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ON SOME GENERALIZED VECTOR EQUILIBRIUM PROBLEMS WITH SET-VALUED MAPS

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ABSTRACT. In this paper we consider the generalized vector equilibrium problem (P_α) of finding a point $(z_0, x_0) \in E \times K$ such that $x_0 \in A(z_0, x_0)$ and

$$\forall \eta \in A(z_0, x_0), \exists z \in B(z_0, x_0, \eta), \quad (F(z, x_0, \eta), C(z, x_0, \eta)) \in \alpha,$$

where α is an arbitrary relation on 2^Y, and A, B, C and F are set-valued maps between finite-dimensional spaces. Existence results are obtained under assumptions different from those of [17]. Some special cases of Problem (P_α) are discussed in detail.

1. INTRODUCTION

Let X, Y and Z be topological vector spaces. Let $K \subset X$ and $E \subset Z$ be nonempty subsets. Let $\mathbb{B} : K \longrightarrow 2^E$, $\mathbb{C} : K \longrightarrow 2^Y$ and $F : E \times K \times K \longrightarrow 2^Y$ be set-valued maps with nonempty values. Under suitable assumptions existence results are obtained in [7] for the following generalized vector equilibrium problem: find a point $x_0 \in K$ such that

$$\forall \eta \in K, \exists v \in \mathbb{B}(x_0),\ F(v, x_0, \eta) \not\subset \mathbb{C}(x_0). \tag{1.1}$$

These results are extensions of those given in [1] and [10]. A generalized version of the above problem is studied in [17]. More precisely, the following Problem (P_α) is considered in [17]:

Problem (P_α) : Find a point $(z_0, x_0) \in E \times K$ such that $x_0 \in A(z_0, x_0)$ and for each $\eta \in A(z_0, x_0)$,

$$\exists v \in B(z_0, x_0, \eta),\ (F(v, x_0, \eta), C(v, x_0, \eta)) \in \alpha \tag{1.2}$$

where α is an arbitrary relation on 2^Y (i.e., a subset of $2^Y \times 2^Y$); $A : E \times K \longrightarrow 2^K$, $B : E \times K \times K \longrightarrow 2^E$, $C : E \times K \times K \longrightarrow 2^Y$ and $F : E \times K \times K \longrightarrow 2^Y$ are set-valued maps with nonempty values.

Obviously, the generalized vector equilibrium problem mentioned above in [7] is a special case of (P_α) with $A(z, x) \equiv K$, $B(z, x, \eta) \equiv \mathbb{B}(x)$, $C(z, x, \eta) \equiv \mathbb{C}(x)$ and $\alpha = \{(a, b) \in 2^Y \times 2^Y : a \not\subset b\}$.

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Existence results for solutions of (P_α) are given in [17], with the help of a fixed point theorem of [16]. In this paper, assuming that all spaces are finite-dimensional, we prove that such results can be obtained under assumptions different from those of [17]. For a detailed comparison, see Remark 3 of Section 3. Our main result is established in Theorem 3.1 of Section 3, with the help of a theorem of existence of continuous selections of [14] and the known Brouwer fixed point theorem. Section 4 is devoted to special cases of Problem (P_α) where B does not depend on z and η, and α is one of the following relations

$$\begin{aligned}
\alpha_1 &= \{(a,b) \in 2^Y \times 2^Y : a \not\subset b\},\\
\alpha_2 &= \{(a,b) \in 2^Y \times 2^Y : a \subset b\},\\
\alpha_3 &= \{(a,b) \in 2^Y \times 2^Y : a \cap b \neq \emptyset\},\\
\alpha_4 &= \{(a,b) \in 2^Y \times 2^Y : a \cap b = \emptyset\}
\end{aligned}$$

($\emptyset$ being the empty set).

A comparison of the results of Section 4 and those given in [7] can be found in Remark 12 of Section 4. The reader who is interested in generalizations of vector equilibrium problems different from those considered in this paper is referred to [22, 23] and references therein.

We conclude this introduction by showing a motivation for considering our general model (Problem (P_α)). Namely, this model provides a unified approach to several vector equilibrium problems with set-valued maps. More precisely, the problems which are investigated in [2, 6, 7, 8, 11, 15, 18] can be interpreted as special cases of our general model with different relations α. We now discuss in more detail how some different quasivariational inclusion problems which appear recently in [19] can reduce to Problem (P_α). The first of them is called the upper quasivariational inclusion problem (shortly, $(UQVIP)$) which is formulated as follows: given topological vector spaces Y, X_i $(i = 1, 2)$, a convex cone $\mathcal{C} \subset Y$, nonempty convex sets $D_i \subset X_i$ $(i = 1, 2)$, set-valued maps $S : D_1 \longrightarrow 2^{D_1}$, $T : D_1 \longrightarrow 2^{D_2}$ and $\mathcal{F} : D_1 \times D_2 \times D_1 \longrightarrow 2^Y$, find a point $(\xi_1^0, \xi_2^0) \in D_1 \times D_2$ such that $\xi_1^0 \in S(\xi_1^0)$, $\xi_2^0 \in T(\xi_1^0)$ and, for all $\eta_1 \in S(\xi_1^0)$,

$$\mathcal{F}(\xi_1^0, \xi_2^0, \eta_1) \subset \mathcal{F}(\xi_1^0, \xi_2^0, \xi_1^0) + \mathcal{C}. \tag{1.3}$$

The second problem considered in [19] is called the lower quasivariational inclusion problem (shortly, $(LQVIP)$) which is to find a point $(\xi_1^0, \xi_2^0) \in D_1 \times D_2$ such that $\xi_1^0 \in S(\xi_1^0)$, $\xi_2^0 \in T(\xi_1^0)$ and, for all $\eta_1 \in S(\xi_1^0)$, instead of condition (1.3) it is required that

$$\mathcal{F}(\xi_1^0, \xi_2^0, \xi_1^0) \subset \mathcal{F}(\xi_1^0, \xi_2^0, \eta_1) - \mathcal{C}.$$

We can see that, though $(UQVIP)$ and $(LQVIP)$ are different problems, they can be treated as special cases of our general model. Indeed, let us set $K := D_1 \times D_2 \subset X := X_1 \times X_2$, $E := D_1 \subset Z := X_1$, $A(z, x) = S(\xi_1) \times T(\xi_1)$,

$$F(z, x, \eta) = \mathcal{F}(\xi_1, \xi_2, \eta_1), \;\; C(z, x, \eta) = \mathcal{F}(\xi_1, \xi_2, z) + \mathcal{C}, \tag{1.4}$$

$B(z,x,\eta)=\{\xi_1\}$ where $z\in E:=D_1, x=(\xi_1,\xi_2)\in K:=D_1\times D_2$ and $\eta=(\eta_1,\eta_2)\in K:=D_1\times D_2$. Then it is clear that $(UQVIP)$ is a special case of Problem (P_α) with $\alpha=\alpha_2$. Similarly, $(LQVIP)$ is also a special case of Problem (P_α) with $\alpha=\alpha_2$ if instead of (1.4) we set

$$F(z,x,\eta)=\mathcal{F}(\xi_1,\xi_2,z),\ C(z,x,\eta)=\mathcal{F}(\xi_1,\xi_2,\eta_1)-\mathcal{C}.$$

Using the same method we can show that all problems studied in [19] can be regarded as special cases of Problem (P_α).

