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THE WEIGHTED RELATIVE EXTREMAL FUNCTIONS AND WEIGHTED CAPACITY

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Abstract. The aim of the present paper is to investigate the weighted relative extremal functions and weighted capacities with weights in the Cegrell classes $\mathcal{F}$ and $\mathcal{E}$. Some results on the connection between the weighted capacities with the weighted relative extremal functions are established. Moreover, we give a characterization of weighted capacities $C_{n,u}$ through u and prove the absolute continuity of $C_{n,u}$ with respect to the Sadullaev's weighted capacity $\mathcal{P}_{n,u}$.

1. Introduction

Pluripotential theory in recent years has seen many important developments. Many results of potential theory on the complex plane were extended successfully to $\mathbb{C}^n$. For example, the Cartan theorem on the polarity of the set $\{u < u^*\}$ on $\mathbb{C}$ was generalized to $\mathbb{C}^n$ by Bedford - Taylor. By constructing the theory of the Monge-Ampère operator for locally bounded plurisubharmonic functions on $\mathbb{C}^n$ they established pluripolarity of the negligible sets (see [3]). The Green function with one pole on $\mathbb{C}$, the main tool solving the Dirichlet problem, also has been extended to the Green functions with one or many poles in $\mathbb{C}^n$. Some authors have tried to extend results of normal pluripotential theory to the weighted pluripotential theory. In 1988-1989 E. Bedford introduced the weighted capacity $C_\varphi(E,\Omega)$ and the weighted relative extremal function $\widetilde{\varphi}_K$ (see [2]) (for details see the precise definitions in the next sections). In 2004, using the notion of weighted capacity of Bedford, U.Cegrell, S.Kolodziej and A.Zeriahi gave a condition under which a negative plurisubharmonic function on a hyperconvex domain Ω in $\mathbb{C}^n$ belongs to the Cegrell class $\mathcal{E}(\Omega)$ (see Proposition 2.2 in [8]). Next, T. Bloom and N. Levenberg in the paper [4] considered the weighted Siciak extremal function $V_{K,Q}$ with the weight Q. They proved that if $K \subset \mathbb{C}^n$ is compact and $\{w_j\}$ is a sequence of admissible weights on K with $w_j \searrow w$, $Q_j = -\log w_j$, $Q = -\log w$, then

$$\lim_j V_{K,Q_j}(z) = V_{K,Q}(z)$$

for $z \in \mathbb{C}^n$. Moreover, the Monge - Ampère measures $(dd^c V^*_{K,Q_j})^n$ converge weakly to $(dd^c V^*_{K,Q})^n$ (see Lemma 7.3 in [4]).

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In this paper we continue to investigate properties of weighted relative extremal functions $h^*_{E,u}$ associated to the function u in the case u is in the Cegrell classes $\mathcal{F}$ and $\mathcal{E}$. Next we look for the relationship between the weighted capacity $C_{n,u}(E,\Omega)$ with $h^*_{E,u}$ and give a capacity characterization of $C_{n,u}$.

The paper is organized as follows. Beside the introdution the paper contains four sections. In section 2 we recall some backgrounds of pluripotential theory and the Cegrell classes $\mathcal{F}$ and $\mathcal{E}$. In section 3 we give the definition of weighted relative extremal functions and study their properties. Section 4 is devoted to present the weighted capacity. We prove that in the case $u \in \mathcal{F}$,

$$C_{n,u}(E) = \int_\Omega (dd^c h^*_{E,u})^n = \inf\{\int (dd^c v)^n : v \leqslant u \ \text{ on E}\}.$$

Finally, in section 5 we give a characterization of capacity $C_{n,u}$ in terms of u and establish the absolute continuity of $C_{n,u}$ and the Sadullaev's weighted capacity $\mathcal{P}_{n,u}$.

2. Some backgrounds of pluripotential theory and the Cegrell classes

Let Ω be a domain in $\mathbb{C}^n$ and by $\mathrm{PSH}(\Omega)$ we denote the convex cone of plurisubharmonic (psh)- functions on Ω. Ω is said to be hyperconvex if there exists a negative exhaustion psh function on Ω.

A subset E of Ω is said to be pluripolar if there exists a $\varphi \in \mathrm{PSH}(\Omega), \varphi \neq -\infty$ and $E \subset \{\varphi = -\infty\}$.

As in [3] the C_n-capacity of a Borel subset $E \subset \Omega$ is given by

$$C_n(E) = C_n(E,\Omega) = \sup\{\int_E (dd^c v)^n : v \in \mathrm{PSH}(\Omega), -1 \leqslant v \leqslant 0\}.$$

Throughout this paper some property on Ω is called to satisfy q.e in C_n-capacity on Ω if it is satisfied outside a pluripolar set of Ω.

We deal with the following classes $\mathcal{F}$ and $\mathcal{E}$ of psh functions introduced and investigated by Cegrell in [6] and [7]:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \{\varphi \in \mathrm{PSH}(\Omega) \cap \mathrm{L}^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0, \ \int_\Omega (dd^c\varphi)^n < \infty\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \big\{\varphi \in \mathrm{PSH}(\Omega) : \exists \ \ \mathcal{E}_0 \ni \varphi_j \searrow \varphi, \ \ \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\big\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \big\{\varphi \in \mathrm{PSH}(\Omega) : \forall z_0 \in \Omega, \exists \ \text{ a neighbourhood } \ \omega \ni z_0,$$
$$\mathcal{E}_0 \ni \varphi_j \searrow \varphi \ \text{ on } \ \omega, \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\big\}.$$

It is obvious that $\mathcal{E}_0 \subset \mathcal{F} \subset \mathcal{E}$.

