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ON LIFTING MODULES RELATIVE TO THE CLASS OF ALL SINGULAR MODULES

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Abstract. Using the class δ of all singular R-modules, we introduce and characterize δ-lifting modules, δ-supplements and δ-coclosed submodules, and give some equivalent conditions to characterize an amply δ-supplemented module $M = \oplus_{i\in I} M_i$ to be δ-lifting. We introduce relatively δ-small projective to give some sufficient conditions that a finite sum of δ-lifting modules is δ-lifting. We also show that R is δ-perfect (δ-semiperfect) if and only if every (finitely generated) projective right R-module is δ-lifting.

1. Introduction

The notion of small submodules (also called superfluous submodules) plays an important role in the theory of modules and rings. Recently, Zhou [8] generalized the concept of small submodules to that of δ-small submodules by considering the class δ of all singular right R-modules in place of the class of all right R-modules, gave various properties of δ-small submodules that are similar to those of small submodules, and then used this concept to generalize the notions of perfect, semiperfect and semiregular rings to those of δ-perfect, δ-semiperfect and δ-semiregular rings.

Small submodules are also important in the theory of lifting (i.e., D_1) modules (see [5, 6]). In this paper, we use the concept of δ-small to introduce some generalizations of lifting modules and investigate the main problem in this theory, that is, when the direct sum of lifting modules is also lifting. It is of interest to know how far the old theories extend to the new situation.

In Section 2 we introduce some basic concepts of δ-lifting modules, δ-supplement, δ-coessential submodule and so on, and give some properties of these notions. In Section 3 we mainly investigate the properties of δ-coclosed submodules. It is proved that if the class δ is closed under module extensions, then every submodule of an amply δ-supplemented module has a δ-s-closure. In Section 4 we show that every direct summand of a δ-lifting module is δ-lifting, and give some equivalent conditions for an amply δ-supplemented module $M = \oplus_{i\in I} M_i$ to be δ-lifting.

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We introduce relatively δ-small projective to give some sufficient conditions that a finite sum of δ-lifting modules is δ-lifting. If every R-module (or simple R-module, resp.) has a projective δ-cover, R is called a δ-perfect (or δ-semiperfect, resp.) ring [8]. In Section 5 we prove that R is δ-perfect (or δ-semiperfect) if and only if every (finitely generated) projective right R-module is δ-lifting. We also give an example to illustrate that a δ-lifting right R-module is not necessarily lifting.

Throughout this paper, R denotes an associative ring with identity, and modules are unitary right R-modules. Let δ be the class of all singular R-modules. According to [4], the class δ is closed under submodules, factor modules and direct sums. Moreover, if R is right nonsingular, then δ is also closed under module extensions, that is, in an exact sequence $0 \to A \to B \to C \to 0$, B is singular whenever A and C are singular. Let $K \subseteq N$ denote that K is a submodule of N, and let $K \subseteq_\oplus N$ denote that K is a direct summand of N. For the other definition and notation in this paper we refer the reader to [1].

2. Basic notions

Let N be a submodule of a module M. N is said to be *small* (or *superfluous*) in M, if $M = N + T$ for a submodule T of M implies $M = T$. As a generalization of small submodules, Zhou [8] gave the concept of δ-small submodules. Let M be a module and $N \subseteq M$, then N is said to be *δ-small* [8] in M, if $M = N + X$ with M/X singular implies $M = X$. We use $N \ll_\delta M$ to indicate that N is a δ-small submodule of M. Every small submodule and every non-singular semisimple submodule of M is δ-small in M. The δ-small submodules of a singular module are small submodules.

We list some properties of δ-small modules from [8] for later use.

Lemma 2.1. *Assume that M is an R-module.*

(1) *Let N, K, L be submodules of M with $K \subseteq N$.*
  (*a*) *$N \ll_\delta M$ if and only if $K \ll_\delta M$ and $N/K \ll_\delta M/K$*
  (*b*) *$N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.*

(2) *If $K \ll_\delta M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_\delta N$. In particular, if $K \ll_\delta M \subseteq N$, then $K \ll_\delta N$.*

(3) *If $K_i \subseteq M_i \subseteq M$ for $i = 1, 2$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ if and only if $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$.*

In order to characterize δ-lifting modules in Section 4, we first give some related definitions.

Definition 2.2. Let N and L be submodules of M.

(1) N is called a *δ-supplement* of L if $M = N + L$ with M/N singular and $N \cap L \ll_\delta N$.

(2) N is called a *δ-supplement* submodule of M if N is a δ-supplement of some submodule of M.

Definition 2.3. M is called an *amply δ-supplemented* module, if for all submodules A, B of M with $M = A + B$ there exists a δ-supplement P of A such that $P \subseteq B$.

Definition 2.4. M is called a *weakly δ-supplemented* module, if for each submodule A of M with M/A singular there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll_\delta M$.

Definition 2.5. Assume that M is a right R-module and $B \subseteq A \subseteq M$.

(1) B is called a *δ-coessential submodule* of A if $A/B \ll_\delta M/B$ with A/B singular. In this case, A is called a δ-coessential extension of B in M (denoted by
$B \subseteq_{\delta ce} A$).