In [12] two variational inclusion problems with constraints are introduced. The first of them, called Problem (V), is formulated as follows : given locally convex topological vector spaces Y, X_i $(i=1,2)$, a convex cone $\mathcal{C}\subset Y$, nonempty convex sets $D_i\subset X_i$ $(i=1,2)$, set-valued maps $S_1,S_2:D_1\longrightarrow 2^{D_1}$, $T:D_1\times D_1\longrightarrow 2^{D_2}$ and $\Phi:D_1\times D_2\times D_1\longrightarrow 2^Y$, find a point $\xi_1^0\in D_1$ such that $\xi_1^0\in S_1(\xi_1^0)$ and, for all $\eta_1\in S_2(\xi_1^0)$ and $\eta_2\in T(\eta_1,\xi_1^0)$,

$$\Phi(\xi_1^0,\eta_2,\eta_1)\subset\Phi(\xi_1^0,\eta_2,\xi_1^0)+\mathcal{C}.$$

The second problem in [12], called Problem (V'), is to find a point $\xi_1^0\in D_1$ such that $\xi_1^0\in S_1(\xi_1^0)$ and, for all $\eta_1\in S_2(\xi_1^0)$ and $\eta_2\in T(\xi_1^0,\xi_1^0)$,

$$\Phi(\xi_1^0,\eta_2,\xi_1^0)\subset\Phi(\xi_1^0,\eta_2,\eta_1)-\mathcal{C}.$$

These problems are also special cases of our general model. Indeed, let us set $K:=D_1\times D_2\times D_1\subset X:=X_1\times X_2\times X_1$, $E:=D_1\subset Z:=X_1$,

$$S(\xi_1)=\{(\eta_2',\eta_1')\in D_2\times D_1:\eta_1'\in S_2(\xi_1),\eta_2'\in T(\eta_1',\xi_1)\},\tag{1.5}$$

$$F(z,x,\eta)=\Phi(\xi_1,\eta_2',\eta_1'),\tag{1.6}$$

$$C(z,x,\eta)=\Phi(z,\eta_2',\xi_1)+\mathcal{C},\tag{1.7}$$

$$A(z,x)=S_1(\xi_1)\times S(\xi_1),$$

$$B(z,x,\eta)=\{\xi_1\},$$

where $z\in E:=D_1$, $x=(\xi_1,\xi_2',\xi_1')\in K:=D_1\times D_2\times D_1$ and $\eta=(\eta_1,\eta_2',\eta_1')\in K:=D_1\times D_2\times D_1$. Then it is clear that Problem (V) is a special case of Problem (P_α) with $\alpha=\alpha_2$. Similarly, Problem (V') is also a special case of Problem (P_α) with $\alpha=\alpha_2$ if instead of (1.5)-(1.7) we set

$$S(\xi_1)=\{(\eta_2',\eta_1')\in D_2\times D_1:\eta_1'\in S_2(\xi_1),\eta_2'\in T(\xi_1,\xi_1)\},$$

$$F(z,x,\eta)=\Phi(z,\eta_2',\xi_1)$$

$$C(z,x,\eta)=\Phi(\xi_1\eta_2',\eta_1')-\mathcal{C}.$$

2. Preliminaries

We first recall some definitions taken from [3]. Let $f:X\longrightarrow 2^Y$ be a set-valued map between topological spaces X and Y. We say that f is upper semicontinuous (usc) at $x\in X$ if for each open set $N\supset f(x)$ there exists a neighbourhood $U(x)$ of x such that $N\supset f(x')$ for each $x'\in U(x)$. A set-valued map f is lower semicontinuous (lsc) at $x\in X$ if for each open set N with $f(x)\cap N\neq\emptyset$ there exists a neighbourhood $U(x)$ of x such that $f(x')\cap N\neq\emptyset$ for each $x'\in U(x)$. A set-valued map f is continuous at $x \in X$ if it is both usc and lsc at this point. A set-valued map f is usc (resp. lsc; continuous) if it is usc (resp. lsc; continuous) at each point $x \in X$. If the graph of f, denoted by gr $f := \{(x,y) \in X \times Y : y \in f(x)\}$, is a closed (resp. open) set of $X \times Y$ then we say that f has closed (resp. open) graph. A map having closed graph is also said to be a closed map.

If X' (resp. Y') is a subset of a topological space X (resp. Y) then we use the symbol $f : X' \longrightarrow 2^{Y'}$ to denote that f is a set-valued map from the topological space X' into the topological space Y' where the topology of X' (resp. Y') is the topology induced by the given topology of X (resp. Y). In this case, the notion of semicontinuity or continuity of f, the notion of closedness or openness of the graph of $f, \ldots$ are considered with respect to the just mentioned topologies of X' and Y'. Thus, if we say that $f : X' \longrightarrow 2^{Y'}$ has closed (resp. open) graph then this means that the set $\{(x', y') \in X' \times Y' : y' \in f(x')\}$ is closed (resp. open) in $X' \times Y'$.

Let $c : W \longrightarrow 2^Y$ and $f : W \longrightarrow 2^Y$ be set-valued maps between some sets W and Y. Let β be a relation on 2^Y, i.e., a subset of the Cartesian product $2^Y \times 2^Y$. For simplicity of notation let us write $(f,c)(w) \in \beta$ instead of $(f(w), c(w)) \in \beta$, where $w \in W$.

We now recall a generalized convexity notion which will be used later. Let $a \subset X$ be a nonempty convex set and $c' \subset Y$ be a convex cone where X and Y are vector spaces. A set-valued map $f : a \longrightarrow 2^Y$ is called natural c'-quasiconvex on a if for all $x_i \in a,\ i = 1, 2$, and $\gamma \in \,]0, 1[$

$$f(\gamma x_1 + (1-\gamma)x_2) \subset \text{co}\,\{f(x_i),\ i = 1,2\} - c',$$

where "co" denotes the convex hull. This definition and other notions of generalized convexity for the single-valued case can be found in [5, 20].

3. Main result

In this paper we assume that X, Y and Z are finite-dimensional spaces, $E \subset Z$ and $K \subset X$ are nonempty sets, and $A : E \times K \longrightarrow 2^K$, $B : W \longrightarrow 2^E$, $C : W \longrightarrow 2^Y$ and $F : W \longrightarrow 2^Y$ are set-valued maps with nonempty values where $W = E \times K \times K$ is the Cartesian product of topological spaces E, K and K. Let α be a relation on 2^Y, and $L_\alpha : E \times K \longrightarrow 2^K$ be a set-valued map defined by

$$L_\alpha(z,x) = \{\eta \in K : \forall v \in B(z,x,\eta), (F,C)(v,x,\eta) \notin \alpha\},\ \forall (z,x) \in E \times K,$$

i.e., for each $(z,x) \in E \times K$, $L_\alpha(z,x)$ is the set of $\eta \in K$ for which condition (1.2) with (z,x) in place of (z_0,x_0) does not hold.

The following result gives sufficient conditions for the existence of a solution of Problem (P_α). This is the main result of this paper.

Theorem 3.1. *Let $E \subset Z$ and $K \subset X$ be nonempty compact convex sets, and let $A : E \times K \longrightarrow 2^K$ be a lsc set-valued map with nonempty convex values such that the set*

$$M := \{(z,x) \in E \times K : x \in A(z,x)\}$$

is closed in $E \times K$. *Assume that there exists a set-valued map* $L : E \times K \longrightarrow 2^K$ *satisfying the following conditions :*

(i) $L_\alpha \subset L$, *i.e.*, $L_\alpha(z,x) \subset L(z,x), \forall (z,x) \in E \times K$.
(ii) L *has open graph.*
(iii) $x \notin \text{co } L(z,x),\ \forall (z,x) \in M$.

Then there exists a solution of Problem (P_α).

