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ON THE PLANCHEREL THEOREM FOR THE OLEVSKII TRANSFORM

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ABSTRACT. We deal here with a class of integral transformations with respect to parameters of hypergeometric functions or the index transforms. In particular, we treat the familiar Olevskii transform, which is associated with the Gauss hypergeometric function as a kernel. It involves, in turn, as particular cases index transforms of the Mehler-Fock type which are used in the mathematical theory of elasticity. Boundedness L_2- properties for the Olevskii transform are investigated. The Plancherel theorem is proved. It shows that the Olevskii transform is an isometric isomorphism between two weighted L_2 - spaces. More examples of such isomorphisms are exhibited for the Mehler-Fock type transforms.

1. Introduction and preliminary results

Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a measurable function. Fixing real positive parameters c, a we will deal with the following Olevskii transformation [15], [22], [23]

$$(1.1)\quad \mathcal{O}_{c,a} f(x) = \frac{x^{-a}}{\Gamma(c)} \int_0^\infty |\Gamma(a+i\tau)|^2 {}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) f(\tau) d\tau,\ x > 0,$$

where the integral in (1.1) is with respect to parameters of the Gauss hypergeometric function ${}_2F_1$ [1, Chapter 2]. It exists in a definite sense, which will be defined below. In the sequel we will use the weighted Lebesgue spaces $L_p(\Omega; \omega(x)dx)$ with respect to the measure $\omega(x)dx$ equipped with the norm

$$||f||_p = \left(\int_\Omega |f(x)|^p \omega(x) dx\right)^{1/p},\ 1 \leqslant\ p < \infty,$$

$$||f||_\infty = \text{ess sup}|f(x)|.$$

We note that $\Gamma(z)$ in (1.1) is Euler's Gamma-function [1] and i is the imaginary unit. The operator (1.1) is called also the Jacobi transform, the Fourier-Jacobi transform, the generalized Fourier transform, the index hypergeometric transform, the ${}_2F_1$ - index transform (see [2], [7], [8], [11], [13], [14], [24]). It is not

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difficult to verify that under conditions on the parameters, the Gauss hypergeometric function in (1.1) is represented by the power series for $x \geqslant 1, \tau \in \mathbb{R}_+$

$$ {}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) = \sum_{n=0}^{\infty} \frac{(a+i\tau)_n (a-i\tau)_n}{(c)_n} \frac{(-1)^n}{x^n n!}. \tag{1.2} $$

When $0 < x < 1$ this function is understood by the following relation (cf. [1], [12])

$$ \begin{aligned} &{}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) \\ =&\frac{\Gamma(c)\Gamma(-2i\tau)}{\Gamma(a-i\tau)\Gamma(c-a-i\tau)} x^{a+i\tau} {}_2F_1(a+i\tau, 1-c+a+i\tau; 1+2i\tau; -x) \\ +&\frac{\Gamma(c)\Gamma(2i\tau)}{\Gamma(a+i\tau)\Gamma(c-a+i\tau)} x^{a-i\tau} {}_2F_1(a-i\tau, 1-c+a-i\tau; 1-2i\tau; -x). \end{aligned} \tag{1.3} $$

On the other hand, we consider the Gauss function as the following Mellin-Barnes integral [1, Ch. I] (cf. formula (8.4.50.2) from [17])

$$ \begin{aligned} &\frac{|\Gamma(a+i\tau)|^2}{\Gamma(c)} x^{-a} {}_2F_1(a+i\tau, a-i\tau; c; -1x) \\ =&\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+i\tau)\Gamma(s-i\tau) \frac{\Gamma(a-s)}{\Gamma(c-a+s)} x^{-s} ds, \ x>0, \ 0<\gamma<a. \end{aligned} \tag{1.4} $$

Series (1.2) can be reobtained if we evaluate integral (1.4) as the sum of residues of the right-hand simple poles $s = a+n, \ n = 0, 1, 2, ...$ of Gamma-functions of the integrand, which are separated from the left-hand ones $s = \pm i\tau - n, \ n = 0, 1, 2, ...$. However, evaluating the same integral as the sum of residues at the left-hand simple poles we obtain series (1.3). We put down here some of important properties of the Gauss function [1], [20], [23]

$$ {}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z), $$

$$ {}_2F_1(a, b; b; z) = (1-z)^{-a}, $$

$$ {}_2F_1(a, b; c; 0) = {}_2F_1(0, b; c; z) = 1, $$

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0, $$

$$ {}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{1.5} $$

$$ {}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z). \tag{1.6} $$

Formula (1.5) is called the Boltz formula and relation (1.6) is called the self-transformation formula.

