ACTA MATHEMATICA VIETNAMICA
Volume 31, Number 3, 2006, pp. 291-300

SOME INEQUALITIES FOR NORMAL OPERATORS IN HILBERT SPACES

S. S. DRAGOMIR

Abstract. Some inequalities for normal operators in Hilbert spaces are given. For this purpose, some results for vectors in inner product spaces due to Buzano, Dunkl-Williams, Hile, Goldstein-Ryff-Clarke, Dragomir-Sándor and the author are employed.

1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $T : H \to H$ a bounded linear operator on H. Recall that T is a *normal operator* if $T^*T = TT^*$. Normal operators may be regarded as a generalisation of self-adjoint operator T in which T^* needs not be exactly T but commutes with T [8, p. 15].

The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [8, p. 1]:

$$W(T) = \{\langle Tx, x \rangle, \ x \in H, \ \|x\| = 1\}.$$

For various properties of the numerical range see [8].

We recall here some of the ones related to normal operators.

Theorem 1.1. *If $W(T)$ is a line segment, then T is normal.*

We denote by $r(T)$ the operator *spectral radius* [8, p. 10] and by $w(T)$ its *numerical radius* [8, p. 8]. The following result may be stated as well [8, p. 15].

Theorem 1.2. *If T is normal, then $\|T^n\| = \|T\|^n$, $n = 1, \ldots$. Moreover, we have:*

$$r(T) = w(T) = \|T\|. \tag{1.1}$$

An important property of the normal operators that will be used frequently in the sequel is the following [9, p. 42]:

Theorem 1.3. *A necessary and sufficient condition for an operator T to be normal is that $\|Tx\| = \|T^*x\|$ for every vector $x \in H$.*

Received June 19, 2006
2000 Mathematics Subject Classification. 47A12.
Key words and phrases. Numerical radius, bounded linear operators, normal operators, Hilbert spaces, norm inequalities.

We observe that, if one uses the Schwarz inequality

$$|\langle u, v\rangle| \leqslant \|u\| \|v\|, u, v \in H,$$

for the choices $u = Tx, v = T^*x$ with $x \in H$, then one gets the following simple inequality for the normal operator T :

$$\|Tx\|^2 \geqslant \left|\left\langle T^2x, x\right\rangle\right|, \ x \in H. \tag{1.2}$$

It is then natural to look for upper bounds for the quantity $\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right|$, under various assumptions for the normal operator T, which would give a measure of the closeness of the terms involved in (1.2).

Motivated by this problem, the aim of the paper is to establish some reverse inequalities for (1.2). Norm inequalities for various expressions with normal operators and their adjoints are also provided. For both purposes, some inequalities for vectors in inner product spaces due to Buzano, Dunkl-Williams, Hile, Goldstein-Ryff-Clarke, Dragomir-Sándor and the author are employed.

2. Inequalities for vectors

The following result may be stated.

Theorem 2.1. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a normal linear operator. Then*

$$\|Tx\|^2 \geqslant \frac{1}{2}\left(\|Tx\|^2 + \left|\left\langle T^2x, x\right\rangle\right|\right) \geqslant |\langle Tx, x\rangle|^2, \tag{2.1}$$

for any $x \in H$, $\|x\| = 1$.The constant $\frac{1}{2}$ is best possible in (2.1).

Proof. The first inequality is obvious.

For the second inequality, we need the following refinement of Schwarz's inequality obtained by the author in 1985 [2, Theorem 2] (see also [5] and [4]):

$$\|a\| \|b\| \geqslant |\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| + |\langle a, e\rangle \langle e, b\rangle| \geqslant |\langle a, b\rangle|, \tag{2.2}$$

provided a, b, e are vectors in H and $\|e\| = 1$.

Observing that

$$|\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| \geqslant |\langle a, e\rangle \langle e, b\rangle| - |\langle a, b\rangle|,$$

then by the first inequality in (2.2) we deduce

$$\frac{1}{2}\left(\|a\| \|b\| + |\langle a, b\rangle|\right) \geqslant |\langle a, e\rangle \langle e, b\rangle|. \tag{2.3}$$

This inequality was obtained in a different way earlier by M.L. Buzano in [1].

