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ON A SUFFICIENT CONDITION FOR THE CENTRAL LIMIT THEOREM

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1. Introduction

In this paper we shall introduce a sufficient condition for the Central Limit Theorem ($C.L.T$) in the case where investigated sequences of random variables ($r.v'.s$) have finite mathematical expectation (Theorem 1). In the classical case with finite second moments we get our sufficient condition by the familiar Lindeberg's condition. We shall introduce an example where our sufficient condition is satisfied but the Lindeberg's condition is not. We also show that our sufficient condition can be used when second moments are not finite.

2. Main results

Theorem 1. *Let $X_n, n = 1, 2, \ldots$ be a sequence of independent r.v'.s with distribution functions $F_n, n = 1, 2, \ldots$ and $EX_n = 0, n = 1, 2, \ldots$; and $B_n, n = 1, 2, \ldots$ be a positive sequence of numbers which increases to infinity. Suppose that*

(i)
$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{-\infty}^{\infty}\frac{x^2}{B_n^2+x^2}dF_k(x) = 1,$$

(ii)
$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{|x|^3}{B_n(B_n^2+x^2)}dF_k(x) = 0;$$

then we have for every x

$$\lim_{n\to\infty} P\Big(\frac{S_n}{B_n}\leqslant x\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt = \Phi(x), \quad x\in\mathbb{R}$$

where $S_n = \sum_{k=1}^{n} X_k$.

We need the following familiar lemma.

Lemma. *Let $Y_n, T_n, U_n, n = 1, 2, \ldots$ be sequences of r.v'.s such that $Y_n = T_n + U_n, n = 1, 2, \ldots$ Suppose that*

($*$) *T_n converges to 0 in probability ($T_n \to 0$ in pr.),*

($**$) *U_n converges in distribution to F ($U_n \to F$ in dist.);*

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then $Y_n \to F$ *in dist.*

Proof of Theorem 1. Put

$$t_{nk} = \frac{X_k^3}{B_n(B_n^2 + X_k^2)}, \qquad u_{nk} = \frac{B_n X_k}{B_n^2 + X_k^2},$$
$$T_n = \sum_{k=1}^{n} t_{nk}, \qquad U_n = \sum_{k=1}^{n} u_{nk},$$
$$Y_n = \frac{S_n}{B_n} = \frac{\sum_{k=1}^{n} X_k}{B_n}.$$

We have

$$Y_n = T_n + U_n, n = 1, 2, \ldots$$

We shall prove that $T_n \to 0$ in pr. and $U_n \to \Phi$ in dist.

By (ii) we have $\lim_{n\to\infty} E|T_n| = 0$; hence $T_n \to 0$ in pr. To prove that $U_n \to \Phi$ in dist. we shall use the general $C.L.T$ in the case of finite second moments. By Theorem 3[3] (pages 101, 102, 103) it suffices to prove that for every $\varepsilon > 0$

(A) $$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|u|\geqslant\varepsilon} u^2 dG_{nk}(u + a_{nk}) = 0,$$

where $G_{nk}(u) = P(u_{nk} \leqslant u), a_{nk} = Eu_{nk}$, and

(B) $$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|u|<\varepsilon} u^2 dG_{nk}(u + a_{nk}) = 1.$$

It is easy to see that (A) and (B) are equivalent to the following conditions, respectively: For every $\varepsilon > 0$

(A') $$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|u|\geqslant 2\varepsilon} u^2 dG_{nk}(u + a_{nk}) = 0,$$

(B') $$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{-\infty}^{+\infty} u^2 dG_{nk}(u + a_{nk}) = 1.$$

Now we prove (A'). For every $\varepsilon > 0$ (we only need to consider $0 < \varepsilon < \frac{1}{2}$), put $\delta = \frac{2\varepsilon}{1+\sqrt{1-4\varepsilon^2}}$. One can easily check that

$$\{\omega \in \Omega : \ |u_{nk}| \geqslant \varepsilon\} = \left\{\omega \in \Omega : \ \frac{B_n|X_n|}{B_n^2 + X_n^2} \geqslant \varepsilon\right\} \subset \{\omega \in \Omega : \ |X_k| \geqslant \delta B_n\}.$$

