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INEQUALITIES FOR MULTILINEAR LITTLEWOOD-PALEY OPERATORS ON CERTAIN HARDY SPACES

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Abstract. In this paper, the boundedness for the multilinear Littlewood-Paley operators on certain Hardy and Herz-Hardy spaces are obtained.

1. Introduction and definitions

Let ψ be a function on R^n which satisfies the following properties:

(1) $\int \psi(x)dx = 0$,
(2) $|\psi(x)| \leqslant C(1+|x|)^{-(n+1)}$,
(3) $|\psi(x+y)-\psi(x)| \leqslant C|y|(1+|x|)^{-(n+2)}$ when $2|y| < |x|$.

Let m be a positive integer and A a function on R^n. The multilinear Littlewood-Paley operator is defined by

$$g_{\mu,*}^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}, \quad \mu > 1,$$

where

$$F_t^A(f)(x,y) = \int_{R^n} \frac{f(z)\psi_t(y-z)}{|x-z|^m} R_{m+1}(A;x,z)dz,$$

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\leqslant m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. We also define

$$g_\mu^*(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |f*\psi_t(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood-Paley operator (see [15]).

Note that when $m = 0$, $g_{\mu,*}^A$ is just the commutator of Littlewood-Paley operator (see [1]). It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-6]). The main purpose of this paper is to consider the continuity of the multilinear

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Littlewood-Paley operators on certain Hardy and Herz-Hardy spaces. Let us first introduce some definitions (see [7], [8] [9], [10], [11], [12]).

Definition 1.1. Let A be a function on R^n, m be a positive integer and $0 < p \leqslant 1$. A bounded measurable function a on R^n is said to be a $(p, D^m A)$ atom if

i) $\operatorname{supp} a \subset B = B(x_0, r)$,
ii) $||a||_{L^\infty} \leqslant |B|^{-1/p}$,
iii) $\int a(y)dy = \int a(y) D^\alpha A(y) dy = 0, |\alpha| = m$.

A tempered distribution f is said to belong to $H^p_{D^m A}(R^n)$ if in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x),$$

where a_j's are $(p, D^m A)$ atoms, $\lambda_j \in C$ and $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. Moreover,

$$||f||_{H^p_{D^m A}} \sim \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p}.$$

Let $B_k = \{x \in R^n : |x| \leqslant 2^k\}$, $C_k = B_k \setminus B_{k-1}, k \in Z$, and $m_k(\lambda, f) = |\{x \in C_k : |f(x)| > \lambda\}|$.

For $k \in N$, let $\tilde{m}_k(\lambda, f) = m_k(\lambda, f)$ and $\tilde{m}_0(\lambda, f) = |\{x \in B_0 : |f(x)| > \lambda\}|$. Denote $\chi_k = \chi_{C_k}$ for $k \in Z$ and $\chi_0 = \chi_{B_0}$, where χ_E is the characteristic function of the set E.

Definition 1.2. Let $0 < p$, $q < \infty$, $\alpha \in R$.

(1) The homogeneous Herz space is defined by

$$\dot{K}^{\alpha,p}_q(R^n) = \{f \in L^q_{loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}^{\alpha,p}_q} < \infty\},$$

where

$$||f||_{\dot{K}^{\alpha,p}_q} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} \right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K^{\alpha,p}_q(R^n) = \{f \in L^q_{loc}(R^n) : ||f||_{K^{\alpha,p}_q} < \infty\},$$

where

$$||f||_{K^{\alpha,p}_q} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q} \right]^{1/p}.$$

Definition 1.3. Let m be a positive integer and A be a function on R^n, $\alpha \in R$, $0 < p < \infty$, $1 < q \leqslant \infty$. A function $a(x)$ on R^n is called a central $(\alpha, q, D^m A)$-atom (or a central $(a, q, D^m A)$-atom of restrict type) if

1) Supp $a \subset B(0, r)$ for some $r > 0$ (or for some $r \geqslant 1$),
2) $||a||_{L^q} \leqslant |B(0, r)|^{-\alpha/n}$,

3) $\int a(x)dx = \int a(x)D^\beta A(x)dx = 0$, $|\beta| = m$.