(2) A is called a *δ-coclosed submodule* of M if M/A is singular and A has no δ-coessential submodule of M (denoted by $A \subseteq_{\delta cc} M$).

(3) B is called a *δ-s-closure* of A in M if B is a δ-coessential submodule of A and B is δ-coclosed of M.

Lemma 2.6. *If P is a δ-supplement of A in M then P is minimal with the property $M = A + P$ and M/P is singular. If the class δ is closed under module extensions, the converse holds.*

Proof. Let $M = A + Q$ with $Q \subseteq P$ and M/Q singular, so that $P = Q + A \cap P$. Since P is a δ-supplement of A in M, then $M = A + P$ and $A \cap P \ll_\delta P$. Note that M/Q is singular, which implies that P/Q is singular. It follows that $P = Q$.

Conversely, let $P = A \cap P + D$ with P/D singular. Then

$$M = A + P = A + A \cap P + D = A + D.$$

Since $0 \to P/D \to M/D \to M/P \to 0$ is exact with P/D and M/P singular, and the class of singular left R-modules is closed under extensions, we have that M/D is singular. Thus, $P = D$ by minimality of P, so $A \cap P \ll_\delta P$. □

Lemma 2.7. *If M is an amply δ-supplemented module, then M/K is an amply δ-supplemented module for each $K \subseteq M$.*

Proof. Let $M/K = A/K + B/K$, thus $M = A + B$. Since M is amply δ-supplemented, there exists $P \subseteq B$ such that M/P is singular, $A \cap P \ll_\delta P$ and $M = A + P$. Hence, $M/K = A/K + (P + K)/K$ and $A/K \cap (P + K)/K = (A \cap P + K)/K$. Now, we claim that

$$(A \cap P + K)/K \ll_\delta (P + K)/K.$$

Let $(A \cap P + K)/K + L/K = (P + K)/K$ with $(P + K)/L$ singular, thus $A \cap P + L = P + K$. Since $A \cap P \ll_\delta P \subseteq P + K$, it follows from Lemma 2.1 that $A \cap P \ll_\delta P + K$. Note that $(P + K)/L$ is singular, so $L = P + K$. Thus, $(A \cap P + K)/K \ll_\delta (P + K)/K$ and $M/(P + K)$ is singular, therefore M/K is amply δ-supplemented. □

Proposition 2.8. *Every direct summand L of an amply δ-supplemented module M is an amply δ-supplemented submodule.*

Proof. Let A, B be submodules of L such that $L = A + B$. Since $L \subseteq_{\oplus} M$, there exists $L^{'}$ such that $M = L \oplus L^{'} = A + B + L^{'} = A + (B + L^{'})$. Since M is an amply δ-supplemented module, $B + L^{'}$ has a δ-supplement submodule P such that $P \subseteq A \subseteq L$, thus $P + B + L^{'} = M$ with M/P singular and $P \cap (B + L^{'}) \ll_{\delta} P$. Since $P + B \subseteq L$ and $L \cap L^{'} = 0$, it follows that $P + B = L$ with L/P singular. Since $P \cap B \subseteq P \cap (B + L^{'})$, it holds $P \cap B \ll_{\delta} P$ by Lemma 2.1. Thus, L is an amply δ-supplemented submodule. □

3. On δ-coclosed submodules

Lemma 3.1. *Let $N \subseteq M$. Consider the following conditions:*

(1) *N is δ-coclosed in M;*
(2) *M/N is singular such that $X \ll_{\delta} M$ implies $X \ll_{\delta} N$ for each $X \subseteq N$;*
(3) *N is a δ-supplement of M.*

Then, (1) $\Rightarrow$ (2). *If M is a weakly δ-supplemented module, then* (2) $\Rightarrow$ (3). *If the class δ is closed under module extensions, then* (3) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Suppose that $N \subseteq_{\delta cc} M$, $X \subseteq N$ and $X \ll_{\delta} M$. Let $N = X + Y$ with N/Y singular. Since N is δ-coclosed, it suffices to show that $N/Y \ll_{\delta} M/Y$. Let $M/Y = N/Y + H/Y$ with $Y \subseteq H \subseteq M$ and M/H singular. Thus,

$$M = N + H = X + Y + H = X + H.$$

Since M/H is singular and $X \ll_{\delta} M$, it follows that $M = H$. Thus, $N/Y \ll_{\delta} M/Y$, hence $N = Y$ for N/Y is singular and N is δ-coclosed. So, $X \ll_{\delta} N$.

(2) $\Rightarrow$ (3) Since M is weakly δ-supplemented, there exists a submodule $L \subseteq M$ such that $M = N + L$ and $N \cap L \ll_{\delta} M$. By hypothesis, $N \cap L \ll_{\delta} N$. Thus N is a δ-supplement of L in M.