We have

$$\begin{aligned}
\sum_{k=1}^{n} a_{nk}^2 &= \sum_{k=1}^{n}\left(\int_{-\infty}^{+\infty} u dG_{nk}(u)\right)^2 = \sum_{k=1}^{n}\left(\int_{-\infty}^{+\infty} \frac{B_n x}{B_n^2+x^2} dF_k(x)\right)^2 \\
&= \sum_{k=1}^{n}\left(\int_{-\infty}^{+\infty} \frac{x}{B_n} dF_k(x) - \int_{-\infty}^{+\infty} \frac{x^3}{B_n(B_n^2+x^2)} dF_k(x)\right)^2 \\
&= \sum_{k=1}^{n}\left(\int_{-\infty}^{+\infty} \frac{x^3}{B_n(B_n^2+x^2)} dF_k(x)\right)^2 \leqslant \sum_{k=1}^{n}\left(\int_{-\infty}^{+\infty} \frac{|x|^3}{B_n(B_n^2+x^2)} dF_k(x)\right)^2 \\
&\leqslant \left(\sum_{k=1}^{n}\int_{-\infty}^{+\infty} \frac{|x|^3}{B_n(B_n^2+x^2)} dF_k(x)\right)^2.
\end{aligned}$$

It follows from (ii) that $\lim_{n\to\infty}\sum_{k=1}^{n} a_{nk}^2 = 0$, hence $\lim_{n\to\infty}\max_{1\leqslant k\leqslant n}\{|a_{nk}|\} = 0$. Thus for sufficiently large n we have $|a_{nk}| < \varepsilon, k = 1, \dots, n$.

We have for sufficiently large n

$$\begin{aligned}
&\sum_{k=1}^{n}\int_{|u|\geqslant 2\varepsilon} u^2 dG_{nk}(u+a_{nk}) \\
=&\sum_{k=1}^{n}\int_{|u-a_{nk}|\geqslant 2\varepsilon} (u-a_{nk})^2 dG_{nk}(u) \\
=&\int_{\substack{|u|\geqslant 2\varepsilon \\ |u-a_{nk}|\geqslant 2\varepsilon}} (u-a_{nk})^2 dG_{nk}(u) + \sum_{k=1}^{n}\int_{\substack{|u|<\varepsilon \\ |u-a_{nk}|\geqslant 2\varepsilon}} (u-a_{nk})^2 dG_{nk}(u) \\
\leqslant&\sum_{k=1}^{n}\int_{|u|\geqslant\varepsilon} (u-a_{nk})^2 dG_{nk}(u) \\
\leqslant&\sum_{k=1}^{n}\int_{|u|\geqslant\varepsilon} u^2 dG_{nk}(u) - 2\sum_{k=1}^{n} a_{nk}\int_{|u|\geqslant\varepsilon} u dG_{nk}(u) + \sum_{k=1}^{n} a_{nk}^2\int_{|u|\geqslant\varepsilon} dG_{nk}(u) \\
\leqslant&\sum_{k=1}^{n}\int_{|x|\geqslant\delta B_n} \frac{B_n^2}{B_n^2+x^2}\frac{x^2}{B_n^2+x^2} dF_k(x) + 2\sum_{k=1}^{n}|a_{nk}|\int_{|x|\geqslant\delta B_n}\frac{B_n|x|}{B_n^2+x^2} dF_k(x) + \sum_{k=1}^{n} a_{nk}^2 \\
\leqslant&\sum_{k=1}^{n}\int_{|x|\geqslant\delta B_n} \frac{x^2}{B_n^2+x^2} dF_k(x) + \frac{2}{\delta^2}\sum_{k=1}^{n}\int_{|x|\geqslant\delta B_n}\frac{|x|^3}{B_n(B_n^2+x^2)} dF_k(x) + \sum_{k=1}^{n} a_{nk}^2 \\
\leqslant&\frac{1}{\delta}\sum_{k=1}^{n}\int_{|x|\geqslant\delta B_n} \frac{|x|^3}{B_n(B_n^2+x^2)} dF_k(x) + \frac{2}{\delta^2}\sum_{k=1}^{n}\int_{|x|\geqslant\delta B_n}\frac{|x|^3}{B_n(B_n^2+x^2)} dF_k(x) + \sum_{k=1}^{n} a_{nk}^2 \\
\leqslant&\frac{\delta+2}{\delta^2}\sum_{k=1}^{n}\int_{|x|\geqslant\delta B_n} \frac{|x|^3}{B_n(B_n^2+x^2)} dF_k(x) + \sum_{k=1}^{n} a_{nk}^2.
\end{aligned}$$