Now, choose in (2.3), $e = x$, $\|x\| = 1$, $a = Tx$ and $b = T^*x$ to get

$$\frac{1}{2}\left(\|Tx\| \|T^*x\| + \left|\left\langle T^2x, x\right\rangle\right|\right) \geqslant |\langle Tx, x\rangle|^2, \tag{2.4}$$

for any $x \in H$, $\|x\| = 1$. Since T is normal, then $\|Tx\| = \|T^*x\|$, and by (2.4) we deduce the inequality (2.1).

The fact that the constant $\frac{1}{2}$ is best possible in (2.1) is obvious since for $T = I$, the identity operator, we get equality in (2.1). □

From a different way, we can state the following reverse inequality for (2.1).

Theorem 2.2. *Let $T : H \to H$ be a normal operator on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$. If $\lambda \in \mathbb{C}$, then*

$$0 \leqslant \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \leqslant \frac{2}{(1+|\lambda|)^2}\|Tx - \lambda T^*x\|^2, \tag{2.5}$$

for any $x \in H$, $\|x\| = 1$.

Proof. By taking the square

$$\|a-b\| \geqslant \frac{1}{2}(\|a\| + \|b\|)\left\|\frac{a}{\|a\|} - \frac{b}{\|b\|}\right\|, \quad a, b \in H\setminus\{0\},$$

the Dunkl-Williams inequality [6]. we have

$$\frac{4\|a-b\|^2}{(\|a\|+\|b\|)^2} \geqslant \left\|\frac{a}{\|a\|} - \frac{b}{\|b\|}\right\|^2 = 2 - 2\cdot\frac{\operatorname{Re}\langle a,b\rangle}{\|a\|\,\|b\|},$$

which shows that (see [3, Eq. (2.5)])

$$\frac{\|a\|\,\|b\| - |\langle a,b\rangle|}{\|a\|\,\|b\|} \leqslant \frac{2\|a-b\|^2}{(\|a\|+\|b\|)^2}.$$

Now, for $x \in H\setminus\ker(T)$, $\|x\| = 1$, choose $a = Tx$ and $b = \lambda T^*x$ $(\lambda \neq 0)$ to obtain

$$\|Tx\|\,\|T^*x\| - \left|\left\langle T^2x, x\right\rangle\right| \leqslant \frac{2\|Tx\|\,\|T^*x\|}{(\|Tx\| + |\lambda|\,\|T^*x\|)^2}\|Tx - \lambda T^*x\|^2. \tag{2.6}$$

Since $\|Tx\| = \|T^*x\|$, T being a normal operator, we get from (2.6) that (2.5) holds for any $x \in H\setminus\ker(T)$, $\|x\| = 1$.

For $\lambda = 0$ the inequality (2.5) is obvious.

Since for normal operators $\ker(T) = \ker(T^*)$ then for $x \in \ker(T)$, $\|x\| = 1$ the inequality (2.5) also holds. □

The following result provides a different upper bound for the nonnegative quantity

$$\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right|, x \in H, \|x\| = 1.$$

Theorem 2.3. *Let $T : H \to H$ be a normal operator and $\alpha, \lambda \in \mathbb{C}\setminus\{0\}$. Then*

$$\begin{aligned} 0 &\leqslant \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \\ &\leqslant \frac{1}{2}\cdot\frac{\left[|Re\alpha|\left\|Tx - \frac{\alpha}{\bar{\alpha}}\lambda T^*x\right\| + |Im\alpha|\left\|Tx + \frac{\alpha}{\bar{\alpha}}\lambda T^*x\right\|\right]^2}{|\lambda|\,|\alpha|^2} \end{aligned} \tag{2.7}$$

for any $x \in H$, $\|x\| = 1$.