(3) $\Rightarrow$ (1) Suppose that N is a δ-supplement of L in M. By Lemma 2.6, N is minimal with the property that $M = N + L$ with M/N singular. Let $K \subseteq N$ and $N/K \ll_{\delta} M/K$ with N/K singular. Then, $M/K = (N + L)/K = N/K + (L + K)/K$. Note that

$$\begin{aligned} M/(L+K) &= (N+L)/(L+K) \\ &= (N+L+K)/(L+K) \\ &\simeq N/(N\cap(L+K)) \\ &= N/(N\cap L+K) \end{aligned}$$

and N/K is singular, thus $M/(L+K)$ is singular. Since $N/K \ll_{\delta} M/K$, we have $M/K = (L + K)/K$, so $M = L + K$. Since $0 \to N/K \to M/K \to M/N \to 0$ is exact with N/K and M/N singular, and the class δ is closed under module extensions, it follows that M/K is singular. By minimality of N it follows that $N = K$. Hence, N is δ-coclosed in M. □

Lemma 3.2. *Let M be a module, and let $M = A + B$ and $M = (A \cap B) + C$. Then, $M = (B \cap C) + A = (A \cap C) + B$.*

Proof. Straightforward. □

Lemma 3.3. *Assume that $M = A+B$ and $B \subseteq C$ with $C/B \ll_\delta M/B$ and C/B singular. Then, $(A \cap C)/(A \cap B) \ll_\delta M/(A \cap B)$.*

Proof. Suppose that $M/(A \cap B) = (A \cap C)/(A \cap B) + X/(A \cap B)$ with M/X singular and $(A \cap B) \subseteq X \subseteq M$. Thus $M = (A \cap C) + X$. By Lemma 3.2, $M = C + (A \cap X)$. Thus, $M/B = (C + A \cap X)/B = C/B + (A \cap X + B)/B$ and

$$\begin{aligned} M/(A \cap X + B) &= (A \cap X + C)/(A \cap X + B) \\ &= (A \cap X + B + C)/(A \cap X + B) \\ &\simeq C/(C \cap (A \cap X + B)) \\ &= C/(C \cap A \cap X + B), \end{aligned}$$

where $M/(A \cap X + B)$ is singular for C/B is singular. Since $C/B \ll_\delta M/B$, it holds $M = B + (A \cap X)$. Again by Lemma 3.2, $M = (A \cap B) + X$, which implies $M = X$. So, $(A \cap C)/(A \cap B) \ll_\delta M/(A \cap B)$. □

Proposition 3.4. (1) *Let $B \subseteq C$ be submodules of M. If B is a δ-supplement submodule of M and C/B is a δ-supplement submodule of M/B, then C is a δ-supplement submodule of M.*

(2) *Assume that M is a weakly δ-supplement module and that the class δ is closed under module extensions. Let $B \subseteq C$ be submodules of M. If $B \subseteq_{\delta cc} M$ and $C/B \subseteq_{\delta cc} M/B$, then $C \subseteq_{\delta cc} M$.*

Proof. (1) Let B be a δ-supplement of B' in M, and let C/B be a δ-supplement of C'/B in M/B. Thus $M/B = C/B + C'/B$ with $C/B \cap C'/B \ll_\delta C/B$, and $M = B + B'$ with $B \cap B' \ll_\delta B$. By Lemma 2.1, $B \cap B' \ll_\delta C$. Since $B \subseteq M = (C \cap C') + B'$ and $M = C + C'$, it follows from Lemma 3.2 that $M = C + (B' \cap C')$. Hence

$$C = C \cap (B + B') = B + (C \cap B')$$

and $(C \cap C')/B \ll_\delta C/B$. Since B is a δ-supplement in M, we have that $(C \cap C')/B$ is singular. By Lemma 3.3,

$$(C \cap C' \cap B')/(B \cap B') \ll_\delta C/(B \cap B').$$

Note that $B \cap B' \ll_\delta C$, thus $(C \cap C' \cap B') \ll_\delta C$ by Lemma 2.1. So, C is a δ-supplement of $B' \cap C'$ in M.

(2) Since M/B is singular and $C/B \subseteq M/B$, it follows that C/B is singular. The conclusion follows from Lemma 3.1 and (1). □

Proposition 3.5. *Assume that the class δ is closed under module extensions and M is an amply δ-supplemented module. Then, every submodule of M has a δ-s-closure.*

Proof. Let $A \subseteq M$. Since M is amply δ-supplemented, there exists a submodule B such that B is minimal with the property $M = A + B$ with M/B singular. Again since M is amply δ-supplemented, there exists $C \subseteq A$ such that $M = C+B$ with M/C singular and $C \cap B \ll_\delta C$. Hence, A/C is singular. Now we claim that

$A/C \ll_\delta M/C$. Let $C \subseteq X \subset M$. If $A/C + X/C = M/C$ with M/X singular, then

$$M = C + (X \cap B) + A = A + (X \cap B).$$

Note that

$$M/(X \cap B) = (A + X \cap B)/(X \cap B) \simeq A/(A \cap X \cap B)$$

and $C \subseteq A$, $C \subseteq X$. Thus, there exists an epimorphism $A/(B \cap C) \twoheadrightarrow A/(A \cap X \cap B)$. Consider the exact sequence

$$0 \to (A \cap B)/(B \cap C) \to A/(B \cap C) \to A/(A \cap B) \to 0,$$

where

$$(A \cap B)/(B \cap C) = (A \cap B)/(A \cap B \cap C) \simeq (A \cap B + C)/C \subseteq M/C$$

is singular. Since $A/(A \cap B) \simeq A + B/B \subseteq M/B$ is singular, we have that $A/(B \cap C)$ is singular. Hence $M/(X \cap B)$ is singular. By minimality of B it holds $X \cap B = B$, and hence $M = X$, a contradiction. Thus $M \neq A + X$. Since C is a δ-supplement, which implies that M/C is singular, and the class of singular left R-modules is closed under extensions, it follows from Lemma 3.1 that C is δ-coclosed. So, C is a δ-s-closure of A in M. □