Proof. Let $M \neq \emptyset$. It is enough to show that there exists a point $(z_0, x_0) \in M$ such that $A(z_0,x_0) \cap L_\alpha(z_0,x_0) = \emptyset$. Indeed, otherwise we get by (i)

$$\emptyset \neq A(z,x) \cap L(z,x) \subset A(z,x) \cap \widehat{L}(z,x)$$

for each $(z,x) \in M$ where $\widehat{L}(z,x) = \text{co } L(z,x)$. Therefore, the map $H : E \times K \longrightarrow 2^K$ defined by

$$H(z,x) = \begin{cases} A(z,x) \cap \widehat{L}(z,x) & \text{if } (z,x) \in M, \\ A(z,x) & \text{if } (z,x) \in [E \times K] \setminus M \end{cases}$$

has nonempty convex values. Since by (ii) L has open graph it follows from [24] that $\widehat{L}$ has open graph. Combining this fact with the lower semicontinuity of A we obtain from [24] that the map

$$(z,x) \in M \longmapsto A(z,x) \cap \widehat{L}(z,x)$$

is lsc. From this and from the definition of H we can verify that H is lsc. This fact can be used to check the lower semicontinuity of the map

$$(z,x) \in E \times K \longmapsto \phi(z,x) := E'(z,x) \times H(z,x) \subset E \times K$$

where $E' : E \times K \longrightarrow 2^E$ is the constant map defined by $E'(z,x) \equiv E$. Since $E \times K$ is a compact convex set it follows from [14, Theorem 3.1‴] that ϕ has a continuous selection, i.e., a continuous single-valued map $\varphi : E \times K \longrightarrow E \times K$ such that $\varphi(z,x) \in \phi(z,x)$ for each $(z,x) \in E \times K$. Applying the Brouwer fixed point theorem to φ proves that φ has a fixed point denoted by $(z_0,x_0) \in E \times K$. Obviously, (z_0,x_0) is also a fixed point of ϕ. Thus, $(z_0,x_0) \in \phi(z_0,x_0)$, i.e., $z_0 \in E$ and $x_0 \in H(z_0,x_0) \subset A(z_0,x_0)$. This yields $(z_0,x_0) \in M$ and hence, by the definition of H, $x_0 \in A(z_0,x_0) \cap \widehat{L}(z_0,x_0) \subset \text{co } L(z_0,x_0)$, a contradiction to condition (iii).

To complete our proof it remains to show that $M \neq \emptyset$. Indeed, let us consider the following set-valued map

$$(z,x) \in E \times K \longmapsto \phi'(z,x) := E'(z,x) \times A(z,x) \subset E \times K$$

which, by [14, Theorem 3.1‴], has a continuous selection $\varphi' : E \times K \longrightarrow E \times K$. Applying the Brouwer fixed point theorem to φ' proves that φ' has a fixed point denoted by (z_0', x_0'). Since $(z_0',x_0') = \varphi'(z_0',x_0') \in \phi'(z_0',x_0') = E'(z_0',x_0') \times A(z_0',x_0')$ we obtain $z_0' \in E'(z_0',x_0') = E$ and $x_0' \in A(z_0',x_0')$. Thus, $(z_0',x_0') \in M$, i.e., $M \neq \emptyset$, as desired. □

Remark 1. Observe that the set M in Theorem 3.1 is closed in $E \times K$ if A has closed graph.

Remark 2. The compactness and convexity of both the sets E and K in Theorem 3.1 can be relaxed if we make some additional assumptions. We delete the detailed discussion of this claim, noting that it is based on the approach of Tian [21] who deals with a similar situation in [21].

Let us consider the set

$$\begin{aligned} W_1 &= \{w = (z, x, \eta) \in W : (z, x) \in M,\ \ \eta \in A(z, x)\} \\ &= \{w = (z, x, \eta) \in W : x \in A(z, x),\ \ \eta \in A(z, x)\}. \end{aligned}$$

We say that condition (ps) (resp. condition (wps)) holds if there exist a relation β on 2^Y and set-valued maps (with nonempty values) $b : W \longrightarrow 2^E, c : W \longrightarrow 2^Y$ and $f : W \longrightarrow 2^Y$ such that for all $(z, x, \eta) \in W_1$

$$[\exists u \in b(z, x, \eta), (f, c)(u, x, \eta) \in \beta] \Rightarrow [\forall v \in B(z, x, \eta), (F, C)(v, x, \eta) \in \alpha]$$

(resp.

$$[\exists u \in b(z, x, \eta), (f, c)(u, x, \eta) \in \beta] \Rightarrow [\exists v \in B(z, x, \eta), (F, C)(v, x, \eta) \in \alpha]).$$

Obviously, condition (ps) $\Rightarrow$ condition (wps), and the converse implication is no longer true. Observe that the above conditions (ps) and (wps) are taken from [17]. It is shown [17] that they are generalized versions of pseudomonotonicity and weak pseudomonotonicity conditions of [7].

From now on we assume that b, c and f are set-valued maps appearing in the definition of condition (ps) or condition (wps). Let us consider the following set-valued maps $\widehat{L}_\alpha : E \times K \longrightarrow 2^K$ and $l_\beta : E \times K \longrightarrow 2^K$ defined by

$$\begin{aligned} \widehat{L}_\alpha(z, x) &= \{\eta \in K : \exists v \in B(z, x, \eta),\ \ (F, C)(v, x, \eta) \notin \alpha\}, \\ l_\beta(z, x) &= \{\eta \in K : \forall u \in b(z, x, \eta),\ \ (f, c)(u, x, \eta) \notin \beta\}. \end{aligned}$$

Thus, for each $(z, x) \in E \times K$, the set $\widehat{L}_\alpha(z, x)$ consists of all points $\eta \in K$ such that condition $(F, C)(v, x, \eta) \in \alpha$ holds not for all $v \in B(z, x, \eta)$. The set $l_\beta(z, x)$ consists of all points $\eta \in K$ such that condition $(f, c)(u, x, \eta) \in \beta$ does not hold for each $u \in b(z, x, \eta)$.

Before providing existence theorems for Problem (P_α) let us introduce the following conditions [17]:

(a) $x \notin \text{co } L_\alpha(z, x),\ \forall (z, x) \in M.$

(b) $x \notin \text{co } \widehat{L}_\alpha(z, x),\ \forall (z, x) \in M.$

(c) Condition (ps) holds and $x \notin \text{co } l_\beta(z, x),\ \forall (z, x) \in M.$

(d) Condition (wps) holds and $x \notin \text{co } l_\beta(z, x),\ \forall (z, x) \in M.$

Obviously, (c) $\Rightarrow$ (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

Sufficient conditions for the validity of conditions (a), (b), (c) and (d) are given in the following lemma whose proof is obvious.

Lemma 3.1. *(see [17]) (a′) If for each $(z,x) \in M$, $L_\alpha(z,x)$ is convex and there exists $v \in B(z,x,x)$ such that $(F,C)(v,x,x) \in \alpha$ then condition (a) holds.*

(b′) If for each $(z,x) \in M$, $\widehat{L}_\alpha(z,x)$ is convex and, for each $v \in B(z,x,x)$, $(F,C)(v,x,x) \in \alpha$ then condition (b) holds.

(c′) If condition (ps) holds and if for each $(z,x) \in M$, $l_\beta(z,x)$ is convex and there exists $v \in b(z,x,x)$ such that $(f,c)(v,x,x) \in \beta$ then condition (c) holds.

(d′) If condition (wps) holds and if for each $(z,x) \in M$, $l_\beta(z,x)$ is convex and there exists $v \in b(z,x,x)$ such that $(f,c)(v,x,x) \in \beta$ then condition (d) holds.

Making use of Theorem 3.1 we obtain the following corollary.