Proof. Using the following inequality (see [3, Theorem 2.11]):

$$\|a\|\,\|b\| - \operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2}\langle a,b\rangle\right] \leqslant \frac{1}{2}\cdot\frac{\left[|\operatorname{Re}\alpha|\,\|a-b\| + |\operatorname{Im}\alpha|\,\|a+b\|\right]^2}{|\alpha|^2} \tag{2.8}$$

for the choices:

$$a = \frac{Tx}{\alpha}, \quad b = \frac{\lambda}{\bar{\alpha}}T^*x, \quad x \in H,$$

we obtain

$$\begin{aligned} &\frac{|\lambda|\,\|Tx\|\,\|T^*x\|}{|\alpha|^2} - \operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2}\cdot\frac{\bar{\lambda}}{\alpha^2}\langle Tx, T^*x\rangle\right] \\ &\leqslant \frac{1}{2}\cdot\frac{\left[|\operatorname{Re}\alpha|\left\|\frac{Tx}{\alpha} - \frac{\lambda}{\alpha}T^*x\right\| + |\operatorname{Im}\alpha|\left\|\frac{Tx}{\alpha} + \frac{\lambda}{\alpha}T^*x\right\|\right]^2}{|\alpha|^2}. \end{aligned} \tag{2.9}$$

Since T is normal, we get from (2.9) the inequality (2.7). The details are omitted. □

A similar result is incorporated in the following theorem.

Theorem 2.4. *Let $T : H \to H$ be a normal operator, $s \in [0,1]$ and $t \in \mathbb{R}$. Then*

$$\begin{aligned} 0 &\leqslant \|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \\ &\leqslant \|Tx\|^2\left[s\,\|tT^*x - Tx\|^2 + (1-s)\,\|T^*x - tTx\|^2\right]. \end{aligned} \tag{2.10}$$

In particular,

$$\begin{aligned} 0 &\leqslant \|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \\ &\leqslant \frac{1}{2}\|Tx\|^2 \inf_{t\in\mathbb{R}}\left[\|tT^*x - Tx\|^2 + \|T^*x - tTx\|^2\right]. \end{aligned}$$

Proof. We use the inequality obtained in [4, Theorem 2] to state that

$$\begin{aligned} &\left[(1-s)\,\|a\|^2 + s\,\|b\|^2\right]\left[(1-s)\,\|b\|^2 + s\,\|a\|^2\right] - |\langle a,b\rangle|^2 \\ &\leqslant \left[(1-s)\,\|a\|^2 + s\,\|b\|^2\right]\left[(1-s)\,\|b - ta\|^2 + s\,\|tb - a\|^2\right] \end{aligned} \tag{2.11}$$

for any $s \in [0,1]$, $t \in \mathbb{R}$ and $a, b \in H$.

If in (2.11) we choose $a = Tx$, $b = T^*x$, $x \in H$ and $\|x\| = 1$, then we get

$$\begin{aligned} &\|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \\ &\leqslant \|Tx\|^2\left[s\,\|tT^*x - Tx\|^2 + (1-s)\,\|T^*x - tTx\|^2\right] \end{aligned}$$

for any $s \in [0,1]$, $t \in \mathbb{R}$, which derives the inequality (2.10). □

Theorem 2.5. *Let $T : H \to H$ be a normal operator on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$. If $\lambda \in \mathbb{C}\setminus\{0\}$ and $r > 0$ are defined such that*

$$\|T - \lambda T^*\| \leqslant r, \tag{2.12}$$

then:

$$0 \leqslant \|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \leqslant \frac{r^2}{|\lambda|^2}\|Tx\|^2 \tag{2.13}$$

for any $x \in H$, $\|x\| = 1$.

Proof. Consider the reverse of the quadratic Schwarz inequality obtained by the author in [4]

$$0 \leqslant \|a\|^2\|b\|^2 - |\langle a, b\rangle|^2 \leqslant \frac{1}{|\alpha|^2}\|a\|^2\|a - \alpha b\|^2, \tag{2.14}$$

provided $a, b \in H$ and $\alpha \in \mathbb{C}\setminus\{0\}$.