Proposition 3.6. *Let $K \subseteq L \subseteq M$.*

(1) *If $L \subseteq_{\delta cc} M$ then $L/K \subseteq_{\delta cc} M/K$.*
(2) *If $K \subseteq_{\delta cc} M$ then $K \subseteq_{\delta cc} L$. Conversely, if $K \subseteq_{\delta cc} L$ and $L \subseteq_{\delta cc} M$ then $K \subseteq_{\delta cc} M$.*

Proof. (1) Let $N/K \subseteq_{\delta ce} L/K$. If $N/K \subseteq L/K$ and $(L/K)/(N/K) \ll_\delta (M/K)/(N/K)$ with $(L/K)/(N/K)$ singular, that is, $L/N \ll_\delta M/N$ with L/N singular. Thus, $N \subseteq_{\delta ce} L$. Since $L \subseteq_{\delta cc} M$, we have $N = L$, hence $N/K = L/K$. So, $L/K \subseteq_{\delta cc} M/K$.

(2) Let $X \subseteq K$ be such that $K/X \ll_\delta L/X \subseteq M/X$ with K/X singular. By Lemma 2.1, $K/X \ll_\delta M/X$. Since $K \subseteq_{\delta cc} M$, we have $K = X$. Thus, $K \subseteq_{\delta cc} L$.

Conversely, let $K \subseteq_{\delta cc} L$ and $L \subseteq_{\delta cc} M$. If $K/X \ll_\delta M/X$ with K/X singular, it follows from (1) that $L/X \subseteq_{\delta cc} M/X$. By Lemma 3.1 (1) $\Rightarrow$ (2), $K/X \ll_\delta L/X$, i.e., $X \subseteq_{\delta ce} K$, so $X = K$. □

Lemma 3.7. *If A is a direct summand of M and M/A is singular, then A is a δ-coclosed submodule of M.*

Proof. Let A be a direct summand of M such that M/A is singular. There exists a submodule A' such that $M = A \oplus A'$. Suppose that $B \subseteq A \subseteq M$ and $A/B \ll_\delta M/B$ with A/B singular. Then,

$$M/B = (A \oplus A')/B = A/B + (A' + B)/B$$

where

$$\begin{aligned} M/(A' + B) &= (A \oplus A')/(A' + B) \\ &= (A + A' + B)/(A' + B) \\ &\simeq A/A \cap (A' + B) \\ &= A/B \end{aligned}$$

is singular. Hence $M = A' \oplus B$ and $B \subseteq A$, thus $A = B$. So A is a δ-coclosed submodule of M. □

Proposition 3.8. *Let $L \subseteq_{\delta cc} M$ and $K \subseteq_{\oplus} L$ with L/K singular. Then, $K \subseteq_{\delta cc} M$.*

Proof. Since $K \subseteq_{\oplus} L$ with L/K singular, it follows that $K \subseteq_{\delta cc} L$ by Lemma 3.7. Since $L \subseteq_{\delta cc} M$, we have $K \subseteq_{\delta cc} M$ by Proposition 3.6. □

Proposition 3.9. *Let M be a weakly δ-supplemented module and let $L \subseteq_{\oplus} M$ with M/L singular. If the class δ is closed under module extensions, then L is a weakly δ-supplemented submodule.*

Proof. Let $A \subseteq L \subseteq M$ with L/A singular. Since the class δ is closed under module extensions and M/L is singular, it follows that M/A is singular. Since M is a weakly
δ-supplemented module, there exists $B \subseteq M$ such that $M = A + B$ and $A \cap B \ll_{\delta} M$. Since $L \subseteq_{\oplus} M$, there exists $L' \subseteq M$ such that $M = L + L'$. Thus,

$$L = L \cap (A + B) = A + (L \cap B)$$

and $A \cap (L \cap B) = A \cap B$. Since $L \subseteq_{\oplus} M$ with M/L singular, by Lemma 3.7 we have $L \subseteq_{\delta cc} M$. So, $A \cap B \ll_{\delta} L$ by Lemma 3.1. □

4. On δ-lifting modules

According to [5], a module M is called a *lifting module* (or is said to satisfy (D_1)), if for every submodule N of M there exists a direct summand K of M such that $K \subseteq N$ and that N/K is small in M/K. As a generalization, we give the following definition.

Definition 4.1. A module M is called a δ-*lifting* module (or is said to satisfy δ-(D_1)), if for every submodule N of M there exists a direct summand K of M such that $K \subseteq N$ and that $N/K \ll_{\delta} M/K$.