Corollary 3.1. *Let $E \subset Z$ and $K \subset X$ be nonempty compact convex sets, and let $A : E \times K \longrightarrow 2^K$ be a lsc set-valued map with nonempty convex values such that the set*

$$M := \{(z,x) \in E \times K : x \in A(z,x)\}$$

is closed in $E \times K$. Then under one of the following conditions there exists a solution of Problem (P_α):

(i) *One of the conditions (a), (b), (c) and (d) holds, and L_α has open graph.*
(ii) *One of the conditions (b) and (c) holds, and $\widehat{L}_\alpha$ has open graph.*
(iii) *One of the conditions (c) and (d) holds, and l_β has open graph.*

Proof. The proof is derived from Theorem 3.1 where we set $L = L_\alpha$ (resp. $L = \widehat{L}_\alpha$; $L = l_\beta$) in case (i) (resp. (ii); (iii)). □

Remark 3. Existence results given in Corollary 3.1 were established in [17] under the assumption that A has open lower sections, i.e., $A^{-1}(\xi) := \{(z,x) \in E \times K : \xi \in A(z,x)\}$ is open in $E \times K$ for each $\xi \in K$. This assumption is stronger than the requirement of the lower semicontinuity of A used in Corollary 3.1. However, in Corollary 3.1 we must assume that the graph of L_α, $\widehat{L}_\alpha$ or l_β is open while in [17] this condition is replaced by the weaker condition that L_α, $\widehat{L}_\alpha$ or l_β has open lower sections. In addition, unlike our Corollary 3.1 where all spaces must be finite-dimensional, the results of [17] are valid in arbitrary topological vector spaces.

The following Example 3.1 will illustrate that Corollary 3.1 can be applied while the corresponding result of [17] cannot (since A has no open lower sections). We denote by $\mathbb{R}^k$ the k-dimensional Euclidean space. The nonnegative orthant of $\mathbb{R}^k$ is denoted by $\mathbb{R}^k_+$.

Example 3.1. Consider Problem (P_α) where $\alpha = \alpha_2$, $X = Z = \mathbb{R}^1 := \mathbb{R}$, $Y = \mathbb{R}^2$, $E = K = [0,1] \subset \mathbb{R}$, and for each $(z,x,\eta) \in E \times K \times K$, $A(z,x) = [0, z - zx]$, $F(z,x,\eta) = [(z, x^2 - \eta^2), (z + \eta^2, x^2)] \subset \mathbb{R}^2$, $C(z,x,\eta) \equiv F(z,x,x) + \mathbb{R}^2_+$, and

$$B(z,x,\eta) = \begin{cases} [0,z] & \text{if } x \neq \eta, \\ \{0\} & \text{if } x = \eta. \end{cases}$$

Obviously, A has no open lower sections but A is a lsc set-valued map with nonempty convex values and the set

$$M = \{(z,x) \in E \times K : x \in A(z,x)\} = \{(z,x) \in [0,1]^2 : x \leqslant z/(z+1)\}$$

is closed in $[0,1]^2 := [0,1] \times [0,1]$. Observe that for each $(z,x) \in E \times K$,

$$\begin{aligned} L_{\alpha_2}(z,x) &= \{\eta \in [0,1] : \forall v \in B(z,x,\eta), F(v,x,\eta) \not\subset C(v,x,\eta)\} \\ &= \{\eta \in [0,1] : \forall v \in B(z,x,\eta), \\ &\qquad [(v,x^2-\eta^2),(v+\eta^2,x^2)] \not\subset [(v,0),(v+x^2,x^2)] + \mathbb{R}^2_+\} \\ &= \{\eta \in [0,1] : \forall v \in B(z,x,\eta), [(v,x^2-\eta^2),(v+\eta^2,x^2)] \not\subset (v,0) + \mathbb{R}^2_+\} \\ &= \{\eta \in [0,1] : x < \eta\} \\ &=]x,1]. \end{aligned}$$

Thus, L_{α_2} has convex values and $x \notin L_{\alpha_2}(z,x)$, for all $(z,x) \in M$. Moreover,

$$\begin{aligned} \text{gr } L_{\alpha_2} &= \{(z,x,\eta) \in [0,1]^3 : \eta \in L_{\alpha_2}(z,x)\} \\ &= \{(z,x,\eta) \in [0,1]^3 : x < \eta\} \end{aligned}$$

is open in $[0,1]^3 := [0,1] \times [0,1] \times [0,1]$. Therefore, all the assumptions of Corollary 3.1(i) are satisfied and hence, Problem (P_α) in Example 3.1 has a solution.

Remark 4. The existence of open lower sections of set-valued maps is also used in [12] for Problem (V) formulated in the introduction. An existence result for Problem (V) is given in [12, Theorem 3.3] where it is assumed that S_2 has open lower sections. Obviously, many problems can be seen as special cases of each of the Problems (V) and (P_α). However, sometimes existence results for such problems cannot be obtained from [12, Theorem 3.3] while they can be derived from Corollary 3.1 of this paper. This remark is illustrated by the following example.

Example 3.2. Consider a special case of Problem (V) where $X_1 = X_2 = \mathbb{R}$, $Y = \mathbb{R}^2$, $D_1 = D_2 = [0,1] \subset \mathbb{R}$, $\mathcal{C} = \mathbb{R}^2_+$, and for each $(\xi,\nu,\gamma) \in D_1 \times D_2 \times D_1$, $S_1(\xi) = S_2(\xi) = S(\xi) := [0, 1-\xi]$, $T(\xi,\gamma) = \{|\xi-\gamma|\}$, $\Phi(\xi,\nu,\gamma) = [(\nu,\xi^2 - \gamma^2),(\nu+\gamma^2,\xi^2)] \subset \mathbb{R}^2$. It is easy to verify that the set-valued map S_2 has no open lower sections. This proves that condition (ii) of Theorem 3.3 in [12] is violated and hence, this theorem cannot be applied. We now show that Corollary 3.1 can be used to derive an existence result for the above problem. Indeed, let us consider Problem (P_α) where $\alpha = \alpha_2$, $K := D_1 \times D_2 \subset X := X_1 \times X_2$, $E := D_1 \subset Z := X_1$, $A(z,x) = S(\xi_1) \times T(S(\xi_1),\xi_1)$,

$$F(z,x,\eta) = \Phi(\xi_1,\eta_2,\eta_1),\ C(z,x,\eta) = \Phi(z,\eta_2,\xi_1) + \mathcal{C},$$

and $B(z,x,\eta) = \{\xi_1\}$ where $z \in E := D_1, x = (\xi_1,\xi_2) \in K := D_1 \times D_2$ and $\eta = (\eta_1,\eta_2) \in K := D_1 \times D_2$.

Arguing as in Example 3.1, we see that all assumptions of Corollary 3.1(i) are satisfied for this Problem (P_α). Therefore, there exist points $z_0 \in E$ and $x_0 = (\xi_1^0,\xi_2^0) \in K = D_1 \times D_2$ such that $x_0 = (\xi_1^0,\xi_2^0) \in S(\xi_1^0) \times T(S(\xi_1^0),\xi_1^0)$ and, for each $\eta = (\eta_1,\eta_2) \in S(\xi_1^0) \times T(S(\xi_1^0),\xi_1^0)$, we can find a point $v \in B(z_0,x_0,\eta) \equiv$

$\{\xi_1^0\}$ with $F(v, x_0, \eta) \subset C(v, x_0, \eta)$. This implies that there exists a point $\xi_1^0 \in D_1$ such that $\xi_1^0 \in S(\xi_1^0)$ and, for all $\eta_1 \in S(\xi_1^0)$ and $\eta_2 \in T(S(\xi_1^0), \xi_1^0)$,

$$\Phi(\xi_1^0, \eta_2, \eta_1) \subset \Phi(\xi_1^0, \eta_2, \xi_1^0) + \mathcal{C}.$$

Now, if we take an arbitrary point (η_1, η_2) with $\eta_1 \in S_2(\xi_1^0)$ and $\eta_2 \in T(\eta_1, \xi_1^0)$, then we get $\eta_1 \in S_2(\xi_1^0) = S(\xi_1^0)$ and $\eta_2 \in T(\eta_1, \xi_1^0) \subset T(S(\xi_1^0), \xi_1^0)$. Hence the above claim shows that

$$\Phi(\xi_1^0, \eta_2, \eta_1) \subset \Phi(\xi_1^0, \eta_2, \xi_1^0) + \mathcal{C}.$$

Since the last inclusion holds for an arbitrary point (η_1, η_2) with $\eta_1 \in S_2(\xi_1^0)$ and $\eta_2 \in T(\eta_1, \xi_1^0)$, and since $\xi_1^0 \in D_1$ and $\xi_1^0 \in S(\xi_1^0) = S_1(\xi_1^0)$ we see that ξ_1^0 is exactly a solution of Problem (V) considered in Example 3.2.