A tempered distribution f is said to belong to $H\dot{K}^{\alpha,p}_{q,D^mA}(R^n)$ (or $HK^{\alpha,p}_{q,D^mA}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, where a_j is a central (α, q, D^mA)-atom (or a central (α, q, D^mA)-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ (or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$). Moreover,

$$||f||_{H\dot{K}^{\alpha,p}_{q,D^mA}} \text{ (or } ||f||_{HK^{\alpha,p}_{q,D^mA}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

2. Theorems and proofs

We begin with some preliminary lemmas.

Lemma 2.1. ([4]) *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leqslant C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2.2. *Let $1 < p < \infty$ and $D^\alpha A \in L^r(R^n)$, $|\alpha| = m$, $1 < r \leqslant \infty$, $1/q = 1/p + 1/r$. Then $g^A_{\mu,*}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is*

$$||g^A_{\mu,*}(f)||_{L^q} \leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{L^r} ||f||_{L^p}.$$

Proof. By the Minkowski inequality and the condition on ψ, we have

$$\begin{aligned}
&g^A_{\mu,*}(f)(x)\\
&\leqslant \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^m}\left(\int_{R^{n+1}_+} |\psi_t(y-z)|^2 \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{dydt}{t^{1+n}}\right)^{1/2} dz\\
&\leqslant C\int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^m}\\
&\quad\times \left(\int_0^\infty \int_{R^n} \frac{t^{-2n}}{(1+|y-z|/t)^{2n+2}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{dydt}{t^{1+n}}\right)^{1/2} dz\\
&\leqslant C\int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^m}\\
&\quad\times \left[\int_0^\infty \left(t^{-n}\int_{R^n} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{dy}{(t+|y-z|)^{2n+2}}\right) tdt\right]^{1/2} dz.
\end{aligned}$$

Noting that

$$t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y-z|)^{2n+2}}\leqslant CM\left(\frac{1}{(t+|x-z|)^{2n+2}}\right)$$
$$\leqslant C\frac{1}{(t+|x-z|)^{2n+2}}$$

(where Mg denotes the Hardy-Littlewood maximal function of g) and

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2}}=C|x-z|^{-2n},$$

we obtain

$$g_{\mu,*}^A(f)(x)\leqslant C\int_{R^n}\frac{|f(z)|}{|x-z|^m}|R_{m+1}(A;x,z)|\left(\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+2}}\right)^{1/2}dz$$
$$=C\int_{R^n}\frac{|f(z)|}{|x-z|^{m+n}}|R_{m+1}(A;x,z)|dz.$$

Thus, the lemma follows from [5], [6]. □

Theorem 2.1. *Let $1\geqslant p>n/(n+1)$ $D^\beta A\in BMO(R^n)$ for $|\beta|=m$. Then $g_{\mu,*}^A$ is bounded from $H^p_{D^mA}(R^n)$ to $L^p(R^n)$.*

Proof. It suffices to show that there exists a constant $c>0$ such that for every (p,D^mA) atom a,

$$||g_{\mu,*}^A(a)||_{L^p}\leqslant C.$$

Let a be a (p,D^mA) atom supported on a ball $B=B(x_0,r)$. We write

$$\int_{R^n}[g_{\mu,*}^A(a)(x)]^pdx=\int_{|x-x_0|\leqslant 2r}[g_{\mu,*}^A(a)(x)]^pdx+\int_{|x-x_0|>2r}[g_{\mu,*}^A(a)(x)]^pdx$$
$$=I+II.$$