Clearly, every lifting module is δ-lifting. We shall give an example to illustrate that a δ-lifting right R-module is not necessarily lifting in Section 5.

Proposition 4.2. *Every direct summand of a δ-lifting module M is δ-lifting.*

Proof. Let L be a direct summand of M and let $H \subseteq L \subseteq M$. Since M is δ-lifting, there exists $K \subseteq_{\oplus} M$ such that $K \subseteq H$ and $H/K \ll_{\delta} M/K$, hence $K \subseteq_{\oplus} L$. Let $H/K + X/K = L/K$ with L/X singular. Then $H + X = L$. Suppose that $M = L \oplus L'$, thus $H + L + L' = M$. Hence $H/K + (X + L')/K = M/K$. Since

$$\begin{aligned} M/(X + L') &= (X + L + L')/(X + L') \\ &\simeq L/L \cap (X + L') \\ &= L/(X + L \cap L') \\ &= L/X \end{aligned}$$

is singular it follows that $(X + L')/K = M/K$, i.e., $X + L' = M$, hence $X = L$ for $X \subseteq L$. Thus, $X/K = L/X$, so $H/K \ll_{\delta} L/K$. □

Lemma 4.3. *Let M be an amply δ-supplemented module such that every δ-supplement of M is a direct summand, then M is a δ-lifting module.*

Proof. Let $A \subseteq M$. Since M is an amply δ-supplemented, there exists a δ-supplement B of A, and B has a δ-supplement M_1 such that $M_1 \subseteq A$ with M/M_1 singular and $M_1 \subseteq_{\oplus} M$. Thus, $M = M_1 \oplus M_2$ for some submodule M_2 of M. Hence, $A = M_1 \oplus A \cap M_2$. Note that $M = M_1 + B$ and $A = M_1 + A \cap B$. Let π denote the projection $M_1 \oplus M_2 \twoheadrightarrow M_2$. Thus, $A \cap M_2 = \pi A = \pi(A \cap B)$. Since B is a δ-supplement of A we have $(A \cap B) \ll_{\delta} M$, and hence $(A \cap M_2) \ll_{\delta} M$ by Lemma 2.1, so $A \cap M_2 \ll_{\delta} M_2$, i.e., $A/M_1 \ll_{\delta} M/M_1$. Since M/M_1 is singular, so is A/M_1, thus M is δ-lifting. $\square$

In the following, if $M = \oplus_{i \in I} M$, we use M_{-i} to denote $\oplus_{j \in I \setminus i} M_j$.

Theorem 4.4. *Assume that the class δ is closed under module extensions. Let $M = \oplus_{i \in I} M_i$ and $|I| \geq 2$. If M is an amply δ-supplemented module, then the following statements are equivalent:*

(1) *M is δ-lifting.*

(2) *For every δ-coclosed submodule K of M, if either $M = K + M_i$ or $M = K + M_{-i}$ for some $i \in I$, then $K \subseteq_{\oplus} M$.*

(3) *For every δ-coclosed submodule K of M, if either $(K + M_i)/K \ll_{\delta} M/K$ or*
$(K + M_{-i})/K \ll_{\delta} M/K$ or $M = K + M_i = K + M_{-i}$ for some $i \in I$, then $K \subseteq_{\oplus} M$.

Proof. (1) $\Rightarrow$ (2) For each $K \subseteq_{\delta cc} M$, since M is δ-lifting, there exists $H \subseteq_{\oplus} M$ such that $H \subseteq K$ with K/H singular and $K/H \ll_{\delta} M/H$. Thus, $K = H$ for $K \subseteq_{\delta cc} M$, so $K \subseteq_{\oplus} M$.

(2) $\Rightarrow$ (1) Let $K \subseteq_{\delta cc} M$. By Lemma 2.7, M/K is an amply δ-supplemented module. By Proposition 3.5, $(K + M_i)/K$ has δ-s-closure in M/K, i.e., there exists a δ-coclosed submodule N/K in M/K such that $N/K \subseteq (K + M_i)/K$ and $(K + M_i)/N \ll_{\delta} M/N$ with $(K + M_i)/N$ singular. Thus, N is a δ-coclosed submodule of M by Proposition 3.4. Note that $M = (K + M_i) + M_{-i}$ and

$$M/N = ((K + M_i) + M_{-i})/N = (K + M_i)/N + (M_{-i} + N)/N.$$

Since M/K is singular, we have that $M/(M_{-i} + N)$ is singular. Thus, $M = N + M_{-i}$ for $(K + M_i)/N \ll_{\delta} M/N$. By (2), $M = N \oplus N'$ for some $N' \subseteq M$. Hence, $K = N \cap (K + N')$ and $M = N + (K + N')$, so $M/K = N/K \oplus (K + N')/K$. By Lemma 3.7, $(K + N')/K$ is δ-coclosed submodule of M/K, and hence $K + N'$ is a δ-coclosed submodule of M by Proposition 3.4. Note that $M = (K + M_i) + N' = (K + N') + M_i$. By (2), $K + N'$ is a direct summand of M. Suppose that $M = (K + N') \oplus K'$ for some $K' \subseteq M$. Thus, $N' = (K + N') \cap (N' + K')$ and

$$K \cap (N' + K') = N \cap (K + N') \cap (N' + K') = 0,$$

so $M = K \oplus (N' + K')$. By Lemma 3.1 and Lemma 4.3, M is δ-lifting.