Remark 5. Observe that Corollary 3.1 fails to hold if the assumption that the set M is closed in $E \times K$ is violated.

This remark is illustrated by the following example.

Example 3.3. Let us consider Problem (P_α) where $\alpha = \alpha_1$, $X = Y = Z = \mathbb{R}$, $E = K = [0, 1] \subset \mathbb{R}$, $B(z, x, \eta) \equiv \{1\}$, $C(z, x, \eta) \equiv -\text{int}\, \mathbb{R}_+$ (the negative half-line), $F(z, x, \eta) = \{z(x - \eta)\}$, and for each $(z, x) \in E \times K$

$$A(z, x) = \begin{cases} [0, 1 - x[& \text{if } x \in [0, 1[\ , \\ \{0\} & \text{if } x = 1. \end{cases}$$

Obviously, in this example the set M is not closed in $E \times K$. Now let us set $\beta = \alpha = \alpha_1$, $f = F, b = B$ and $c = C$. Then it is obvious that both conditions (ps) and (wps) hold trivially. Observe that in our case

$$L_{\alpha_1}(z, x) \equiv \widehat{L}_{\alpha_1}(z, x) \equiv l_\beta(z, x) \equiv e(x)$$

where $e(x) = \{\eta \in [0, 1] : x < \eta\}$. Since $e(x)$ is convex and $x \notin e(x)$ for each $x \in K = [0, 1]$ it follows that each of conditions (a), (b), (c) and (d) holds. Also, each of maps L_{α_1}, $\widehat{L}_{\alpha_1}$ and l_β has open graph. Therefore, each of conditions (i), (ii) and (iii) of Corollary 3.1 is satisfied. However, (P_α) has no solution. Indeed, if (z_0, x_0) is a solution of (P_α) then $x_0 \in A(z_0, x_0) = [0, 1 - x_0[$ and

$$\forall \eta \in A(z_0, x_0), \exists v \in B(z_0, x_0, \eta) \equiv \{1\}, v(x_0 - \eta) \geqslant 0.$$

This means that $x_0 - \eta \geqslant 0$ for all $\eta \in [0, 1 - x_0[$, which is impossible.

Corollary 3.2. *Let $K \subset X$ be a nonempty compact convex set. Let $\mathbb{A} : K \longrightarrow 2^K$ be a lsc set-valued map with nonempty convex values such that the set*

$$M' = \{x \in K : x \in \mathbb{A}(x)\}$$

is closed in K. Let $T : K \longrightarrow 2^X$ be a set-valued map with nonempty values such that for each $(x, \eta) \in K \times K$

$$\zeta(x, \eta) := \inf_{p \in T(x)} \langle p, x - \eta \rangle \in \mathbb{R}$$

and the map

$$\eta \in K \longmapsto \{x \in K : \zeta(x, \eta) \leqslant 0\} \tag{3.1}$$

has the graph closed in $K \times K$. Then there exists $x_0 \in K$ such that $x_0 \in \mathbb{A}(x_0)$ and $\zeta(x_0, \eta) \leqslant 0, \forall \eta \in \mathbb{A}(x_0)$. If in addition $T(x_0)$ is compact and convex then there exists $p_0 \in T(x_0)$ such that $\langle p_0, x_0 - \eta \rangle \leqslant 0$, $\forall \eta \in \mathbb{A}(x_0)$.

Proof. Let us set $Y = \mathbb{R}, Z = X, E = K$, $A(z,x) \equiv \mathbb{A}(x)$, $B(z,x,\eta) \equiv \{z\}$, $F(z,x,\eta) \equiv \{\zeta(x,\eta)\}$, $C(z,x,\eta) \equiv -\mathbb{R}_+$ (the nonpositive half-line) and $\alpha = \alpha_2$. Then Problem (P_α) is to find $(z_0, x_0) \in E \times K = K \times K$ such that $x_0 \in \mathbb{A}(x_0)$ and for each $\eta \in \mathbb{A}(x_0)$, $\zeta(x_0, \eta) \leqslant 0$. Thus to prove the first conclusion of Corollary 3.2 it suffices to show that this Problem (P_α) has a solution. Indeed, first observe that $M = E \times M'$ is closed in $E \times K$ since M' is closed in K. Also, for each $(z,x) \in E \times K$ the set

$$L_\alpha(z,x) = \{\eta \in K : \zeta(x,\eta) > 0\}$$

is convex since $\zeta(x,\cdot)$ is a concave function. On the other hand, $\zeta(x,x) = 0$. Therefore, $x \notin L_\alpha(z,x) = \text{co } L_\alpha(z,x)$ for each $(z,x) \in E \times K$. This proves that condition (a) holds. We now claim that L_α has open graph. Indeed, first observe that the set

$$K \times K \setminus \{(x,\eta) \in K \times K : \zeta(x,\eta) \leqslant 0\}$$

is open in $K \times K$. Using this fact and observing that

$$\begin{aligned} \text{gr } L_\alpha &= E \times K \times K \setminus \{(z,x,\eta) \in E \times K \times K : \zeta(x,\eta) \leqslant 0\} \\ &= E \times [K \times K \setminus \{(x,\eta) \in K \times K : \zeta(x,\eta) \leqslant 0\}], \end{aligned}$$

we see that gr L_α is open in $E \times K \times K$. Thus, all the requirements in condition (i) of Corollary 3.1 are satisfied. Hence, the above Problem (P_α) has a solution, as required. To obtain the second conclusion of Corollary 3.2 it remains to apply a minimax theorem of [9]. Details can be found in the proof of Theorem 3.1 of [4]. □

Remark 6. Results similar to those of Corollary 3.2 are established in Theorems 3.1 and 3.2 of [4]. However, Corollary 3.2 cannot be derived from these results of [4]. As an example illustrating this remark let us take the following example which is a modified version of Example 3.1 of [4].

Example 3.4. Let $K = [0,1] \subset \mathbb{R}$, $\mathbb{A}(x) \equiv \{0\}$,

$$T(x) = \begin{cases} \{1\} & \text{if } x = 0, \\ \{0\} & \text{if } 0 < x \leqslant 1. \end{cases}$$

It is easy to see that the graph of the set-valued map (3.1) is closed in $K \times K$. Other requirements of Corollary 3.2 are also satisfied for Example 3.4. Hence, Corollary 3.2 can be applied to this example. However, Theorem 3.1 of [4] cannot be used since the requirement that aff $\mathbb{A}(x) =$ aff $[0,1]$ in this theorem is not satisfied. (Here aff S denotes the affine hull of the set S.) The fact that Theorem 3.2 of [4] cannot be applied in Example 3.4 can be found in Remark 3.2 of [4].

Remark 7. In Theorem 3.1 of [13] it is shown that Corollary 3.2 remains true for the case X being a separable Banach space if the closedness (in K) of M' is replaced by the closedness of the graph of $\mathbb{A}$. The following Example 3.5 shows that all assumptions of Corollary 3.2 can be satisfied, while the graph of $\mathbb{A}$ is not closed. This means that Corollary 3.2 is not a special case of Theorem 3.1 of [13].