For I, taking $q>1$, by Hölder's inequality and the L^q-boundedness of $g_{\mu,*}^A$ (see Lemma 2.2), we see that

$$I\leqslant C||g_{\mu,*}^A(a)||_{L^q}^p\cdot|B(x_0,2r)|^{1-p/q}\leqslant C||a||_{L^q}^p|B|^{1-p/q}\leqslant C.$$

To obtain the estimate of II, we need to estimate $g_{\mu,*}^A(a)(x)$ for $x\in(2B)^c$.
Let $\tilde B=5\sqrt{n}B$ and $\tilde A(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde B}\cdot x^\alpha$, where $(A)_B$ are the mean values of A on B. Then $R_m(A;x,y)=R_m(\tilde A;x,y)$. We write, by the vanishing moment of a,

$$F_t^A(a)(x,y)=\int_B\left[\frac{\psi_t(y-z)}{|x-z|^m}-\frac{\psi_t(y-x_0)}{|y-x_0|^m}\right]R_m(\tilde A;x,z)a(z)dz$$
$$+\int_B\frac{\psi_t(y-x_0)}{|y-x_0|^m}[R_m(\tilde A;x,z)-R_m(\tilde A;x,x_0)]a(y)dy$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_B\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m}(D^\alpha A(z)-(D^\alpha A)_B)a(z)dz,$$

thus

$$
\begin{aligned}
&g_{\mu,*}^{A}(a)(x) \\
\leqslant\ & \left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\right. \\
&\left.\times\left(\int_B\left|\frac{\psi_t(y-z)}{|x-z|^m}-\frac{\psi_t(y-x_0)}{|x-x_0|^m}\right||R_m(\tilde{A};x,z)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2} \\
&+\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\right. \\
&\left.\times\left(\int_B\frac{|\psi_t(y-x_0)|}{|x-x_0|^m}|R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2} \\
&+\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\right. \\
&\left.\times\left|\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_B\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m}(D^\alpha A(z)-(D^\alpha A)_B)a(z)dz\right|^2\frac{dydt}{t^{n+1}}\right]^{1/2} \\
\equiv\ & II_1+II_2+II_3.
\end{aligned}
$$

By Lemma 2.1, for $z\in B$ and $x\in 2^{k+1}B\setminus 2^kB$, we know

$$|R_m(\tilde{A};x,z)|\leqslant\ C|x-z|^m\sum_{|\alpha|=m}|D^\alpha A(x)-(D^\alpha A)_{2^kB}|.$$

By the condition on ψ and Minkowski's inequality and similar to the proof of Lemma 2.2, we note that $|x-z|\sim|x-x_0|$ for $z\in B$ and $x\in R^n\setminus B$ and we obtain

$$
\begin{aligned}
II_1\leqslant\ & C\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\right. \\
&\left.\times\left(\int_B\frac{|x_0-z|}{|x-x_0|^{m+1}}\frac{t}{(t+|y-x_0|)^{n+1}}|R_m(A;x,z)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2} \\
&+C\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\right. \\
&\left.\times\left(\int_B\frac{|x_0-z|}{|x-x_0|^{m}}\frac{t}{(t+|y-x_0|)^{n+2}}|R_m(A;x,z)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}
\end{aligned}
$$

$$\begin{aligned}
&\leqslant C\frac{|B|^{1/n-1/p}}{|x-x_0|^{m+1}}\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{t^{1-n}dydt}{(t+|y-x_0|)^{2n+2}}\right]^{1/2}\\
&\quad\times\left(\int_B|R_m(A;x,z)|dz\right)\\
&\quad+C\frac{|B|^{1/n-1/p}}{|x-x_0|^{m}}\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{t^{1-n}dydt}{(t+|y-x_0|)^{2n+4}}\right]^{1/2}\\
&\quad\times\left(\int_B|R_m(A;x,z)|dz\right)\\
&\leqslant C|x-x_0|^{-(m+n+1)}|B|^{1/n-1/p}\left(\int_B|R_m(A;x,z)|dz\right)\\
&\leqslant Ck|x-x_0|^{-n-1}|B|^{1/n-1/p+1}\sum_{|\alpha|=m}|D^\alpha A(x)-(D^\alpha A)_{2^{k+1}B}|.
\end{aligned}$$