One can mention also the integral representation of the Gauss function in terms of the product of Bessel functions (see [16, relation (2.16.21.1)], [23, formula (1.101)])
(1.7)

$$ {}_2F_1\left(a+i\tau, a-i\tau; c; -x^2\right) = \frac{2^{1-2a+c}x^{1-c}\Gamma(c)}{|\Gamma(a+i\tau)|^2}\int_0^\infty y^{2a-c}J_{c-1}(xy)K_{2i\tau}(y)dy,\ x>0. $$

It is easily seen by the asymptotic behavior of the Bessel functions near origin and at the infinity (cf. [12]) that integral (1.7) absolutely converges for any $c, a > 0$. We recall that the Gauss function in (1.1) has the following asymptotic behavior for each $\tau \in \mathbb{R}_+$, when $x \to 0+$ (cf. [1], [12], [23])

$$ {}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) = O\left(x^a \log x\right),\ x \to 0+\,. \tag{1.8} $$

We note that the kernel (1.8) is a continuous function with respect to $\tau > 0$. Furthermore, via [23, Theorem 1.12] we see that when $\tau \to +\infty$ it behaves for each $x > 0$ as

$$ {}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) = O\left(\tau^{1/2-c}\right),\ \tau \to +\infty. \tag{1.9} $$

We mention here that the modified Bessel function $K_{2i\tau}(2\sqrt{x})$ is real-valued and it represents the kernel of the Kontorovich-Lebedev transform [18], [19], [22], [23]

$$ [KLf](x) = \int_0^\infty K_{2i\tau}(2\sqrt{x})f(\tau)d\tau. \tag{1.10} $$

At the same time it can be given by the Mellin-Barnes integral (see [23, relation (1.113)])

$$ K_{2i\tau}(2\sqrt{x}) = \frac{1}{4\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(s+i\tau\right)\Gamma\left(s-i\tau\right)x^{-s}ds, \tag{1.11} $$

where $x > 0, \gamma > 0, \tau \in \mathbb{R}$. As it is known [12], [17], theory of the Mellin - Barnes integrals is based on the Mellin direct and inverse transforms, which are defined by the formulas

$$ f^{\mathcal{M}}(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{1.12} $$

$$ f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^{\mathcal{M}}(s)x^{-s}ds,\ s=\gamma+it,\ x>0, \tag{1.13} $$

where integrals (1.12)- (1.13) exist as Lebesgue integrals or, in particular, in mean with respect to the norm of spaces $L_2(\gamma - i\infty, \gamma + i\infty)$ and $L_2(\mathbb{R}_+; x^{2\gamma-1})$, respectively. In the latter case, the Parseval equality holds

$$\int_0^\infty |f(x)|^2 x^{2\gamma-1}dx = \frac{1}{2\pi}\int_{-\infty}^\infty |f^{\mathcal{M}}(\gamma+it)|^2 dt. \tag{1.14}$$

The Kontorovich-Lebedev transformation (1.10) (cf. [23], [24]), in turn, is an isomorphism between the spaces $L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2d\tau)$ and $L_2(\mathbb{R}_+;x^{-1}dx)$ with the Parseval equality of the form

$$\int_0^\infty |[KLf](x)|^2\frac{dx}{x} = \frac{\pi}{2}\int_0^\infty |f(\tau)|^2|\Gamma(2i\tau)|^2 d\tau. \tag{1.15}$$

The corresponding inverse operator in the mean convergence sense is written in the form

$$f(\tau) = \frac{2}{\pi|\Gamma(2i\tau)|^2}\int_0^\infty K_{2i\tau}(2\sqrt{x})[KLf](x)\frac{dx}{x}. \tag{1.16}$$

The aim of this paper is to prove the Plancherel type theorem for the Olevskii transformation (1.1) and its particular cases. We note that the case $c = 2a$ was considered in [22, p. 136]. The Olevskii transformation for some particular values of c has been treated also in [13], [14], [28]. About the distributional analog of the Olevskii transform see in [6], [9], [10]. Some mapping properties for these index operators have been investigated in [3], [4], [5], [21]. Finally, we will exhibit the related results for the Mehler-Fock type transforms (see also in [24], [26]).