Choosing in (2.14) $a = Tx$, $\alpha = \lambda$, $b = T^*x$, we get

$$\begin{aligned} \|Tx\|^4 &\leqslant \left|\left\langle T^2x, x\right\rangle\right|^2 + \frac{1}{|\lambda|^2}\|Tx\|^2\|Tx - \lambda T^*x\|^2 \\ &\leqslant \left|\left\langle T^2x, x\right\rangle\right|^2 + \frac{1}{|\lambda|^2}r^2\|Tx\|^2, \end{aligned} \tag{2.15}$$

which implies the inequality (2.13). □

Lemma 2.1. *Let* $a, b \in H\setminus\{0\}$ *and* $\varepsilon \in \left(0, \frac{1}{2}\right]$. *If*

$$0 \leqslant 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leqslant \frac{\|a\|}{\|b\|} \leqslant 1 - \varepsilon + \sqrt{1 - 2\varepsilon}, \tag{2.16}$$

then

$$0 \leqslant \|a\|\|b\| - Re\langle a, b\rangle \leqslant \varepsilon\|a - b\|^2. \tag{2.17}$$

Theorem 2.6. *Let* $T : H \to H$ *be a normal operator. If* $\lambda \in \mathbb{C}$ *is defined such that*

$$0 \leqslant 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leqslant |\lambda| \leqslant 1 - \varepsilon + \sqrt{1 - 2\varepsilon}, \quad \varepsilon \in \left(0, \frac{1}{2}\right], \tag{2.18}$$

then

$$0 \leqslant \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \leqslant \frac{\varepsilon}{|\lambda|}\|Tx - \lambda T^*x\|^2 \tag{2.19}$$

for any $x \in H$, $\|x\| = 1$.

Proof. Utilising Lemma 2.1 for $a = \lambda T^*x$, $b = Tx$, $x \in H\setminus\ker(T)$, $\|x\| = 1$, we have

$$|\lambda|\|Tx\|^2 - |\lambda|\left|\left\langle T^2x, x\right\rangle\right| \leqslant \varepsilon\|Tx - \lambda T^*x\|^2. \tag{2.20}$$

For $x \in \ker(T)$, $\|x\| = 1$ the inequality (2.19) holds, and the proof follows. □

3. Inequalities for operator norm

The purpose of this section is to point out some norm inequalities for normal operators that can be naturally obtained from various vector inequalities in inner product spaces, such as the ones due to Hile, Goldstein-Ryff-Clarke, Dragomir-Sándor and the author.

Theorem 3.1. *Let $T : H \to H$ be a normal operator. If $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$, then*

$$\left\| T - |\lambda|^{v+1} T^* \right\| \leqslant \frac{1 - |\lambda|^{v+1}}{1 - |\lambda|} \| T - \lambda T^* \| , \tag{3.1}$$

for any $v > 0$.

Proof. Consider the Hile inequality [10]:

$$\| \|a\|^v a - \|b\|^v b \| \leqslant \frac{\|a\|^{v+1} - \|b\|^{v+1}}{\|a\| - \|b\|} \| a - b \| , \tag{3.2}$$

provided $v > 0$ and $\|a\| \neq \|b\|$.

If we choose in (3.2) $a = Tx$, $b = \lambda T^* x$, since T is normal, we have $\|a\| = \|Tx\|$, $\|b\| = |\lambda| \|Tx\|$ and by (3.2) we get

$$\|Tx\|^v \left\| Tx - |\lambda|^{v+1} T^* x \right\| \leqslant \|Tx\|^v \frac{\left(1 - |\lambda|^{v+1}\right)}{1 - |\lambda|} \| Tx - \lambda T^* x \| , \tag{3.3}$$

for any $x \in H \setminus \ker(T)$.

If $x \notin \ker(T)$, then from (3.3) we get

$$\left\| Tx - |\lambda|^{v+1} T^* x \right\| \leqslant \frac{1 - |\lambda|^{v+1}}{1 - |\lambda|} \| Tx - \lambda T^* x \| . \tag{3.4}$$

If $x \in \ker(T)$ and since $\ker(T) = \ker(T^*)$, T being normal, then the inequality (3.4) holds. Therefore, (3.4) holds for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get the desired inequality (3.1). □

Remark 1. *For $v = 1$, we get the inequality:*

$$\left\| T - |\lambda|^2 T^* \right\| \leqslant (1 + |\lambda|) \| T - \lambda T^* \| . \tag{3.5}$$