On the other hand, by the formula (see [4])

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};z,x_0)(x-x_0)^\beta$$

and Lemma 2.1, we get

$$|R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)|\leqslant C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}||D^\alpha A||_{BMO},$$

so that

$$\begin{aligned}
II_2&\leqslant C\int_B|x-x_0|^{-(n+m)}\sum_{|\beta|<m}\left|R_{m-|\beta|}(D^\beta\tilde{A};z,x_0)\right||x-x_0|^{|\beta|}|a(z)|dz\\
&\leqslant C\int_B|x-x_0|^{-(n+m)}\sum_{|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|a(z)|dz\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\int_B\frac{|x_0-z|}{|x-x_0|^{n+1}}|a(z)|dz\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|x-x_0|^{-n-1}|B|^{1/n-1/p+1}.
\end{aligned}$$

For II_3 , we write

$$\begin{aligned}
&\int_B\frac{\psi_t(y-z)(x-z)^\alpha}{|x-y|^m}(D^\alpha A(z)-(D^\alpha A)_B)a(z)dz\\
&=\int_B\left[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m}-\frac{\psi_t(y-x_0)(y-x_0)^\alpha}{|x-x_0|^m}\right][D^\alpha A(z)-(D^\alpha A)_B]a(z)dz,
\end{aligned}$$

Similar to the estimate of II_1, we obtain

$$\begin{aligned} II_3 &\leqslant C \sum_{|\alpha|=m} |x-x_0|^{-(n+1)} \int_B |x_0 - z||D^\alpha A(z) - (D^\alpha A)_B||a(z)|dz \\ &\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} |B|^{1/n-1/p+1} |x-x_0|^{-n-1}. \end{aligned}$$

Therefore, recalling that $p > n/(n+1)$,

$$\begin{aligned} II &\leqslant \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} [g_{\mu,*}^A(a)(x)]^p dx \\ &\leqslant C \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} k^p |x-x_0|^{-p(n+1)} |B|^{p(1+1/n-1/p)} \\ &\quad \times \left(\sum_{|\alpha|=m} |D^\alpha A(x) - (D^\alpha A)_{2^{k+1}B}| \right)^p dx \\ &\quad + C \left(\sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \right)^p \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |x-x_0|^{-p(n+1)} |B|^{p(1+1/n-1/p)} dx \\ &\leqslant C \left(\sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \right)^p \sum_{k=1}^{\infty} k^p 2^{k(n-p-pn)} \\ &\leqslant C \left(\sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \right)^p, \end{aligned}$$

which together with the estimate for I yields the desired result. This finishes the proof of Theorem 2.1. □

Theorem 2.2. *Let* $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \leqslant \alpha < n(1-1/q)+1$ *and* $D^\beta A \in BMO(R^n)$ *for* $|\beta| = m$. *Then* $g_{\mu,*}^A$ *is bounded from* $H\dot{K}_{q,D^m A}^{\alpha,p}(R^n)$ *to* $\dot{K}_q^{\alpha,p}(R^n)$.

Proof. Let $f \in H\dot{K}_{q,D^m A}^{\alpha,p}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 1.3. We write

$$\begin{aligned} ||g_{\mu,*}^A(f)||_{\dot{K}_q^{\alpha,p}} &\leqslant C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \, ||g_{\mu,*}^A(a_j)\chi_k||_{L^q} \right)^p \right]^{1/p} \\ &\quad + C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \, ||g_{\mu,*}^A(a_j)\chi_k||_{L^q} \right)^p \right]^{1/p} = I + II. \end{aligned}$$