Hence, by (ii) we get (A').

Next, we prove (B'). We have

$$\sum_{k=1}^{n}\int_{-\infty}^{+\infty}u^2dG_{nk}(u+a_{nk})=\sum_{k=1}^{n}\int_{-\infty}^{+\infty}(u-a_{nk})^2dG_{nk}(u)$$

$$=\sum_{k=1}^{n}\int_{-\infty}^{+\infty}u^2dG_{nk}(u)-\sum_{k=1}^{n}a_{nk}^2 \tag{2.1}$$

$$\sum_{k=1}^{n}\int_{-\infty}^{+\infty}u^2dG_{nk}(u)=\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{B_n^2}{B_n^2+x^2}\frac{x^2}{B_n^2+x^2}dF_k(x)$$

$$\leqslant\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{x^2}{B_n^2+x^2}dF_k(x). \tag{2.2}$$

For every $\tau>0$ we also have

$$\sum_{k=1}^{n}\int_{-\infty}^{+\infty}u^2dG_{nk}(u)=\frac{B_n^2}{B_n^2+x^2}\int_{-\infty}^{+\infty}\frac{x^2}{B_n^2+x^2}dF_k(x)$$

$$\geqslant\int_{|x|<\tau B_n}\frac{B_n^2}{B_n^2+x^2}\frac{x^2}{B_n^2+x^2}dF_k(x)$$

$$\geqslant\frac{1}{1+\tau^2}\sum_{k=1}^{n}\int_{|x|<\tau B_n}\frac{x^2}{B_n^2+x^2}dF_k(x). \tag{2.3}$$

Note that by (ii) we have $\lim_{n\to\infty}\sum_{k=1}^{n}\int_{|x|\geqslant\tau B_n}\frac{x^2}{B_n^2+x^2}dF_k(x)=0$ and therefore $\lim_{n\to\infty}\sum_{k=1}^{n}\int_{|x|<\tau B_n}\frac{x^2}{B_n^2+x^2}dF_k(x)=1$ by (i).

From (2.1), (2.2) and (2.3) we get (B') by letting first $n\to\infty$ and then $\tau\to 0$. Theorem 1 is completely proved. □

Notes.

1. It is not hard to see that the conditions (i) and (ii) in Theorem 1 are equivalent to the following conditions, respectively: For every $\varepsilon>0$:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{|x|<\varepsilon B_n}\frac{x^2}{B_n^2+x^2}dF_k(x)=1, \tag{i'}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{|x|\geqslant\varepsilon B_n}\frac{|x|^3}{B_n(B_n^2+x^2)}dF_k(x)=0. \tag{ii'}$$

2. In the classical case we assume that $B_n^2=\sum_{k=1}^{n}Var(X_k)$. Then from Lindeberg's condition we get the conditions (i') and (ii') because for every

$\varepsilon > 0:$

$$\frac{1}{1+\varepsilon^2}\sum_{k=1}^{n}\int_{|x|<\varepsilon B_n}\frac{x^2}{B_n^2}dF_k(x) \leqslant \sum_{k=1}^{n}\int_{|x|<\varepsilon B_n}\frac{x^2}{B_n^2+x^2}dF_k(x) \leqslant \sum_{k=1}^{n}\int_{|x|<\varepsilon B_n}\frac{x^2}{B_n^2}dF_k(x),$$

and

$$\sum_{k=1}^{n}\int_{|x|\geqslant\varepsilon B_n}\frac{|x|^3}{B_n(B_n^2+x^2)}dF_k(x) \leqslant \frac{1}{2}\sum_{k=1}^{n}\int_{|x|\geqslant\varepsilon B_n}\frac{x^2}{B_n^2}dF_k(x).$$

3. When using Chebyshev inequality to prove $T_n \to 0$ in pr. and $U_n \to 0$ in pr. (U_n, T_n as in the proof of Theorem 1) we obtain the next result for the Large Number Law.