Utilising the second inequality due to Hile (see [10, Eq. (5.2)]):

$$\left\| \frac{a}{\|a\|^{v+2}} - \frac{b}{\|b\|^{v+2}} \right\| \leqslant \frac{\|a\|^{v+2} - \|b\|^{v+2}}{\|a\| - \|b\|} \cdot \frac{\|a - b\|}{\|a\|^{v+1} \cdot \|b\|^{v+1}},$$

for $a, b \in H$, $a, b \neq 0$ and $\|a\| \neq \|b\|$, and by similar arguments used in the proof of the above theorem, we can prove the following result:

Theorem 3.2. *Let $T : H \to H$ be a normal operator. If $\lambda \in \mathbb{C}$, $|\lambda| \neq 0, 1$, then*

$$\left\| T - \frac{\lambda}{|\lambda|^{v+2}} T^* \right\| \leqslant \frac{1 - |\lambda|^{v+1}}{(1 - |\lambda|)\, |\lambda|^{v+1}} \left\| T - \lambda T^* \right\|, \tag{3.6}$$

where $v > 0$.

Theorem 3.3. *Let $T : H \to H$ be a normal operator. If $|\lambda| \leqslant 1$, then*

$$(1 - |\lambda|^\rho)^2 \|T\|^2 \leqslant \begin{cases} \rho^2 \|T - \lambda T^*\|^2 & \text{if } \rho \geqslant 1, \\ |\lambda|^{2\rho-2} \|T - \lambda T^*\|^2 & \text{if } \rho < 1. \end{cases} \tag{3.7}$$

Proof. We use the following inequality due to Goldstein, Ryff and Clarke [7]

$$\begin{aligned} &\|a\|^{2\rho} + \|b\|^{2\rho} - 2\|a\|^{\rho-1}\|b\|^{\rho-1} \operatorname{Re}\langle a, b\rangle \\ &\qquad \leqslant \begin{cases} \rho^2 \|a\|^{2\rho-2} \|a - b\|^2 & \text{if } \rho \geqslant 1, \\ \|b\|^{2\rho-2} \|a - b\|^2 & \text{if } \rho < 1, \end{cases} \end{aligned} \tag{3.8}$$

provided $\rho \in \mathbb{R}$ and $a, b \in H$ with $\|a\| \geqslant \|b\|$.

Since $\operatorname{Re}\langle a, b\rangle \leqslant |\langle a, b\rangle|$, then from (3.8), we have

$$\begin{aligned} \|a\|^{2\rho} + \|b\|^{2\rho} &\leqslant 2\|a\|^{\rho-1}\|b\|^{\rho-1} |\langle a, b\rangle| \\ &\qquad + \begin{cases} \rho^2 \|a\|^{2\rho-2} \|a - b\|^2 & \text{if } \rho \geqslant 1, \\ \|b\|^{2\rho-2} \|a - b\|^2 & \text{if } \rho < 1. \end{cases} \end{aligned} \tag{3.9}$$

We choose $a = Tx$, $b = \lambda T^* x$ and since $|\lambda| \leqslant 1$, we have $\|a\| \geqslant \|b\|$. From (3.9), taking into account the fact that $\|Tx\| = \|T^* x\|$, we deduce

$$\begin{aligned} \|Tx\|^{2\rho} + |\lambda|^{2\rho} \|Tx\|^{2\rho} &\leqslant 2\|Tx\|^{2\rho-2} |\lambda|^\rho \left|\left\langle T^2 x, x\right\rangle\right| \\ &\qquad + \begin{cases} \rho^2 \|Tx\|^{2\rho-2} \|Tx - \lambda T^* x\|^2 & \text{if } \rho \geqslant 1, \\ |\lambda|^{2\rho-2} \|Tx\|^{2\rho-2} \|Tx - \lambda T^* x\|^2 & \text{if } \rho < 1, \end{cases} \end{aligned}$$

which implies that

$$\begin{aligned} &\left(1 + |\lambda|^{2\rho}\right) \|Tx\|^2 \\ &\qquad \leqslant 2|\lambda|^\rho \left|\left\langle T^2 x, x\right\rangle\right| + \begin{cases} \rho^2 \|Tx - \lambda T^* x\|^2 & \text{if } \rho \geqslant 1, \\ |\lambda|^{2\rho-2} \|Tx - \lambda T^* x\|^2 & \text{if } \rho < 1, \end{cases} \end{aligned} \tag{3.10}$$

for any $x \in H$, $\|x\| = 1$.