For II, by the boundedness of $g_{\mu,*}^A$ on $L^q(R^n)$ (see Lemma 2.2), we have

$$
\begin{aligned}
II &\leqslant C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\ ||a_j||_{L^q}\right)^p\right]^{1/p} \\
&\leqslant C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j| 2^{-j\alpha}\right)^p\right]^{1/p} \\
&\leqslant C\begin{cases}\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}, \quad 0<p\leqslant 1 \\ \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p/2}\right)\left(\sum_{j=k-2}^{\infty} 2^{-j\alpha p'/2}\right)^{p/p'}\right]^{1/p}, \ p>1\end{cases} \\
&\leqslant C\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\right)\right]^{1/p}, \quad 0<p\leqslant 1 \\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2}\right)\right]^{1/p}, \quad p>1\end{cases} \\
&\leqslant C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p} \leqslant C||f||_{H\dot{K}_{q,D^mA}^{\alpha,p}}.
\end{aligned}
$$

For I, similar to the proof of Theorem 2.1, we have, for $x\in C_k$, $j\leqslant k-3$,

$$
\begin{aligned}
g_{\mu,*}^A(a_j)(x) &\leqslant C|x-x_0|^{-n-m-1}|B_j|^{1/n}\left(\int_{B_j}|a_j(y)||R_m(\tilde{A};x,y)|dy\right) \\
&\quad + C\sum_{|\beta|=m}||D^\beta A||_{BMO}(k-j)|x-x_0|^{-n-1}|B_j|^{1/n}\int_{B_j}|a(y)|dy \\
&\leqslant C2^{-k(n+1)}2^{j(1+n(1-1/q)-\alpha)}\left(\sum_{|\beta|=m}|D^\beta A(x)-(D^\beta A)_{B_k}|\right) \\
&\quad + C\sum_{|\beta|=m}||D^\beta A||_{BMO}(k-j)2^{-k(n+1)}2^{j(1+n(1-1/q)-\alpha)},
\end{aligned}
$$

thus

$$
\begin{aligned}
I \leqslant C\Bigg[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Bigg(&\sum_{j=-\infty}^{k-3}|\lambda_j| 2^{-k(n+1)+j(1+n(1-1/q)-\alpha)} \\
&\times\sum_{|\beta|=m}\left(\int_{B_k}|D^\beta A(x)-(D^\beta A)_{B_k}|^q dx\right)^{1/q}\Bigg)^p\Bigg]^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&+C[\sum_{k=-\infty}^{\infty}2^{k\alpha p}(\sum_{j=-\infty}^{k-3}|\lambda_j|(k-j)2^{-k(n+1)+j(1+n(1-1/q)-\alpha)}2^{kn/q}\\
&\times\sum_{|\beta|=m}||D^\beta A||_{BMO})^p]^{1/p}\\
\equiv&I_1+I_2.
\end{aligned}
$$

To estimate I_1 and I_2, we consider two cases.

Case 1: $0<p\leqslant 1$.

$$
\begin{aligned}
I_1\leqslant&\ C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{[-k(n+1)+j(1+n(1-1/q)-\alpha)]p}2^{knp/q}\right.\\
&\left.\times\left(\sum_{|\beta|=m}||D^\beta A||_{BMO}\right)^p\right]^{1/p}\\
=&C\sum_{|\beta|=m}||D^\beta A||_{BMO}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)(1+n(1-1/q)-\alpha)p}\right]^{1/p}\\
\leqslant&\ C\sum_{|\beta|=m}||D^\beta A||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\leqslant\ C||f||_{H\dot{K}^{\alpha,p}_{q,D^\alpha A}},\\
I_2\leqslant&\ C\sum_{|\beta|=m}||D^\beta A||_{BMO}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}(k-j)^p2^{(j-k)(1+n(1-1/q)-\alpha)p}\right]^{1/p}\\
\leqslant&\ C\sum_{|\beta|=m}||D^\beta A||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\leqslant\ C||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}}.
\end{aligned}
$$