(2) $\Rightarrow$ (3) Note that $(K+M_i)/K+(K+M_{-i})/K = M/K$ with $M/(K+M_i)$ and $M/(K+M_{-i})$ singular.

(3) $\Rightarrow$ (2) Let K be a δ-coclosed submodule in M such that $M = K+M_i$. (The case $M = K+M_{-i}$ is similar). Since M/K is an amply δ-supplemented module, $(K+M_{-i})/K$ has a δ-s-closure in M/K, i.e., there exists a δ-coclosed submodule N/K in M/K such that $(K+M_{-i})/N \ll_\delta M/N$. It is clear that $N+M_{-i} = K+M_{-i}$ and $(N+M_{-i})/N \ll_\delta M/N$. Thus, N is a δ-coclosed submodule in M by Proposition3.4. By (3), $M = N \oplus N^{'}$ for some $N^{'} \subseteq M$. It is obvious that $M = K+N^{'}+N$ and $K = (K+N^{'})\cap N$, hence $M/K = N/K \oplus (K+N^{'})/K$, that is, $(K+N^{'})/K$ is a direct summand in M/K, and so $(K+N^{'})/K$ is δ-coclosed in M/K. By Proposition 3.4, $(K+N^{'}) \subseteq_{\delta cc} M$. By (3), $K+N^{'} = K \oplus N^{'}$ is a direct summand in M, thus K is a direct summand in M. □

Lemma 4.5. *Let $M = M_1 \oplus M_2$. If $N \subseteq M$ such that $(N+M_1)/M_1 \ll_\delta M/M_1$ and $M = N+M_2$, then $(N+M_1)/N \ll_\delta M/N$.*

Proof. Let $N \subseteq M$ be such that $(N+M_1)/M_1 \ll_\delta M/M_1$ and $M = N+M_2$. Consider the homomorphism

$$\phi : M/M_1 \simeq M_2 \xrightarrow{f} M_2/(M_2 \cap N) \simeq (M_2+N)/N = M/N.$$

Then, ϕ maps $(N+M_1)/M_1$ to $(N+M_1)/N$, so $(N+M_1)/N \ll_\delta M/N$ by Lemma 2.1(2). □

Definition 4.6. Let M_1 and M_2 be modules. M_1 is called *δ-small M_2-projective* if every homomorphism $f : M_1 \to M_2/A$, where $A \subseteq M_2$ with M_2/A singular and $Imf \ll_\delta M_2/A$, can be lifted to a homomorphism $\varphi : M_1 \to M_2$. M_1 and M_2 are *relatively δ-small projective*, if M_i is δ-small M_j-projective for every $i \neq j \in \{1,2\}$. It is clear that M_1 is δ-small M_2-projective if M_1 is M_2-projective.

Lemma 4.7. *Let $M = M_1 \oplus M_2$. The following statements are equivalent.*

(1) *M_1 is δ-small M_2-projective.*
(2) *For every submodule $N \subseteq M$ with M/N singular and $(N+M_1)/N \ll_\delta M/N$ there exists $N^{'} \subseteq N$ such that $M = N^{'} \oplus M_2$.*

Proof. (1) $\Rightarrow$ (2) Let $N \subseteq M$ with M/N singular be such that $(N+M_1)/N \ll_\delta M/N$. Thus,

$$M/N = (M_1+M_2)/N = (N+M_1)/N + (N+M_2)/N.$$

Since M/N is singular, we have that $M/(N+M_2)$ is singular, so $M = N+M_2$. Consider the homomorphism

$$g : M_1 \to M/N, m_1 \mapsto m_1 + N$$

and the epimorphism

$$f : M_2 \to M/N = (N+M_2)/N, m_2 \mapsto m_2 + N.$$

It follows that $Img = (N+M_1)/N \ll_\delta M/N$. Since M_1 is δ-small M_2-projective, there exists a homomorphism $\varphi : M_1 \to M_2$ such that $f\varphi = g$. Define

$$N^{'} = \{a - \varphi(a) \mid a \in M_1\},$$

so that $N' \subseteq N$ and $M = N' \oplus M_2$.

(2) $\Rightarrow$ (1) Suppose that $A \subseteq M_2$ with M_2/A singular, that $f : M_1 \to M_2/A$ with $Imf \ll_\delta M_2/A$, and let $\pi : M_2 \to M_2/A$ denote the canonical epimorphism. Define

$$N = \{a + b \in M_1 \oplus M_2 \mid f(a) = -\pi(b), a \in M_1, b \in M_2\}.$$

Clearly, $A \subseteq N$ and $M = N + M_2$. Let $Imf = X/A$ with $X \subseteq M_2$. Consider

$$h : M_2/A \to M/N, m_2 \mapsto m_2 + N.$$

Then $h(X/A) = (X + N)/N$. Since $X/A = Imf \ll_\delta M_2/A$, it follows from Lemma 2.1 that $(X + N)/N \ll_\delta M/N$. Note that $(N + M_1)/N \subseteq (X + N)/N$, thus

$$(N + M_1)/N \ll_\delta M/N$$

by Lemma 2.1. Since

$$M/N = (N + M_2)/N \simeq M_2/(N \cap M_2),$$

$A \subseteq N \cap M_2$ and M_2/A is singular, it follows that M/N is singular. Hence there exists $N' \subseteq N$ such that $M = N' \oplus M_2$. Consider the canonical projection $\alpha : N' \oplus M_2 \to M_2$. Then, the homomorphism f can be lifted to the homomorphism $\alpha |_{M_1}: M_1 \to M_2$. So, M_1 is δ-small M_2-projective. □