This inequality is of interest in itself.

Taking the supremum over $x \in H$, $\|x\| = 1$, and using the fact that

$$\sup_{\|x\|=1} \left|\left\langle T^2 x, x\right\rangle\right| = w\left(T^2\right) = \|T\|^2,$$

we get the desired inequality (3.7). □

Remark 2. *If $|\lambda| > 1$, choosing in (3.9) $a = \lambda T^* x$, $b = Tx$ we get*

$$\left(|\lambda|^{2\rho} + 1\right) \|Tx\|^2 \leqslant 2 |\lambda|^{\rho} \left|\left\langle T^2 x, x\right\rangle\right| + \begin{cases} \rho^2 |\lambda|^{2\rho-2} \|Tx - \lambda T^* x\|^2 & \text{if } \rho \geqslant 1, \\ \|Tx - \lambda T^* x\|^2 & \text{if } \rho < 1, \end{cases}$$

which implies the "dual" inequality:

$$(1 - |\lambda|^{\rho})^2 \|T\|^2 \leqslant \begin{cases} \rho^2 |\lambda|^{2\rho-2} \|T - \lambda T^*\|^2 & \text{if } \rho \geqslant 1, \\ \|T - \lambda T^*\|^2 & \text{if } \rho < 1, \end{cases} \tag{3.11}$$

for any $\lambda \in \mathbb{C}$, $|\lambda| > 1$.

The following result concerning operator norm inequalities may be stated as follows.

Theorem 3.4. *Let $T : H \to H$ be a normal operator on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$\|T\|^p \left[(|\alpha| + |\beta|)^p + ||\alpha| - |\beta||^p\right] \leqslant \|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \tag{3.12}$$

if $p \in (1, 2)$ and

$$\|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \geqslant 2 \left(|\alpha|^p + |\beta|^p\right) \|T\|^p \tag{3.13}$$

if $p \geqslant 2$.

Proof. We use the following inequality obtained by Dragomir and Sándor [5]:

$$\|a + b\|^p + \|a - b\|^p \geqslant (\|a\| + \|b\|)^p + |\|a\| - \|b\||^p \tag{3.14}$$

if $p \in (1, 2)$ and

$$\|a + b\|^p + \|a - b\|^p \geqslant 2 \left(\|a\|^p + \|b\|^p\right) \tag{3.15}$$

if $p \geqslant 2$, where a, b are arbitrary vectors in the inner product space $(H; \langle \cdot, \cdot \rangle)$.

We choose $a = \alpha Tx$, $b = \beta T^* x$ to get

$$\begin{aligned} &\|(\alpha T + \beta T^*)(x)\|^p + \|(\alpha T - \beta T^*)(x)\|^p \\ &\geqslant (|\alpha| + |\beta|)^p \|Tx\|^p + ||\alpha| - |\beta||^p \|Tx\|^p \\ &= \left[(|\alpha| + |\beta|)^p + ||\alpha| - |\beta||^p\right] \|Tx\|^p \end{aligned} \tag{3.16}$$

if $p \in (1, 2)$ and

$$\|(\alpha T + \beta T^*)(x)\|^p + \|(\alpha T - \beta T^*)(x)\|^p \geqslant 2 \left(|\alpha|^p + |\beta|^p\right) \|Tx\|^p \tag{3.17}$$

if $p \geqslant 2$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we deduce (3.12) and (3.13). $\square$

Remark 3. *The case $p = 2$ gives the following inequality:*

$$\|\alpha T + \beta T^*\|^2 + \|\alpha T - \beta T^*\|^2 \geqslant 2\left(|\alpha|^2 + |\beta|^2\right)\|T\|^2,$$

that can also be obtained by utilising the parallelogram identity.

The following general result may be stated as follows.