Proposition 2. *Let $X_n, n = 1, 2, \dots$ be a sequence of independent r.v'.s with distribution functions $F_n, n = 1, 2, \dots$, and $EX_n = 0, n = 1, 2, \dots$; and $B_n, n = 1, 2, \dots$ be a positive sequence of numbers which increases to infinity. Assume that*

(i*)
$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{x^2}{B_n^2+x^2}dF_k(x) = 0,$$

(ii*)
$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{|x|^3}{B_n(B_n^2+x^2)}dF_k(x) = 0.$$

Then $\frac{S_n}{B_n} \to 0$ in pr.

3. Examples

3.1. We introduce an example where the conditions (*i*) and (*ii*) in Theorem 1 are satisfied but Lindeberg's condition is not.

Let

$$X_k = \begin{cases} \pm k^2 & \text{with probability } \frac{1}{12k^2} \text{ each;} \\ \pm k & \text{with probability } \frac{1}{12} \text{ each;} \\ 0 & \text{with probability } 1 - \frac{1}{6} - \frac{1}{6k^2}. \end{cases}$$

It is known that Lindeberg's condition is not satisfied (see [1], pages 194, 195). We now show that the conditions (*i*) and (*ii*) in Theorem 1 are satisfied.

For every $c > 0$, we have

$$\begin{aligned}
&\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{|x|^3}{\sqrt{c}n^{3/2}(cn^3+x^2)}dF_k(x)\\
=&\frac{1}{6\sqrt{c}n^{3/2}}\sum_{k=1}^{n}\frac{k^4}{cn^3+k^4}+\frac{1}{6\sqrt{c}n^{3/2}}\sum_{k=1}^{n}\frac{k^3}{cn^3+k^2}\\
\leqslant&\frac{1}{6\sqrt{c}n^{3/2}}n+\frac{1}{6c\sqrt{c}n^{3/2}}n\leqslant\frac{1}{6\sqrt{c}\sqrt{n}}\Big(1+\frac{1}{c}\Big).
\end{aligned}$$

Hence (ii) is satisfied for $B_n^2 = cn^3$.

On the other hand we have

$$\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{x^2}{cn^3+x^2}dF_(x)=\frac{1}{6}\sum_{k=1}^{n}\frac{k^2}{cn^3+k^4}+\frac{1}{6}\sum_{k=1}^{n}\frac{k^2}{cn^3+k^2}. \tag{3.1}$$

By investigating the function $f(x) = x^4 - \sqrt{n}x^3 + cn^3$, it is easy to check that $\frac{k^2}{cn^3+k^4} \leqslant \frac{1}{k\sqrt{n}}$ for $n > \frac{27}{256c}$. Hence

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{cn^3+k^4}=0. \tag{3.2}$$

We also have

$$\frac{1}{cn^3+n^2}\sum_{k=1}^{n}k^2\leqslant\sum_{k=1}^{n}\frac{k^2}{cn^3+k^2}\leqslant\frac{1}{cn^3}\sum_{k=1}^{n}k^2.$$

Letting $n\to\infty$ we get

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{cn^3+k^2}=\frac{1}{3c}. \tag{3.3}$$

From $(3.1), (3.2)$ and (3.3) we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{x^2}{cn^3+x^2}dF_k(x)=\frac{1}{18c}.$$

Thus for $B_n^2 = \frac{n^3}{18}$ the conditions (i) and (ii) in Theorem 1 are satisfied.