2. The plancherel theorem

We have

Theorem. *Let $c > a > 0$. The Olevskii transformation* (1.1) *is the isomorphism*

$$\mathcal{O}_{c,a} : L_2\left(\mathbb{R}_+;\left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 d\tau\right) \leftrightarrow L_2\left(\mathbb{R}_+;(1+x)^{2a-c}\frac{dx}{x}\right), \tag{2.1}$$

where integral (1.1) *converges in mean with respect to the norm in*

$$L_2\left(\mathbb{R}_+;(1+x)^{2a-c}\frac{dx}{x}\right).$$

The inverse operator is given by the formula

$$\begin{aligned} f(\tau) = &\underset{N\to\infty}{\text{l.i.m.}}\frac{|\Gamma(c-a+i\tau)|^2}{2\pi\Gamma(c)|\Gamma(2i\tau)|^2} \\ &\times\int_{1/N}^N (1+x)^{2a-c}x^{-a-1}{}_2F_1\left(a+i\tau,a-i\tau;c;-\frac{1}{x}\right)\mathcal{O}_{c,a}f(x)dx, \end{aligned} \tag{2.2}$$

where the limit is in mean square with respect to the norm in the space

$$L_2\left(\mathbb{R}_+; \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 d\tau\right).$$

Besides, if $f, g \in L_2\left(\mathbb{R}_+; \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 d\tau\right)$ *then the Plancherel formula holds*

$$\int_0^\infty \mathcal{O}_{c,a}f(x)\overline{\mathcal{O}_{c,a}g(x)}(1+x)^{2a-c}\frac{dx}{x} = 2\pi\int_0^\infty \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 f(\tau)\overline{g(\tau)}d\tau \tag{2.3}$$

with the Parseval equality

$$\int_0^\infty \left|\mathcal{O}_{c,a}f(x)\right|^2(1+x)^{2a-c}\frac{dx}{x} = 2\pi\int_0^\infty \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 |f(\tau)|^2 d\tau. \tag{2.4}$$

Proof. Let $f \in C_0^\infty(\mathbb{R}_+)$. Then we use integral representation (1.4) to substitute it in (1.1) and to invert the order of integration via Fubini's theorem. This is indeed possible due to the absolute and uniform convergence of the integral (1.4) with respect to $\tau \in \mathbb{R}_+$. Denoting

$$\Phi_f(z) = \int_0^\infty \Gamma(z+i\tau)\Gamma(z-i\tau)f(\tau)d\tau, \tag{2.5}$$

we arrive then at the representation

$$\mathcal{O}_{c,a}f(x) = 12\pi\int_{-\infty}^\infty \Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha-iy)}{\Gamma(c-a+\alpha+iy)}x^{-\alpha-iy}dy, x>0,\ 0<\alpha<a. \tag{2.6}$$

On the other hand, employing the self-transformation formula (1.6) for the Gauss function we represent the Olevskii transform in the form

$$\mathcal{O}_{c,a}f(x) = \frac{x^{a-c}(1+x)^{c-2a}}{\Gamma(c)}\int_0^\infty |\Gamma(a+i\tau)|^2 {}_2F_1\left(-a+i\tau, c-a-i\tau; c; -\frac{1}{x}\right)f(\tau)d\tau,$$

which gives the following operational relation

$$\mathcal{O}_{c,a}f(x) = (1+x)^{c-2a}\mathcal{O}_{c,c-a}h(x), \tag{2.7}$$

where $h(\tau) = \left|\frac{\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 f(\tau)$. Hence, taking into account (2.6), (2.7), as the consequence of the Parseval equality for the Mellin transform (1.14) with the parallelogram identity, we obtain

$$\int_0^\infty \left|\mathcal{O}_{c,a}f(x)\right|^2(1+x)^{2a-c}x^{2\alpha-1}dx = \int_0^\infty \mathcal{O}_{c,c-a}h(x)\overline{\mathcal{O}_{c,a}f(x)}x^{2\alpha-1}dx$$

$$(2.8) \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_h(\alpha+iy)\overline{\Phi_f(\alpha+iy)}\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)}dy.$$