Case 2: $p>1$. By Hölder's inequality, we deduce that

$$
\begin{aligned}
I_1\leqslant&\ C\sum_{|\beta|=m}||D^\beta A||_{BMO}\Bigg[\sum_{j=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{(j-k)p(1(1-1/q)-\alpha)/2}\right)\\
&\left(\sum_{j=-\infty}^{k-3}2^{(j-k)p'(1+n(1-1/q)-\alpha)/2}\right)^{p/p'}\Bigg]^{1/p}\\
\leqslant&\ C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\leqslant\ C||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}},
\end{aligned}
$$

$$I_2 \leqslant C \sum_{|\beta|=m} ||D^\beta A||_{BMO} \Bigg[\sum_{j=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p (k-j)^p 2^{(j-k)p(1+n(1-1/q)-\alpha)/2} \right)$$

$$\left(\sum_{j=-\infty}^{k-3} 2^{(j-k)p'(1+n(1-1/q)-\alpha)/2} \right)^{p/p'} \Bigg]^{1/p}$$

$$\leqslant C \sum_{|\beta|=m} ||D^\beta A||_{BMO} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} (k-j)^p 2^{(j-k)p(1+n(1-1/q)-\alpha)/2} \right]^{1/p}$$

$$\leqslant C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leqslant C ||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}}.$$

This finishes the proof of Theorem 2.2. □

Remark. Theorem 2.2 also holds for nonhomogeneous Herz-type space.

Theorem 2.3. *Let $D^\beta A \in BMO(R^n)$ for $|\beta| = m$ and $0 < p \leqslant 1 \leqslant q < \infty$, $\alpha = n(1-1/q)+1$. Then, for any $\lambda > 0$ and $f \in HK^{\alpha,p}_{q,D^m A}(R^n)$, we have*

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, g^A_{\mu,*}(f))^{p/q} \right]^{1/p}$$

$$\leqslant C\lambda^{-1} ||f||_{HK^{\alpha,p}_{q,D^m A}(R^n)} \left(1 + \log^+(\lambda^{-1} ||f||_{HK^{\alpha,p}_{q,D^m A}(R^n)}) \right).$$

Proof. Let $f \in HK^{\alpha,p}_{q,D^m A}(R^n)$ and $f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 1.3. We write

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, g^A_{\mu,*}(f))^{p/q} \right]^{1/p} \leqslant C \left[\sum_{k=0}^{3} 2^{k\alpha p} \tilde{m}_k(\lambda, g^A_{\mu,*}(f))^{p/q} \right]^{1/p}$$

$$+ C \left[\sum_{k=4}^{\infty} 2^{k\alpha p} \tilde{m}_k \left(\lambda/2, \sum_{j=0}^{k-3} |\lambda_j| g^A_{\mu,*}(a_j) \right)^{p/q} \right]^{1/p}$$

$$+ C \left[\sum_{k=4}^{\infty} 2^{k\alpha p} \tilde{m}_k \left(\lambda/2, g^A_{\mu,*} \left(\sum_{j=k-2}^{\infty} \lambda_j a_j \right) \right)^{p/q} \right]^{1/p} = I_1 + I_2 + I_3.$$

For I_1, I_3, by the weak (q,q) type boundedness of $g_{\mu,*}^A$ and $0<p\leqslant 1$, we have