3.2. We can use the conditions (i) and (ii) in Theorem 1 when second moments are not finite.

Let $X_k, k = 1, 2, \dots$ be a sequence of independent $r.v'.s$ with the following densities

$$f_k(x)=\begin{cases}\frac{k^2}{|x|^3} & \text{if } |x|\geqslant k,\\ 0 & \text{if } |x|<k.\end{cases}$$

Then $\frac{S_n}{B_n} \to \Phi$ in dist. for $B_n^2 = \frac{n^3\ln n}{3}$.

We first note that

$$\int_{-\infty}^{+\infty} \frac{dx}{x(B_n^2+x^2)} = \frac{1}{B_n^2}(\ln\sqrt{B_n^2+k^2} - \ln k).$$

Then

$$\begin{aligned}\sum_{k=1}^{n}\int_{-\infty}^{+\infty} \frac{x^2}{B_n^2+x^2} f_k(x)dx &= 2\sum_{k=1}^{n} k^2 \int_{-\infty}^{\infty} \frac{dx}{x(B_n^2+x^2)} \\ &= \frac{2}{B_n^2}\sum_{k=1}^{n} k^2 \ln\sqrt{B_n^2+k^2} - \frac{2}{B_n^2}\sum_{k=1}^{n} k^2 \ln k. \qquad (3.4)\end{aligned}$$

But

$$\begin{aligned}\frac{6}{n^3\ln n}\sum_{k=1}^{n} k^2 \ln\sqrt{\frac{n^3\ln n}{3}} &\leqslant \frac{2}{B_n^2}\sum_{k=1}^{n} k^2\ln\sqrt{B_n^2+k^2} \\ &\leqslant \frac{6}{n^3\ln n}\sum_{k=1}^{n} k^2 \ln\sqrt{n^3\ln n + n^2}.\end{aligned}$$

Letting $n \to \infty$ we get

$$\lim_{n\to\infty} \frac{2}{B_n^2}\sum_{k=1}^{n} k^2\ln\sqrt{B_n^2+k^2} = 3. \qquad (3.5)$$

We also have

$$\frac{2}{B_n^2}\sum_{k=1}^{n} k^2 \ln k = \frac{6}{n\ln n}\sum_{k=1}^{n}\frac{k^2}{n^2}\ln\frac{k}{n} + \frac{6\ln n}{n^3\ln n}\sum_{k=1}^{n} k^2.$$

Note that $\lim_{n\to\infty} \frac{6}{n\ln n}\sum_{k=1}^{n}\frac{k^2}{n^2}\ln\frac{k}{n} = 0$, we have

$$\lim_{n\to\infty}\frac{2}{B_n^2}\sum_{k=1}^{n} k^2\ln k = 2. \qquad (3.6)$$

From (3.4), (3.5) and (3.6) we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{x^2}{B_n^2+x^2} f_k(x)dx = 1.$$

We have

$$\begin{aligned}\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{|x|^3}{B_n(B_n^2+x^2)} f_k(x)dx &= \frac{2}{B_n}\sum_{k=1}^{n} k^2 \int_{k}^{+\infty}\frac{dx}{B_n^2+x^2} \\ &\leqslant \frac{2\sqrt{3}}{\sqrt{n^3\ln n}}\sum_{k=1}^{n} k^2\int_{k}^{+\infty}\frac{dx}{B_n^2+x^2} \\ &\leqslant \frac{2\sqrt{3}}{\sqrt{n^3\ln n}}\frac{\sqrt{3}}{\sqrt{n^3\ln n}}\frac{\pi}{2}\sum_{k=1}^{n} k^2.\end{aligned}$$

Hence we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{-\infty}^{+\infty}\frac{|x|^3}{B_n(B_n^2+x^2)}f_k(x)dx=0.$$

References

[1] K. L. Chung, *A course in Probability Theory*, Harcourt, Barace & World, Inc., 1968.

[2] W. Feller, *An intruduction to Probability Theory and Its Applications,* John Wiley & Sons, Inc., NewYork, 1968.

[3] B. V. Gnedenko and A. N. Kolmogoroff, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley Publishing Company, Inc., 1954.

[4] A. N. Shyraev, *Probability*, Springer-Verlag, NewYork, 1996.

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