Equality (2.6) yields (see (1.13)) that

$$\mathcal{O}_{c,a}f(x) \leftrightarrow \Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha-iy)}{\Gamma(c-a+\alpha+iy)},$$

$$\mathcal{O}_{c,c-a}h\,(x) \leftrightarrow \Phi_h(\alpha+iy)\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)},$$

where $0<\alpha\leqslant\ b<\min(a,c-a)$ are Mellin's L_2-pairs and all integrals in (2.8) are finite. In fact, we will show that for any $f\in C_0^{\infty}(\mathbb{R}_+)$

$$(2.9) \qquad \sup_{0<\alpha\leqslant b}\int_{-\infty}^{\infty}\left|\Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|^2 dy<\infty,$$

$$(2.10) \qquad \sup_{0<\alpha\leqslant b}\int_{-\infty}^{\infty}\left|\Phi_h(\alpha+iy)\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)}\right|^2 dy<\infty.$$

Then, since the integrands in (2.9), (2.10) are analytic in the strip $0<\alpha<\min(a,c-a)$ we will get immediately that each one belongs to the Hardy space $\mathbb{H}_2^{(0,b]}$ (cf. [19]). Thus, almost everywhere one admits the limit L_2- values, which are equal correspondingly to

$$(2.11) \qquad [\mathcal{G}f](y)\frac{\Gamma(a+iy)}{\Gamma(c-a-iy)},$$

and

$$(2.12) \qquad [\mathcal{G}h](y)\frac{\Gamma(c-a-iy)}{\Gamma(a+iy)},$$

where by $[\mathcal{G}f](x)$ we denote the so-called Gamma-product transform, which has been introduced and studied by the author in [25]

$$(2.13) \qquad [\mathcal{G}f](x)=\text{P.V.}\int_0^{\infty}\Gamma\left(i(x+\tau)\right)\Gamma\left(i(x-\tau)\right)f(\tau)d\tau,\ x\in\mathbb{R}.$$

Furthermore, it is not difficult to conclude that $\mathcal{O}_{c,a}f(x),\ \mathcal{O}_{c,c-a}h\,(x)$ are reciprocal Mellin's transforms (1.13) from $L_2(\mathbb{R}_+;x^{-1}dx)$. Moreover, by (1.14) we have the Parseval equalities

$$(2.14) \qquad \int_0^{\infty}|\mathcal{O}_{c,a}f(x)|^2\frac{dx}{x}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|[\mathcal{G}f](y)\frac{\Gamma(a+iy)}{\Gamma(c-a-iy)}\right|^2 dy,$$

$$(2.15) \qquad \int_0^{\infty}|\mathcal{O}_{c,c-a}h(x)|^2\frac{dx}{x}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|[\mathcal{G}h](y)\frac{\Gamma(c-a-iy)}{\Gamma(a+iy)}\right|^2 dy.$$

Hence, returning to (2.8) and employing the Cauchy-Schwarz inequality we find by using Fatou's lemma that

$$\int_0^\infty |\mathcal{O}_{c,a}f(x)|^2 (1+x)^{2a-c}\frac{dx}{x} \leqslant \liminf_{\alpha\to 0+} \int_0^\infty |\mathcal{O}_{c,a}f(x)|^2 (1+x)^{2a-c} x^{2\alpha-1}dx$$

$$= \frac{1}{2\pi}\liminf_{\alpha\to 0+} \int_{-\infty}^\infty \Phi_h(\alpha+iy)\overline{\Phi_f(\alpha+iy)}\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)}dy$$

$$\leqslant \sup_{0<\alpha\leqslant b}\frac{1}{2\pi}\int_{-\infty}^\infty \left|\Phi_h(\alpha+iy)\Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)}\right| dy$$

$$\leqslant \sup_{0<\alpha\leqslant b}\frac{1}{2\pi}\left[\left(\int_{-\infty}^\infty \left|\Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|^2 dy\right)^{1/2}\right.$$

$$\left.\times\left(\int_{-\infty}^\infty \left|\Phi_h(\alpha+iy)\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)}\right|^2 dy\right)^{1/2}\right] < \infty.$$

