n^+, \tag{3.2}$$

$$\begin{aligned} \lambda \mathrm{P}_{\mathcal{F}} \left\{ \min_{k \leqslant n} X_k \leqslant -\lambda \right\} &\leqslant \mathrm{E}^{\mathcal{F}}[X_n I_{[\min_{k\leqslant n} X_k > -\lambda]}] - \mathrm{E}^{\mathcal{F}} X_0 \\ &\leqslant \mathrm{E}^{\mathcal{F}} X_n^+ - \mathrm{E}^{\mathcal{F}} X_0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \lambda \mathrm{P}_{\mathcal{F}} \left\{ \max_{k \leqslant n} Y_k \geqslant \lambda \right\} &\leqslant \mathrm{E}^{\mathcal{F}} Y_0 - \mathrm{E}^{\mathcal{F}}[Y_n I_{[\max_{k\leqslant n} Y_k < \lambda]}] \\ &\leqslant \mathrm{E}^{\mathcal{F}} Y_0 - \mathrm{E}^{\mathcal{F}} Y_n^-, \end{aligned} \tag{3.4}$$

$$\lambda \mathrm{P}_{\mathcal{F}} \left\{ \min_{k \leqslant n} Y_k \leqslant -\lambda \right\} \leqslant -\mathrm{E}^{\mathcal{F}}[Y_n I_{[\min_{k\leqslant n} Y_k \leqslant -\lambda]}] \leqslant \mathrm{E}^{\mathcal{F}} Y_n^-. \tag{3.5}$$

Proof. We define stopping times $\sigma = \inf\{k \leqslant n : Y_k \leqslant -\lambda\}$, where $\sigma = n$ if $\min_{k\leqslant n} Y_k > -\lambda$ and $\tau = \inf\{k \leqslant n : Y_k \geqslant \lambda\}$, where $\tau = n$ if $\max_{k\leqslant n} Y_k < \lambda$.

To prove the inequality (3.4) we use Theorem 2.1.

$$\begin{aligned} \mathrm{E}^{\mathcal{F}} Y_0 \geqslant \mathrm{E}^{\mathcal{F}} Y_\tau &= \mathrm{E}^{\mathcal{F}} \left[Y_\tau I_{[\max_{k\leqslant n} Y_k \geqslant \lambda]} \right] + \mathrm{E}^{\mathcal{F}} \left[Y_\tau I_{[\max_{k\leqslant n} Y_k < \lambda]} \right] \\ &\geqslant \lambda \mathrm{P}_{\mathcal{F}} \left\{ \max_{k\leqslant n} Y_k \geqslant \lambda \right\} + \mathrm{E}^{\mathcal{F}} \left[Y_n I_{[\max_{k\leqslant n} Y_k < \lambda]} \right] \end{aligned}$$

which completes the proof of (3.4) because the last part of it is a consequence of $-Y_n^- \leqslant Y_n$.

The proof of (3.5) is also based on Theorem 2.1.

$$\begin{aligned} E^{\mathcal{F}} Y_n \leqslant E^{\mathcal{F}} Y_\sigma &= E^{\mathcal{F}}\left[Y_\sigma I_{[\min_{k\leqslant n} Y_k \leqslant -\lambda]}\right] + E^{\mathcal{F}}\left[Y_\sigma I_{[\min_{k\leqslant n} Y_k > -\lambda]}\right] \\ &\leqslant -\lambda P_{\mathcal{F}}\left\{\min_{k\leqslant n} Y_k \leqslant -\lambda\right\} + E^{\mathcal{F}}\left[Y_n I_{[\min_{k\leqslant n} Y_k > -\lambda]}\right], \end{aligned}$$

where the last part of (3.5) is obvious.

To prove (3.2), (3.3) it should be noted that $X_n = -Y_n$ is a conditional submartingale and by (3.4), (3.5) we obtain

$$\begin{aligned} \lambda P_{\mathcal{F}}\left\{\max_{k\leqslant n} X_k \geqslant \lambda\right\} &= \lambda P_{\mathcal{F}}\left\{\min_{k\leqslant n} Y_k \leqslant -\lambda\right\} \leqslant -E^{\mathcal{F}}\left[Y_n I_{[\min_{k\leqslant n} Y_k \leqslant -\lambda]}\right] \\ &= E^{\mathcal{F}}\left[X_n I_{[\max_{k\leqslant n} X_k \geqslant \lambda]}\right] \leqslant E^{\mathcal{F}}\left[X_n I_{[\max_{k\leqslant n} X_k^+ \geqslant \lambda]}\right] \\ &\leqslant E^{\mathcal{F}} X_n^+, \end{aligned}$$

which competes the proof of (3.2).