Proposition 4.8. *Let $M = M_1 \oplus M_2$ be an amply δ-supplemented module. The following statements are equivalent.*

(1) *If $M = N + M_2$ such that M/N is singular and $(N + M_1)/M_1 \ll_\delta M/M_1$, then there exists $N' \subseteq N$ such that $M = N' \oplus M_2$.*
(2) *If $M = N + M_2$ such that M/N is singular and $(N + M_1)/N \ll_\delta M/N$, then there exists $N' \subseteq N$ such that $M = N' \oplus M_2$.*
(3) *M_1 is δ-small M_2-projective.*

Proof. (1) $\Rightarrow$ (2) Suppose that $M = N + M_2$ with M/N singular and

$$(N + M_1)/N \ll_\delta M/N.$$

Since M is an amply δ-supplemented module, M/M_1 is an amply δ-supplemented module by Lemma 2.7. Thus there exists $M_1 \subseteq X \subseteq M$ such that

$$M/M_1 = X/M_1 + (N + M_1)/M_1 \text{ and } (X \cap (N + M_1))/M_1 \ll_\delta X/M_1,$$

hence

$$M = N + X = N + (X \cap (M_1 + M_2)) = (N + M_1) + (X \cap M_2).$$

Note that $M/N = (N + M_1)/N + (X \cap M_2 + N)/N$, where

$$\begin{aligned} M/(X \cap M_2 + N) &= (X \cap M_2 + N + X)/(X \cap M_2 + N) \\ &\simeq X/(X \cap (X \cap M_2 + N)) \\ &= X/(X \cap M_2 + X \cap N), \end{aligned}$$

and there is an epimorphism $X/(M_1 \cap M_2 + X \cap N) \to X/(X \cap M_2 + X \cap N)$ where

$$X/(M_1 \cap M_2 + X \cap N) = X/X \cap N = (X + N)/N = M/N$$

singular, so $M/(X \cap M_2 + N)$ is singular, thus $M/N = (X \cap M_2 + N)/N$, so $M = X \cap M_2 + N$. By Lemma 3.2, $M = M_2 + (X \cap N)$. Since $(N \cap X) + M_1 = X \cap (N + M_1)$, it follows that

$$((N \cap X) + M_1)/M_1 \ll_\delta X/M_1.$$

By Lemma 2.1, $((N \cap X) + M_1)/M_1 \ll_\delta M/M_1$. Thus there exists $N^{'} \subseteq M$ such that $N^{'} \subseteq N \cap X$ and $M = N^{'} \oplus M_2$.

(2) $\Rightarrow$ (1) By Lemma 4.5.

(2) $\Leftrightarrow$ (3) By Lemma 4.7. $\square$

Lemma 4.9. *Assume that $M = M_1 \oplus M_2$ is such that M_2 is a δ-lifting module and M_1 is δ-small M_2-projective. If $N \subseteq_{\delta cc} M$ and $(N + M_1)/N \ll_\delta M/N$, then N is a direct summand.*

Proof. Suppose that $N \subseteq_{\delta cc} M$ and that $(N + M_1)/N \ll_\delta M/N$. By Lemma 4.7, there exists $N^{'} \subseteq N$ such that $M = N^{'} \oplus M_2$. Clearly, $M/N^{'} \simeq M_2$ is δ-lifting. Since $N \subseteq_{\delta cc} M$, we have $N/N^{'} \subseteq_{\delta cc} M/N^{'}$ by Lemma 3.6. Note that $N/N^{'} + (M_2 + N^{'})/N^{'} = M/N^{'}$. It follows from Theorem 4.4(1) $\Rightarrow$ (2) that $N/N^{'} \subseteq_\oplus M/N^{'}$. Thus, $N \subseteq_\oplus M$. $\square$

The following lemma is known from [7, 41.14].

Lemma 4.10. *Let $M = M_1 \oplus M_2$. The following statements are equivalent.*

(1) *M_1 is M_2-projective.*
(2) *For every $N \subseteq M$ with $M = N + M_2$ there exists $N^{'} \subseteq N$ such that $M = N^{'} \oplus M_2$.*

Theorem 4.11. *Assume that the class δ is closed under module extensions, that $M = M_1 \oplus M_2$ is an amply δ-supplemented module, and that M_1 and M_2 are δ-lifting. If one of the following conditions holds, then M is δ-lifting.*

(1) *M_1 is δ-small M_2-projective and every $N \subseteq_{\delta cc} M$ with $M = N + M_1$ is a direct summand.*
(2) *M_1 and M_2 are relatively δ-small projective, and every $N \subseteq_{\delta cc} M$ with $M = N + M_1 = N + M_2$ is a direct summand.*
(3) *M_2 is M_1-projective, and M_1 is δ-small M_2-projective.*
(4) *M_1 is semisimple and δ-small M_2-projective.*

Proof. (1) and (2) follow from Theorem 4.4 and Lemma 4.9.