So in order to prove (2.9), (2.10) we appeal to the following integral representation for the product of Gamma-functions (cf. [23, relation (1.104)]

$$\Gamma\left(\alpha+i(x+\tau)\right)\Gamma\left(\alpha+i(x-\tau)\right) = \frac{\Gamma\left(2(\alpha+ix)\right)}{2^{2(\alpha+ix)-1}}\int_0^\infty \frac{\cos\tau y}{\cosh^{2(\alpha+ix)}(y/2)}dy.$$

We substitute it into (2.5), change the order of integration and the result we write in the form

(2.16)

$$\Phi_f(\alpha+iy) = \sqrt{\frac{\pi}{2}}\frac{\Gamma(2(\alpha+iy))}{2^{2(\alpha+iy)-1}}\int_0^\infty \frac{d\hat{f}}{dt}\frac{dt}{\cosh^{2(\alpha+iy)} t},\ 0<\alpha\leqslant\ b<\min(a,c-a),$$

where $\hat{f}(t)$ denotes the following Fourier sine integral

$$\hat{f}(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(\tau)\frac{\sin\tau t}{\tau}d\tau.$$

After integration by parts and elimination of the outintegrated terms in (2.16) we use the relation $\Gamma(2z)2z = \Gamma(1+2z)$ and the substitution $e^{\xi} = \cosh^2 t$. Thus, we arrive at the following Fourier integral

$$\Phi_f(\alpha+iy) = \sqrt{\frac{\pi}{2}}\frac{\Gamma(1+2(\alpha+iy))}{2^{2(\alpha+iy)}}\int_0^\infty e^{-(\alpha+iy)\xi}\hat{f}(\text{arccosh } e^{\xi/2})d\xi.$$

Hence

$$\int_{-\infty}^{\infty}\left|\Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|^2 dy=\frac{\pi}{2^{4\alpha+1}}\int_{-\infty}^{\infty}\left|\frac{\Gamma(1+2(\alpha+iy))\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|^2$$

$$\times\left|\int_0^{\infty}e^{-(\alpha+iy)\xi}\hat{f}(\text{arccosh } e^{\xi/2})d\xi\right|^2 dy. \tag{2.17}$$

However, the Gamma-ratio in (2.17) is bounded on $(\alpha-i\infty,\alpha+i\infty),\ 0\leqslant\ \alpha\leqslant b<\min(a,c-a)$ since via Stirling's asymptotic formula [1] we have

$$\left|\frac{\Gamma(1+2(\alpha+iy))\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|=O(e^{-\pi|y|}|y|^{2a-c+1/2}),\ |y|\to\infty.$$

Consequently, applying twice the Parseval equality for the Fourier transform and making elementary substitutions we obtain from (2.17)

$$\int_{-\infty}^{\infty}\left|\Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|^2 dy\leqslant\ C_1\int_{-\infty}^{\infty}\left|\int_0^{\infty}e^{-(\alpha+iy)\xi}\hat{f}(\text{arccosh } e^{\xi/2})d\xi\right|^2 dy$$

$$=C_2\int_0^{\infty}\alpha|\hat{f}(\text{arccosh } e^{\xi/2})|^2d\xi\leqslant\ C_2\int_0^{\infty}|\hat{f}(\text{arccosh } e^{\xi/2})|^2d\xi$$

$$=2C_2\int_0^{\infty}|\hat{f}(y)|^2\tanh y\ dy\leqslant\ 2C_2\int_{\text{supp}f}|f(\tau)|^2\frac{d\tau}{\tau^2}<\infty,$$