Similarly we proceed in the proof of inequality (3.3)

$$\begin{aligned} \lambda P_{\mathcal{F}}\left\{\min_{k\leqslant n} X_k \leqslant -\lambda\right\} &= \lambda P_{\mathcal{F}}\left\{\max_{k\leqslant n} Y_k \geqslant \lambda\right\} \leqslant E^{\mathcal{F}} Y_0 - E^{\mathcal{F}}[Y_n I_{[\max_{k\leqslant n} Y_k < \lambda]}] \\ &= -E^{\mathcal{F}} X_0 + E^{\mathcal{F}}[X_n I_{[\min_{k\leqslant n} X_k > -\lambda]}] \leqslant E^{\mathcal{F}} X_n^+ - E^{\mathcal{F}} X_0. \end{aligned}$$

□

A natural consequence of the above theorem is the following corollary.

Corollary 3.3. *Let $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ be a conditional submartingale and a supermartingale respectively. Moreover, let $\lambda > 0$ be an $\mathcal{F}$-measurable random variable, then*

$$\lambda P_{\mathcal{F}}\left\{\max_{k\leqslant n} |X_k| \geqslant \lambda\right\} \leqslant 3\max_{k\leqslant n} E^{\mathcal{F}}|X_n|, \tag{3.6}$$

$$\lambda P_{\mathcal{F}}\left\{\max_{k\leqslant n} |Y_k| \geqslant \lambda\right\} \leqslant 3\max_{k\leqslant n} E^{\mathcal{F}}|Y_n|. \tag{3.7}$$

Proof. The proof of (3.7) is based on (3.4) and (3.2). Note that $Y_n^- = (-Y_n)^+$ is a conditional submartingale. Thus,

$$\begin{aligned} P_{\mathcal{F}}\left\{\max_{k\leqslant n} |Y_k| \geqslant \lambda\right\} &\leqslant P_{\mathcal{F}}\left\{\max_{k\leqslant n} Y_k^+ \geqslant \lambda\right\} + P_{\mathcal{F}}\left\{\max_{k\leqslant n} Y_k^- \geqslant \lambda\right\} \\ &= P_{\mathcal{F}}\left\{\max_{k\leqslant n} Y_k \geqslant \lambda\right\} + P_{\mathcal{F}}\left\{\max_{k\leqslant n} Y_k^- \geqslant \lambda\right\} \\ &\leqslant E^{\mathcal{F}} Y_0 + 2E^{\mathcal{F}} Y_n^- \leqslant 3E^{\mathcal{F}}|Y_n|. \end{aligned}$$

If we assume that $X_n = -Y_n$ and use (3.7), then (3.6) holds. □

In the next example we want to present applications of maximal inequalities in conditional version.

Example 5. Let $\{X_n\}_{n=1}^{\infty}$ and ϕ fulfill the assumptions of Theorem 3.1(a). Then the sequence $\{\phi(X_n)\}_{n=1}^{\infty}$ is a conditional submartingale, so it fulfills the assumptions of Theorem 3.2. In the particular case, when $\phi(t) = |t|$, by (3.2) it is evident that

$$\lambda \mathrm{P}_{\mathcal{F}} \left\{ \max_{k \leqslant n} |X_k| \geqslant \lambda \right\} \leqslant \mathrm{E}^{\mathcal{F}} |X_n|.$$

If $\phi(t) = t^2$ and $X_k = U_1 + \cdots + U_k,\ k = 1, 2, \ldots, n$, where U_i are $\mathcal{F}$-independent random variables such that $\mathrm{E}^{\mathcal{F}} U_i = 0$ and $\sigma^2_{\mathcal{F}} U_i < \infty$, then the conditional version of Kolmogorov's inequality [3]

$$\mathrm{P}_{\mathcal{F}} \left\{ \max_{k \leqslant n} |X_k| \geqslant \lambda \right\} \leqslant \frac{1}{\lambda^2} \sum_{k=1}^{n} \mathrm{E}^{\mathcal{F}} U_k^2$$

holds.

References

[1] O. Kallenberg, *Foundations of Modern Probability*, Springer 1997.

[2] D. Majerek, W. Nowak, and W. Zięba, *Conditional strong law of large numbers*, Inter. J. Pure and Appl. Math. **20** (2) (2005), 143-157.

[3] D. Majerek, W. Nowak, and W. Zięba, *On uniform integrability of random variables*, Stati. Prob. Letters **74** (2005), 272-280.

[4] J. Neveu, *Discrete-parameter Martingales*, North-Holland Publishing Company, 1975.

[5] W. Nowark and W. Zięba, *Types of conditional convergence*, Anna. Univ. Mariae Curie-Skłlodowska **59** (2005), 97-105.

[6] A. R. Padmanabhan, *Convergence in probability and allied result*, Math. Japon. **15** (2) (1970), 111-117.

[7] A. N. Shiryaev, *Probability*, Springer Verlag, 1995.

Department of Mathematics,
Technical University, Lublin, Poland

E-mail address: majerek@antenor.pol.lublin.pl

Institute of mathematics,
Maria Curie-Skłodowska University,
Lublin 20-031, pl. Marii Curie-Skłodowskiej 1, Poland

E-mail address: zieba@golem.umcs.lublin.pl