Example 3.5. Let K and T be as in Example 3.4 and let

$$\mathbb{A}(x)=\begin{cases}[0,\frac{1}{2}] & \text{if } x\in[0,\frac{1}{2}],\\ [0,1] & \text{if } x\in]\frac{1}{2},1].\end{cases}$$

It is obvious that all assumptions of Corollary 3.2 hold while gr $\mathbb{A}$ is not closed in $K\times K=[0,1]\times[0,1]$. Observe also that in this example $\mathbb{A}$ is not continuous.

Remark 8. Corollary 3.1 requires that the graph of L_α, $\widehat{L}_\alpha$ or l_β is open. Propositions 4.1 and 4.2 in [17] give sufficient conditions for the validity of this property.

4. Some special cases

This section is devoted to examples illustrating the result obtained in the previous section for Problem (P_α) with $\alpha=\alpha_i,\ i=1,2,3,4$. We begin by the following technical lemma.

Lemma 4.1. *Let* $a\subset K$ *be a nonempty convex set. Let*

$$G_\alpha(z,x,\eta)=\{\xi\in[x,\eta]:\forall v\in B(z,\xi,\eta),(F,C)(v,x,\eta)\notin\alpha\},(z,x,\eta)\in W, \tag{4.1}$$

$$g_\beta(z,x,\xi)=\{\eta\in a:\forall v\in b(z,x,\xi),\ (f,c)(v,x,\eta)\notin\beta\},(z,x,\xi)\in W. \tag{4.2}$$

Let the following conditions be satisfied:

(i) *For each* $(z,x,\eta)\in E\times a\times a$ *with* $x\neq\eta$, *if* $x\in G_\alpha(z,x,\eta)$ *then* $\xi\in G_\alpha(z,x,\eta)$ *for some* $\xi\in\,]x,\eta[$.
(ii) *For each* $(z,x,\eta)\in E\times a\times a$ *with* $x\neq\eta$ *and for each* $\xi\in\,]x,\eta[$, *if* $\xi\in G_\alpha(z,x,\eta)$ *then* $\xi\in g_\beta(z,x,\xi)$.
(iii) *For each* $(z,x)\in E\times a$ *there exists* $v\in B(z,x,x)$ *such that* $(F,C)(v,x,x)\in\alpha$.

Then for each $(z,x)\in E\times a$

$$[\forall\eta\in a,\exists v\in b(z,x,\eta),\ (f,c)(v,x,\eta)\in\beta] \Longrightarrow [\forall\eta\in a,\exists v\in B(z,x,\eta),\ (F,C)(v,x,\eta)\in\alpha]. \tag{4.3}$$

Proof. We need to prove that if $(z,x)\in E\times a$ does not satisfy the statement in the right-hand side of implication (4.3) then it does not satisfy the statement in the left-hand side of (4.3). Indeed, the negation of the statement in the right-hand side of (4.3) means that

$$\exists\eta\in a,\forall v\in B(z,x,\eta),\ (F,C)(v,x,\eta)\notin\alpha,$$

i.e., $x\in G_\alpha(z,x,\eta)$. By (iii) we get $x\neq\eta$. By (i) there exists $\xi\in\,]x,\eta[$ with $\xi\in G_\alpha(z,x,\eta)$. By (ii) $\xi\in g_\beta(z,x,\xi)$, i.e.,

$$\forall v\in b(z,x,\xi),\ (f,c)(v,x,\xi)\notin\beta. \tag{4.4}$$

Observe now that $\xi \in a$ by the convexity of a. Thus the point $\xi \in a$ satisfies (4.4). This proves that (z, x) does not satisfy the statement in the left-hand side of (4.3). □

Remark 9. Let us consider the following problem:

Problem (p_β): Find $(z_0, x_0) \in E \times K$ such that $x_0 \in A(z_0, x_0)$ and

$$\forall \eta \in A(z_0, x_0), \exists v \in b(z_0, x_0, \eta),\ (f, c)(v, x_0, \eta) \in \beta.$$

Then, under conditions of Lemma 4.1 with $a = A(z_0, x_0)$ every solution of Problem (p_β) is also a solution of Problem (P_α).

Remark 10. If for all $(z, x, \eta) \in E \times a \times a$ the set $G_\alpha(z, x, \eta)$ is open in $[x, \eta]$ then condition (i) of Lemma 4.1 holds.

Before providing other sufficient assumptions for condition (i) of Lemma 4.1 to hold, let us introduce some definitions.

Let $a \subset K$ be a convex set. We say [7, Def.1(iii)] that F is u-hemicontinuous on $E \times a \times a$ with respect to B if for any point $(z, x, \eta) \in E \times a \times a$ the set-valued map

$$\lambda \in [0,1] \longmapsto F(B(z, x_\lambda, \eta), x, \eta) := \bigcup_{v \in B(z, x_\lambda, \eta)} F(v, x, \eta), \tag{4.5}$$

where $x_\lambda := x + \lambda(\eta - x)$, is upper semicontinuous at $\lambda = 0$.

Clearly, F is u-hemicontinuous on $E \times a \times a$ with respect to B if and only if for any point $(z, x, \eta) \in E \times a \times a$ and any open set $U \subset Y$ with

$$F(v, x, \eta) \subset U,\ \forall v \in B(z, x, \eta), \tag{4.6}$$

there exists $\delta \in]0, 1[$ such that

$$F(v, x, \eta) \subset U,\ \forall v \in B(z, x_\lambda, \eta),\ \forall \lambda \in]0, \delta[\ . \tag{4.7}$$

We say that F is l-hemicontinuous on $E \times a \times a$ with respect to B if for any point $(z, x, \eta) \in E \times a \times a$ and any open set $U \subset Y$ with

$$F(v, x, \eta) \cap U \neq \emptyset,\ \forall v \in B(z, x, \eta), \tag{4.8}$$

there exists $\delta \in]0, 1[$ such that

$$F(v, x, \eta) \cap U \neq \emptyset,\ \forall v \in B(z, x_\lambda, \eta),\ \forall \lambda \in]0, \delta[\ , \tag{4.9}$$

where $x_\lambda := x + \lambda(\eta - x)$.

Obviously, if B is single-valued then the l-hemicontinuity coincides with the lower semicontinuity of the set-valued map

$$\lambda \in [0,1] \longmapsto F(B(z, x_\lambda, \eta), x, \eta),$$

at $\lambda = 0$.

Lemma 4.2. *Assume that $a \subset X$ is a nonempty convex set and $\mathbb{C} : a \longrightarrow 2^Y$ is a set-valued map with nonempty values. Then condition* (i) *of Lemma 4.1 holds under one of the following conditions:*

(i) $\alpha = \alpha_1$, *F is u-hemicontinuous on* $E \times a \times a$ *with respect to* B, *and for all* $(z, x, \eta) \in W$, $C(z, x, \eta) \equiv \text{int}\, \mathbb{C}(x) \neq \emptyset$.

(ii) $\alpha = \alpha_2$, *F is l-hemicontinuous on* $E \times a \times a$ *with respect to* B, *and for all* $(z, x, \eta) \in W$, $C(z, x, \eta) \equiv \mathbb{C}(x)$ *and* $\mathbb{C}(x)$ *is closed.*

(iii) $\alpha = \alpha_3$, *F is u-hemicontinuous on* $E \times a \times a$ *with respect to* B, *and for all* $(z, x, \eta) \in W$, $C(z, x, \eta) \equiv \mathbb{C}(x)$ *and* $\mathbb{C}(x)$ *is closed.*

(iv) $\alpha = \alpha_4$, *F is l-hemicontinuous on* $E \times a \times a$ *with respect to* B, *and for all* $(z, x, \eta) \in W$, $C(z, x, \eta) \equiv \text{int}\, \mathbb{C}(x) \neq \emptyset$.

Proof. Let us prove Lemma 4.2 in cases (i) and (ii). The cases (iii) and (iv) can be considered similarly.