where C_1,C_2 are absolute positive constants. Thus, we have proved (2.9). In the same manner we establish (2.10). Combining now with (2.7), (2.14), (2.15) we apply (1.14) and the Plancherel identity for the Gamma-product transform [25]. Therefore, as a consequence of (2.8) we derive the chain of equalities

$$\int_0^{\infty}|\mathcal{O}_{c,a}f(x)|^2\,(1+x)^{2a-c}\frac{dx}{x}=\int_0^{\infty}\mathcal{O}_{c,c-a}h(x)\overline{\mathcal{O}_{c,a}f(x)}\frac{dx}{x}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\mathcal{G}h](y)\overline{[\mathcal{G}f](y)}dy=2\pi\int_0^{\infty}\left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2|f(\tau)|^2d\tau,$$

which prove (2.4) for any $f\in C_0^{\infty}(\mathbb{R}_+)$. Moreover, it gives the validity of the Plancherel identity (2.3). Then we continuously extend these equalities from the dense set of smooth functions with compact support on the whole weighted L_2-spaces to obtain the desired isomorphism (2.1). The Olevskii transform (1.1) is understood as a limit in the mean square with respect to the norm in the space

$L_2\left(\mathbb{R}_+;(1+x)^{2a-c}\frac{dx}{x}\right)$. The reciprocal formula (2.2) can be proved as follows. From the Plancherel identity (2.3), it is not difficult to arrive at the equality

$$f(\tau)=\left|\frac{\Gamma(c-a+i\tau)}{\Gamma(a+i\tau)\Gamma(2i\tau)}\right|^2\frac{1}{2\pi\Gamma(c)\tau}\frac{d}{d\tau}\int\limits_0^\infty\mathcal{O}_{c,a}f(x)\int\limits_0^\tau t\left|\Gamma(a+it)\right|^2$$

$$\times{}_2F_1\left(a+it,a-it;c;-\frac{1}{x}\right)(1+x)^{2a-c}x^{-a-1}dtdx.$$

Hence, $f(\tau)=\underset{N\to\infty}{\text{l.i.m.}}f_N(\tau)$, where the limit is in the mean square sense with respect to the norm in the space $L_2\left(\mathbb{R}_+;\left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2\right)$ and

$$f_N(\tau)=\left|\frac{\Gamma(c-a+i\tau)}{\Gamma(a+i\tau)\Gamma(2i\tau)}\right|^2\frac{1}{2\pi\Gamma(c)\tau}\frac{d}{d\tau}\int\limits_{1/N}^N\mathcal{O}_{c,a}f(x)\int\limits_0^\tau t\left|\Gamma(a+it)\right|^2$$

$$\times{}_2F_1\left(a+it,a-it;c;-\frac{1}{x}\right)(1+x)^{2a-c}x^{-a-1}dtdx$$

$$=\frac{|\Gamma(c-a+i\tau)|^2}{2\pi\Gamma(c)|\Gamma(2i\tau)|^2}\int\limits_{1/N}^N(1+x)^{2a-c}x^{-a-1}{}_2F_1\left(a+i\tau,a-i\tau;c;-\frac{1}{x}\right)\mathcal{O}_{c,a}f(x)dx,$$

since we can put the derivative under the sign of the latter integral via its uniform convergence with respect to τ. Theorem is proved. □

Let us consider particular cases of the Olevskii transform (1.1), which are associated with the Mehler-Fock integrals [18], [22], [23]. Precisely, putting in (1.4) $a=\frac{1}{2},c=1-\mu,\ \mu<\frac{1}{2}$, we employ relation (8.4.41.12) in [17] to obtain

$${}_2F_1\left(\frac{1}{2}+i\tau,\frac{1}{2}-i\tau;1-\mu;-\frac{1}{x}\right)=\Gamma(1-\mu)(1+x)^{-\mu/2}P^\mu_{-1/2+i\tau}\left(\frac{2}{x}+1\right),$$

where $P^\mu_\nu(z)$ is the associated Legendre function of the first kind [20]. Thus, we arrive at the formula of the generalized Mehler-Fock transform