$$\begin{aligned}
I_1 &\leqslant C\lambda^{-1}\left[\sum_{k=0}^{3}2^{k\alpha p}||f||_{L^q}^p\right]^{1/p} \leqslant C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p||a_j||_{L^q}^p\right)^{1/p}\\
&\leqslant C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\cdot 2^{-j\alpha p}\right)^{1/p}\\
&\leqslant C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \leqslant C\lambda^{-1}||f||_{HK_{q,D^mA}^{\alpha,p}},\\
I_3 &\leqslant C\lambda^{-1}\left[\sum_{k=4}^{\infty}2^{k\alpha p}\left|\left|\sum_{j=k-2}^{\infty}\lambda_j a_j\right|\right|_{L^q}^p\right]^{1/p}\\
&\leqslant C\lambda^{-1}\left[\sum_{k=4}^{\infty}2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}\\
&\leqslant C\lambda^{-1}\left[\sum_{j=0}^{\infty}|\lambda_j|^p\sum_{k=0}^{j+2}2^{(k-j)\alpha p}\right]^{1/p}\\
&\leqslant C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\\
&\leqslant C\lambda^{-1}||f||_{HK_{q,D^mA}^{\alpha,p}}.
\end{aligned}$$

For I_2, by the argument of the proof of Theorems 2.1 and 2.2, we have

$$g_{\mu,*}^A(a_j)(x)\leqslant C2^{-k(n+1)}\left(\sum_{|\beta|=m}|D^\beta A(x)-(D^\beta A)_{B_k}|+k\sum_{|\beta|=m}||D^\beta A||_{BMO}\right).$$

Therefore

$$\begin{aligned}
I_2 \leqslant\ & C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-k(n+1)}\sum_{|\beta|=m}|D^\beta A(x)-(D^\beta A)_{B_k}|\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p}\\
&+C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-k(n+\varepsilon)}k\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p}\\
\equiv\ & I_2^{(1)}+I_2^{(2)}.
\end{aligned}$$

For $I_2^{(1)}$, using the John-Nirenberg inequality (see [15]), we gain

$$
\begin{aligned}
I_2^{(1)} \leqslant\; & C\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\left(\exp\left(-\frac{C2^{k(n+1)}\lambda}{\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)2^{kn}\right)^{p/q}\right]^{1/p} \\
\leqslant\; & C\left[\sum_{k=0}^{\infty} 2^{k(n+1)p}\exp\left(-\frac{C\lambda 2^{k(n+1)}}{\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)\right]^{1/p} \\
\leqslant\; & C\left[\int_0^{\infty} x^{p-1}\exp\left(-\frac{c\lambda x}{\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)dx\right]^{1/p} \\
=\; & C\lambda^{-1}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\left(\int_0^{\infty} t^{p-1}e^{-t}dt\right)^{1/p} \\
\leqslant\; & C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \leqslant\; C\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}}.
\end{aligned}
$$

For $I_2^{(2)}$, we use the following fact: If there exists $u > 1$ such that $2^x/x \leqslant u$ for $x \geqslant 3$, then $2^x \leqslant cu\log^+ u$. We have, if

$$
\left|\left\{x \in C_k : C2^{-k(n+1)}k\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j| > \lambda/4\right\}\right| \neq 0,
$$

then

$$
1 < 2^{k(n+1)/k(n+1)} < C\lambda^{-1}\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|.
$$

Thus

$$
2^{k(n+1)} \leqslant\; C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\log^+\left(\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\right).
$$

Let K_λ be the maximal integer k which satisfies this estimate. Then

$$
\begin{aligned}
I_2^{(2)} \leqslant\; & C\left(\sum_{k=4}^{K_\lambda} 2^{k\alpha p}2^{knp/q}\right)^{1/p} \leqslant\; C2^{K_\lambda(n+1)} \\
\leqslant\; & C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\log^+\left(\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\right) \\
\leqslant\; & C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\log^+\left(\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\right) \\
\leqslant\; & C\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}}\log^+\left(\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}}\right).
\end{aligned}
$$

Now, summing up the above estimates, we have

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p}\tilde{m}_k(\lambda, g_{\mu,*}^A(f))^{p/q}\right]^{1/p} \leqslant C\lambda^{-1}||f||_{H^{\alpha,p}_{q,D^mA}}\left(1+\log^+\left(\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}}\right)\right).$$

This completes the proof of Theorem 2.3. □

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