(3) Let $N \subseteq_{\delta cc} M$ be such that $M = N + M_1$. By Lemma 4.10, there exists $N^{'} \subseteq N$ such that $M = N^{'} \oplus M_2$. Note that

$$N/N^{'} + (M_2 + N^{'})/N^{'} = M/N^{'}$$

and that M/N is singular. Since $M/N^{'}$ is δ-lifting and $N/N^{'} \subseteq_{\delta cc} M/N^{'}$, it follows that $N/N^{'} \subseteq_\oplus M/N^{'}$ by Theorem 4.4. So, $N \subseteq_\oplus M$. Now, (3) follows from (1).

(4) follows from (3). $\square$

5. Projective δ-covers

Recall that a pair (P,p) is called a *projective δ-cover* [8] of a module M if P is projective and p is an epimorphism of P onto M with $\ker(p) \ll_\delta P$. In this section, we mainly use δ-lifting modules to characterize rings R such that every R-module (or every simple R-module, resp.) has a projective δ-cover.

A ring R is called *δ-perfect* (or *δ-semiperfect*) [8], if every R-module (or every simple R-module, resp.) has a projective δ-cover. It is proved in [8, Theorem 3.6] that R is δ-semiperfect if and only if every finitely generated right module has a projective δ-cover. Thus, every right perfect ring is clearly δ-perfect. Semiperfect rings and δ-perfect rings are δ-semiperfect. Zhou [8] gives some examples to illustrate that a δ-perfect ring is not necessarily semiperfect, and that a δ-semiperfect ring is not necessarily semiperfect.

According to [2], a ring R is right perfect (or semiperfect) if and only if every (or every finitely generated) projective right R-module is lifting. We generalize this as follows.

Theorem 5.1. *A ring R is δ-perfect (or δ-semiperfect) if and only if every (or every finitely generated, resp.) projective right R-module is δ-lifting.*

Proof. Let R be δ-perfect (or δ-semiperfect) and let P be a (finitely generated) projective R-module. For a submodule A of P, consider the canonical epimorphism $\varphi : P \to P/A$. Since R is δ-perfect, P/A has a projective δ-cover, hence there exists a decomposition $P = P_1 \oplus P_2$ such that $P_2 \subseteq A$ and $P_1 \cap A \ll_\delta P$ by [8, Lemma 2.4]. If $A/P_2 + L/P_2 = P/P_2$ with P/L singular, then $A + L = P$. Since

$$A = A \cap (P_1 + P_2) = P_2 + A \cap P_1,$$

it follows that $P_2 + A \cap P_1 + L = P$. Thus, $A \cap P_1 + L = P$ with P/L singular. Since $P_1 \cap A \ll_\delta P$, we have that $P = L$. Hence $P/P_2 = L/P_2$. So $A/P_2 \ll_\delta P/P_2$, therefore P is δ-lifting.

Conversely, since every (finitely generated) module M is an epimorphic image of a (finitely generated) free module, we can consider an epimorphism $f : P \to M$, where P is (finitely generated) projective. Since P is δ-lifting, there exists a decomposition $P = P^* \oplus P^{**}$ such that $\ker f/P^* \ll_\delta P/P^*$. Thus, we get an epimorphism $(f\mid_{P^{**}}) : P^{**} \to M$ with $\ker(f\mid_{P^{**}}) = \ker f \cap P^{**}$ and

$$\begin{aligned} \ker f/P^* &= \ker f \cap (P^* + P^{**})/P^* \\ &= Ker f \cap P^{**} + P^*/P^* \\ &\simeq \ker f \cap P^{**}. \end{aligned}$$

Note that $\ker f/P^* \ll_\delta P/P^*$. It follows that $\ker(f\mid_{P^{**}}) \ll_\delta P^{**}$. Hence

$$\ker(f\mid_{P^{**}}) \ll_\delta P.$$

Therefore, $(f\mid_{P^{**}})$ is a projective δ-cover. □

Example 5.2. (1) Let F be a field, let $I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, and let

$$R = \{(x_1, \cdots, x_n, x, x, \cdots) \mid n \in \mathbb{N}, x_i \in M_2(F), x \in I\}.$$

With component wise operations, R is a ring. According to [8, Example 4.3], R is δ-perfect but not right perfect. Thus, every projective right R-module is δ-lifting, but not necessarily lifting.

(2) Let $Q = \prod_{i=1}^{\infty} F_i$, where $F_i = \mathbb{Z}_2$ for all i. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. According to [8, Example 4.1], R is δ-semiperfect but not semiperfect. Thus, every finitely generated projective right R-module is δ-lifting, but not necessarily lifting.

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