$$[P_\mu f](x)=x^{-1/2}(1+x)^{-\mu/2}\int\limits_0^\infty\left|\Gamma(1/2+i\tau)\right|^2P^\mu_{i\tau-1/2}\left(1+\frac{2}{x}\right)f(\tau)d\tau, \tag{2.18}$$

where integral (2.18) converges with respect to the norm in $L_2\left(\mathbb{R}_+;(1+x)^\mu\frac{dx}{x}\right)$. According to our Theorem, it forms the isometric isomorphism

$$[P_\mu f]:L_2\left(\mathbb{R}_+;\left|\frac{\Gamma(2i\tau)\Gamma(1/2+i\tau)}{\Gamma(1/2-\mu+i\tau)}\right|^2d\tau\right)\leftrightarrow L_2\left(\mathbb{R}_+;(1+x)^\mu\frac{dx}{x}\right)$$

with the Parseval equality

$$\int\limits_0^\infty|[P_\mu f](x)|^2(1+x)^\mu\frac{dx}{x}=2\pi\int\limits_0^\infty\left|\frac{\Gamma(2i\tau)\Gamma(1/2+i\tau)}{\Gamma(1/2-\mu+i\tau)}\right|^2|f(\tau)|^2d\tau.$$

The reciprocal inverse operator is written in the form

$$f(\tau) = \frac{1}{2\pi} \left| \frac{\Gamma(1/2 - \mu + i\tau)}{\Gamma(2i\tau)} \right|^2 \int_0^\infty (1+x)^{\mu/2} x^{-3/2} P^{\mu}_{-1/2+i\tau} \left(\frac{2}{x} + 1 \right) [P_\mu f](x) dx,$$

where the latter integral converges with respect to the norm in $L_2 \left(\mathbb{R}_+; \left| \frac{\Gamma(2i\tau)\Gamma(1/2+i\tau)}{\Gamma(1/2-\mu+i\tau)} \right|^2 \right)$.

Finally, if we set $c = a + 1$, then by virtue of the formula (7.3.1.52) in [17] we have

$$ {}_2F_1 \left(a + i\tau, a - i\tau; a + 1; -\frac{1}{x} \right) = \frac{\Gamma(a+1) x^a (1+x)^{(1-a)/2}}{2i\tau} \times \left[P^{1-a}_{i\tau} \left(1 + \frac{2}{x} \right) - P^{1-a}_{i\tau-1} \left(1 + \frac{2}{x} \right) \right].$$

Thus we obtain the following transformation of the Mehler-Fock type

$$[P^a f](x) = \frac{i}{2} (1+x)^{(1-a)/2} \times \int_0^\infty |\Gamma(a+i\tau)|^2 \left[P^{1-a}_{i\tau-1} \left(1 + \frac{2}{x} \right) - P^{1-a}_{i\tau} \left(1 + \frac{2}{x} \right) \right] f(\tau) \frac{d\tau}{\tau}.$$

It isomorphically maps the space $L_2 \left(\mathbb{R}_+; \left| \frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(1+i\tau)} \right|^2 d\tau \right)$ onto the space $L_2 \left(\mathbb{R}_+; (1+x)^{a-1} \frac{dx}{x} \right)$. Moreover, the Parseval equality

$$\int_0^\infty |[P^a f](x)|^2 (1+x)^{a-1} \frac{dx}{x} = 2\pi \int_0^\infty \left| \frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(1+i\tau)} \right|^2 |f(\tau)|^2 d\tau$$

holds. The inverse operator is given by the formula

$$f(\tau) = \frac{|\Gamma(1+i\tau)|^2}{4\pi i\tau |\Gamma(2i\tau)|^2} \int_0^\infty (1+x)^{(a-1)/2} \left[P^{1-a}_{i\tau} \left(1 + \frac{2}{x} \right) - P^{1-a}_{i\tau-1} \left(1 + \frac{2}{x} \right) \right] [P^a f](x) \frac{dx}{x},$$

where the convergence is with respect to the norm in $L_2 \left(\mathbb{R}_+; \left| \frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(1+i\tau)} \right|^2 d\tau \right)$.

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