Observe that in case (i)

$$G_\alpha(z, x, \eta) = \{\xi \in [x, \eta] : \forall v \in B(z, \xi, \eta),\ F(v, x, \eta) \subset \text{int}\, \mathbb{C}(x)\},$$

and $x \in G_\alpha(z, x, \eta)$ means that (4.6) holds with $U = \text{int}\, \mathbb{C}(x)$. Hence from (4.7) we can find $\lambda \in]0, 1[$ such that

$$\forall v \in B(z, x_\lambda, \eta),\ F(v, x, \eta) \subset U = \text{int}\, \mathbb{C}(x).$$

Setting $\xi = x_\lambda$ we see that $\xi \in G_\alpha(z, x, \eta)$, i.e., condition (i) of Lemma 4.1 holds.

In case (ii) we have

$$\begin{aligned} G_\alpha(z, x, \eta) &= \{\xi \in [x, \eta] : \forall v \in B(z, \xi, \eta),\ F(v, x, \eta) \not\subset \mathbb{C}(x)\} \\ &= \{\xi \in [x, \eta] : \forall v \in B(z, \xi, \eta),\ F(v, x, \eta) \cap U \neq \emptyset\} \end{aligned} \tag{4.10}$$

where $U := Y \setminus \mathbb{C}(x)$ is an open set. Hence, from (4.10) $x \in G_\alpha(z, x, \eta)$ means that (4.8) holds. From (4.9) we can find $\lambda \in]0, 1[$ such that

$$\forall v \in B(z, x_\lambda, \eta),\ F(v, x, \eta) \cap U \neq \emptyset.$$

Setting $\xi = x_\lambda$ we conclude that $\xi \in G_\alpha(z, x, \eta)$, as was to be shown. ☐

We will assume that $B(z, x, \eta)$ and $C(z, x, \eta)$ do not depend on (z, η). So, for the sake of simplicity let us write $\mathbb{B}(x)$ and $\mathbb{C}(x)$ instead of $B(z, x, \eta)$ and $C(z, x, \eta)$:

$$B(z, x, \eta) \equiv \mathbb{B}(x), \tag{4.11}$$

$$C(z, x, \eta) \equiv \mathbb{C}(x). \tag{4.12}$$

Under the above assumptions Problem (P_α) with $\alpha = \alpha_i$ is to find a point $(z_0, x_0) \in E \times K$ such that $x_0 \in A(z_0, x_0)$ and

$$\forall \eta \in A(z_0, x_0), \exists v \in \mathbb{B}(x_0),\ (F(v, x_0, \eta), \mathbb{C}(x_0)) \in \alpha_i.$$

We will refer to this problem as Problem (P^i). Basing on Corollary 3.1, Remark 9 and Lemma 4.2 we can derive existence results in Problem (P^i) for $i = 1, 2, 3, 4$. Since these results are established by the same approach we can restrict ourselves to the case $i = 1$. Other cases can be considered similarly.

Let us reformulate Problem (P^1):

Problem (P^1): Find a point $(z_0, x_0) \in E \times K$ such that $x_0 \in A(z_0, x_0)$ and

$$\forall \eta \in A(z_0, x_0), \exists v \in \mathbb{B}(x_0),\ F(v, x_0, \eta) \not\subset \mathbb{C}(x_0).$$

Together with Problem (P^1) we will consider the following problem:

Problem $(\widehat{P}^1)$: Find a point $(z_0, x_0) \in E \times K$ such that $x_0 \in A(z_0, x_0)$ and

$$\forall \eta \in A(z_0, x_0), \exists u \in \mathbb{B}(\eta),\ F(u, x_0, \eta) \not\subset \mathbb{C}(x_0).$$

We will assume that $\mathbb{C}(x)$ is of the form

$$\mathbb{C}(x) = \sigma(x) + \text{int } \mathbb{C}'(x) \tag{4.13}$$

where for each $x \in K$ $\sigma(x) \subset Y$ is a nonempty convex set and $\mathbb{C}'(x) \subset Y$ is a convex cone with $\emptyset \neq \text{int } \mathbb{C}'(x)$ (the interior of $\mathbb{C}'(x)$).

Proposition 4.1. *Let* $\mathbb{C}$ *be of the form* (4.13). *Let* (z_0, x_0) *be a solution of Problem* $(\widehat{P}^1)$ *and let* $a := A(z_0, x_0)$. *Then* (z_0, x_0) *is also a solution of Problem* (P^1) *if*

(i) F *is u-hemicontinuous on* $E \times a \times a$ *with respect to* B *defined by* (4.11).
(ii) *For each* $(z, x) \in E \times a, F(z, x, \cdot)$ *is natural* $[-\mathbb{C}'(x)]$*-quasiconvex on* a.
(iii) *For each* $(z, x) \in E \times a$,

$$F(z, x, x) \subset \sigma(x) + \mathbb{C}'(x). \tag{4.14}$$

(iv) *For each* $(z, x) \in E \times a$ *there exists* $v \in \mathbb{B}(x)$ *such that*

$$F(v, x, x) \not\subset \sigma(x) + \text{int } \mathbb{C}'(x).$$

Proof. Let us set

$$\beta = \alpha_1, f(z, x, \eta) \equiv F(z, x, \eta), c(z, x, \eta) \equiv \mathbb{C}(x), b(z, x, \eta) \equiv \mathbb{B}(\eta). \tag{4.15}$$

Then Problem $(\widehat{P}^1)$ is exactly Problem (p_β) mentioned in Remark 9. Also, as we remarked above, Problem (P^1) is exactly Problem (P_α) where

$$\alpha = \alpha_1, B(z, x, \eta) \equiv \mathbb{B}(x), C(z, x, \eta) \equiv \mathbb{C}(x). \tag{4.16}$$

So, to prove Proposition 4.1 it suffices to show that assumptions (i), (ii) and (iii) of Lemma 4.1 are satisfied for $\alpha = \beta = \alpha_1$ and $a = A(z_0, x_0)$ (see Remark 9). Let us begin by assumption (i). We have

$$G_{\alpha_1}(z, x, \eta) = \{\xi \in [x, \eta] : \forall v \in B(z, \xi, \eta) \equiv \mathbb{B}(\xi),\ F(v, x, \eta) \subset \mathbb{C}(x)\}.$$

Observe that $\mathbb{C}(x) = \text{int } [\sigma(x) + \mathbb{C}'(x)]$. Therefore, making use of the first claim of Lemma 4.2 with $\sigma(x) + \mathbb{C}'(x)$ instead of $\mathbb{C}(x)$ we infer that assumption (i) of Lemma 4.1 holds. Now let us verify assumption (ii) of this lemma. Indeed, first observe from (4.15) that

$$\begin{aligned} g_\beta(z, x, \xi) &= g_{\alpha_1}(z, x, \xi) \\ &= \{\eta \in a : \forall v \in b(z, x, \xi) \equiv \mathbb{B}(\xi), F(v, x, \eta) \subset \mathbb{C}(x)\} \\ &= \bigcap_{v \in \mathbb{B}(\xi)} \{\eta \in a : F(v, x, \eta) \subset \mathbb{C}(x)\}. \end{aligned} \tag{4.17}$$

So, to verify assumption (ii) of Lemma 4.1 we assume that $\xi \in]x,\eta[$ and $\xi \in G_{\alpha_1}(z,x,\eta)$, i.e., $\forall v \in \mathbb{B}(\xi)$

$$F(v,x,\eta) \subset \sigma(x) + \text{int } \mathbb{C}'(x). \tag{4.18}$$

By (4.14), (4.18) and the natural quasiconvexity of $F(v,x,\cdot)$ we get for each $v \in \mathbb{B}(\xi)$

$$\begin{aligned} F(v,x,\xi) &\subset \text{co } [F(v,x,\eta), F(v,x,x)] + \mathbb{C}'(x) \\ &\subset \sigma(x) + \text{int } \mathbb{C}'(x) + \mathbb{C}'(x) \\ &\subset \mathbb{C}(x). \end{aligned}$$

This proves that $\xi \in g_\beta(z,x,\xi) = g_{\alpha_1}(z,x,\xi)$ (see (4.17)). Thus condition (ii) of Lemma 4.1 holds, as desired. To complete our proof it suffices to note that condition (iv) of Proposition 4.1 is exactly condition (iii) of Lemma 4.1 with $\alpha = \alpha_1$. □

From now on we assume that $E \subset Z$ and $K \subset X$ are nonempty compact convex sets, $A : E \times K \longrightarrow 2^K$ is a lsc set-valued map with nonempty convex values such that the set M (see Theorem 3.1) is closed in $E \times K$.