Theorem 3.5. *Let $T : H \to H$ be a normal operator. If $\alpha, \beta \in \mathbb{C}$ and $r, \rho > 0$ are given such that*

$$\|T - \bar{\alpha} I\| \leqslant r \quad \text{and} \quad \|T^* - \beta I\| \leqslant \rho, \tag{3.18}$$

then

$$\|T\|^2 + \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) \leqslant \frac{1}{2}\left(r^2 + \rho^2\right) + \|\alpha T + \beta T^*\|. \tag{3.19}$$

Proof. The condition (3.18) obviously implies that

$$\|Tx\|^2 + |\alpha|^2 \leqslant 2\mathrm{Re}\,\langle(\alpha T)\,x, x\rangle + r^2, \tag{3.20}$$

and

$$\|T^* x\|^2 + |\beta|^2 \leqslant 2\mathrm{Re}\,\langle(\beta T)^*\,x, x\rangle + \rho^2, \tag{3.21}$$

for any $x \in H$, $\|x\| = 1$.

Adding (3.20) and (3.21) and taking into account the fact that $\|Tx\| = \|T^* x\|$, we obtain

$$\begin{aligned} 2\|Tx\|^2 + |\alpha|^2 + |\beta|^2 &\leqslant 2\mathrm{Re}\,\langle(\alpha T + \beta T^*)\,x, x\rangle + r^2 + \rho^2 \\ &\leqslant 2\,|\langle(\alpha T + \beta T^*)\,x, x\rangle| + r^2 + \rho^2. \end{aligned} \tag{3.22}$$

Taking the supremum on (3.22) over $x \in H$, $\|x\| = 1$, and utilising the fact that for the normal operator T

$$w(\alpha T + \beta T^*) = \|\alpha T + \beta T^*\|$$

we get the desired inequality (3.19). $\square$

Remark 4. *If $\alpha, \beta \in \mathbb{C}$ and $r, \rho > 0$ are given such that $|\alpha|^2 + |\beta|^2 = \rho^2 + r^2$, then from (3.19) we have*

$$\|T\|^2 \leqslant \|\alpha T + \beta T^*\|. \tag{3.23}$$

References

[1] M. L. Buzano, *Generalizzatione della disiguaglianza di Cauchy-Schwaz* (Italian), Rend. Sem. Mat. Univ. e Politech. Torino **31** (1971/73) (1974), 405-409 .

[2] S. S. Dragomir, *Some refinements of Schwarz inequality*, Simposional de Matematică şi Aplicaţii, Polytechnical Institute Timişoara, Romania, 1-2 Nov., 1985, 13-16.

[3] S. S. Dragomir, *A potpourri of Schwarz related inequalities in inner product spaces (I)*, J. Ineq. Pure Appl. Math. **6**(3) (2005), Art. 59.
[Online http://jipam.vu.edu.au/article.php?sid=532].

[4] S. S. Dragomir, *A potpourri of Schwarz related inequalities in inner product spaces (II)*, RGMIA Res. Rep. Coll. **8** (Supp) (2005), Art. 5.
[Online http://rgmia.vu.edu.au/v8(E).html].

[5] S. S. Dragomir and J. Sándor, *Some inequalities in prehilbertian spaces*, Studia Univ. "Babeş-Bolyai"- Mathematica **32** (1) (1987), 71-78.

[6] C. F. Dunkl and K. S. Williams, *A simple norm inequality*, Amer. Math. Monthly **71** (1) (1964), 43-44.

[7] A. A. Goldstein, J. V. Ryff and L. E. Clarke, *Problem 5473*, Amer. Math. Monthly **75** (3) (1968), 309.

[8] K. E. Gustafson and D. K. M. Rao, *Numerical Range*, Springer-Verlag, New York, Inc., 1997.

[9] P. R. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity,* Chelsea Pub. Comp, New York, 1972.

[10] G. H. Hile, *Entire solutions of linear elliptic equations with Laplacian principal part*, Pacific J. Math. **62** (1) (1976), 127-140.

School of Computer Science and Mathematics
Victoria University of Technology
PO Box 14428, Melbourne City, Victoria 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au