We say that the triplet $(F, \mathbb{C}, \mathbb{B})$ satisfies the α-pseudomonotonicity assumption if for each $(z,x,\eta) \in W_1$ (see Section 3)

$$[\exists u \in \mathbb{B}(x), \alpha(F(u,x,\eta), \mathbb{C}(x))] \Longrightarrow [\forall v \in \mathbb{B}(\eta), \alpha(F(v,x,\eta), \mathbb{C}(x))].$$

Theorem 4.1. *Let $\mathbb{C}$ be of the form* (4.13). *Assume that*

(i) *For each $(z,x) \in E \times K$, F is u-hemicontinuous on $E \times A(z,x) \times A(z,x)$ with respect to B defined by* (4.11).
(ii) *For each $(z,x) \in E \times K$, $F(z,x,\cdot)$ is natural $[-\mathbb{C}'(x)]$-quasiconvex on K.*
(iii) *For each $(z,x) \in E \times K$,*

$$F(z,x,x) \subset \sigma(x) + \mathbb{C}'(x).$$

(iv) *For each $(z,x) \in E \times K$ there exists $v \in \mathbb{B}(x)$ such that*

$$F(v,x,x) \not\subset \sigma(x) + \text{int } \mathbb{C}'(x). \tag{4.19}$$

(v) *The triplet $(F, \mathbb{C}, \mathbb{B})$ satisfies the α_1-pseudomonotonicity assumption.*
(vi) *The set-valued map F is usc and compact-valued, the set-valued map $\mathbb{B}$ is lsc and the set-valued map $\mathbb{C}$ has open graph.*

Then there exists a solution of Problem (P^1).

Proof. By Proposition 4.1 it suffices to prove that Problem ($\widehat{P}^1$) has a solution. Since ($\widehat{P}^1$) can be interpreted as Problem (P_α) with

$$\alpha = \alpha_1,\ B(z,x,\eta) \equiv \mathbb{B}(\eta),\ C(z,x,\eta) \equiv \mathbb{C}(x), \tag{4.20}$$

we can use Corollary 3.1 to prove the existence of a solution of ($\widehat{P}^1$). Indeed, since $\alpha = \alpha_1$ we have

$$\widehat{L}_{\alpha_1}(z,x) = \{\eta \in K : \exists v \in \mathbb{B}(\eta),\ F(v,x,\eta) \subset \mathbb{C}(x)\}.$$

Making use of (vi) we can show that $\widehat{L}_{\alpha_1}$ has open graph (see e.g. [17, Proposition 4.2]). By Corollary 3.1(ii) it remains to verify condition (c) of Section 3. By Lemma 3.1 we need to verify condition (c′). Indeed, let us set

$$\beta = \alpha_1,\ f(z,x,\eta) \equiv F(z,x,\eta),\ b(z,x,\eta) \equiv \mathbb{B}(x),\ c(z,x,\eta) \equiv \mathbb{C}(x). \tag{4.21}$$

Then (v) proves that condition (ps) holds. In addition, since

$$\begin{aligned} l_\beta(z,x) &= l_{\alpha_1}(z,x) \\ &= \{\eta \in K : \forall v \in \mathbb{B}(x),\ F(v,x,\eta) \subset \mathbb{C}(x) := \sigma(x) + \text{int } \mathbb{C}'(x)\} \\ &= \bigcap_{v \in \mathbb{B}(x)} \{\eta \in K : F(v,x,\eta) \subset \sigma(x) + \text{int } \mathbb{C}'(x)\}, \end{aligned} \tag{4.22}$$

and since

$$\{\eta \in K :\ F(v,x,\eta) \subset \sigma(x) + \text{int } \mathbb{C}'(x)\}$$

is convex (see (ii)) it follows that $l_\beta(z,x)$ is convex. This together with (iv) proves the validity of (c′). □

Remark 11. If for each $(z,x) \in E \times K$ $\sigma(x) \equiv \{0\}$, $\mathbb{C}'(x) \neq Y$ and $0 \in F(z,x,x)$ then condition (iv) of Theorem 4.1 holds. Indeed, take an arbitrary point $v \in \mathbb{B}(x)$. If (4.9) does not hold then

$$F(v,x,x) \subset \sigma(x) + \text{int } \mathbb{C}'(x) = \text{int } \mathbb{C}'(x).$$

Since by assumption $0 \in F(v,x,x)$ it follows that $0 \in \text{int } \mathbb{C}'(x)$, a contradiction to condition $\mathbb{C}'(x) \neq Y$. The conditions mentioned in Remark 11 are used in [7].

Observe that Theorem 4.1 gives the existence of a solution of Problem (P^1) under a generalized monotonicity assumption. The following theorem does not require such an assumption.

Theorem 4.2. *Let* $\mathbb{C}$ *be of the form* (4.13). *Assume that*

(i) *For each* $(z,x) \in E \times K, F(z,x,\cdot)$ *is natural* $[-\mathbb{C}'(x)]$*-quasiconvex on* K.

(ii) *For each* $(z,x) \in E \times K$ *there exists* $v \in \mathbb{B}(x)$ *such that*

$$F(v,x,x) \not\subset \sigma(x) + \text{int } \mathbb{C}'(x).$$

(iii) *The set-valued maps* F *and* $\mathbb{B}$ *are usc and compact-valued, and the set-valued map* $\mathbb{C}$ *has open graph.*

Then there exists a solution of Problem (P^1).

Proof. This is an immediate consequence of Corollary 3.1. Indeed, let us interpret Problem (P^1) as Problem (P_α) with α, B and C being as in (4.16). Then, since $\alpha = \alpha_1$ we have

$$\begin{aligned} L_{\alpha_1}(z,x) = \{\eta \in K : \forall v \in B(z,x,\eta) \equiv \mathbb{B}(x), \\ F(v,x,\eta) \subset \mathbb{C}(x) := \sigma(x) + \text{int } \mathbb{C}'(x)\}. \end{aligned}$$

Making use of (iii) we can prove that L_{α_1} has open graph (see e.g. [17, Proposition 4.1]). In addition, as in the proof of Theorem 4.1 we can derive from conditions (i) and (ii) (i.e., conditions (ii) and (iv) of Theorem 4.1) that condition (a′) (and hence, condition (a)) holds. To complete our proof it remains to apply Corollary 3.1(i). □

Remark 12. The reader who is interested in existence theorems for Problem (P^1) in topological vector spaces is referred to [7, Theorems 3 and 5]. It is worth noticing that existence theorems of [7] are established only for the case when $\sigma(x) \equiv \{0\}$ and $A(z,x) \equiv K$. Finally, observe that Problem (P^i) with $i \neq 1$ is not considered in [7] while, as we remarked above, existence results for the case $i \neq 1$ can be obtained from Corollary 3.1 and Lemma 4.2. It is worth noticing that, when dealing with Problem (P^i), $i \neq 1$, we must use generalized convexity notions different from the natural quasiconvexity.

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