Theorem 2.1. *([7]) The class $\mathcal{E}$ has the following properties:*

(1) *$\mathcal{E}$ is a convex cone.*
(2) *If $u \in \mathcal{E}$, $v \in PSH^{-}(\Omega) = \{\varphi \in PSH(\Omega) : \varphi \leqslant 0\}$ then* $\max(u, v) \in \mathcal{E}$.
(3) *If $u \in \mathcal{E}$ then $(dd^c u)^n$ is defined as a positive Borel measure on Ω and $PSH(\Omega) \cap L^{\infty}_{\text{loc}} \ni u_j \searrow u$ then $(dd^c u_j)^n$ weakly converges to $(dd^c u)^n$.*
(4) *$\forall u \in \mathcal{E}$ and $\forall K \Subset \Omega$ $\exists \widetilde{u} \in \mathcal{F}$ such that $\widetilde{u} = u$ on K.*

We say that $u \in \mathcal{F}^a$ if $u \in \mathcal{F}$ and for every pluripolar set $E \subset \Omega$ we have $\int\limits_E (dd^c u)^n = 0$.

3. Weighted relative extremal functions and their basic properties

We recall the following definition of weighted relative extremal functions and study their basic properties.

Definition 3.1. Let E be a subset of a bounded hyperconvex domain Ω in $\mathbb{C}^n$ and $u \in \mathcal{E}(\Omega)$. Put

$$h_{E,u} = \sup\Big\{v : v \in \mathrm{PSH}^{-}(\Omega), \quad v \leqslant u \ \text{ on } \ E\Big\}.$$

The function $h_{E,u}$ is called the weighted relative extremal function associated to E and u.

As usual we denote by $h^*_{E,u}$ the upper-semicontinuous regularization of $h_{E,u}$. Now we give some properties of $h^*_{E,u}$.

Proposition 3.1. (i) *$h^*_{E,u} \in \mathcal{E}(\Omega)$ and $h^*_{E,u} = h_{E,u}$ q.e in C_n-capacity.*

(ii) *$h^*_{E\cup F,u} = h^*_{E,u}$ for all pluripolar sets $F \subset \Omega$.*

(iii) *supp$(dd^c h^*_{E,u})^n \subset \overline{E}$.*

Proof. (i) Because of the equality $h^*_{E,u} = \max(h^*_{E,u}, u)$ and $u \in \mathcal{E}$ then Theorem 4.5 in [7] implies that $h^*_{E,u} \in \mathcal{E}$. On the other hand, by [3] $h^*_{E,u} = h_{E,u}$ q.e - C_n-capacity.

(ii) Take $\varphi \in \mathrm{PSH}^{-}(\Omega)$, $\varphi \not\equiv -\infty$ such that $\varphi = -\infty$ on F. Let $v \in \mathrm{PSH}^{-}(\Omega)$ and $v \leqslant u$ on E. Then $v + \varepsilon\varphi \leqslant u$ on $E \cup F$ for every $\varepsilon > 0$. It follows that $v(z) \leqslant u(z)$ for $z \in (E \cup F), \varphi(z) > -\infty$. Hence $v \leqslant u$ on $E \cup F$. Take the supremum over all $v \in \mathrm{PSH}^{-}(\Omega)$, $\ v \leqslant u$ on E we deduce that $h^*_{E,u} \leqslant h^*_{E\cup F,u}$. The opposite inequality is obvious and the desired equality follows.

(iii) First we consider the case $E \Subset \Omega$. The proof of Theorem 4.2 in [7] implies that there exists $v \in \mathcal{F}$ such that $v = u$ on E. Hence by [7, Theorem 4.5] it follows that $h^*_{E,u} = \max(h^*_{E,u}, v) \in \mathcal{F}$. Thus $h_{E,u}$ can be defined by

$$h_{E,u} = \sup\{v : v \in \mathcal{F}, \quad v \leqslant u \ \text{ on } \ E\}.$$

By Choquet's lemma [5] we can find an increasing sequence $\{v_j\} \subset \mathcal{F}$ which converges to $h^*_{E,u}$ q.e-C_n-capacity. Proposition 1.4.10 in [5] implies that for each $j \geqslant 1$ we can find $\widehat{v_j} \in \mathrm{PSH}^{-}(\Omega)$ such that $v_j \leqslant \widehat{v_j}$, $\ v_j = \widehat{v_j}$ on $\Omega \setminus \mathbb{B}(a, r)$ and

$\widehat{v_j}$ is maximal in $\mathbb{B}(a,r)$ where $\mathbb{B}(a,r)$ is an arbitrary ball of radius $r > 0$ with center at a in $\Omega \setminus \overline{E}$.

Note that $\widehat{v_j} \in \mathcal{F}$ and because $\Omega \setminus \mathbb{B}(a,r) \supset E$ then $\widehat{v_j} \leqslant u$ on E. It follows that $\widehat{v_j} \nearrow h^*_{E,u}$ q.e- C_n- capacity. Remark at page 175 in [7] implies that $(dd^c\widehat{v_j})^n \longrightarrow (dd^c h^*_{E,u})^n$ weakly. Since $\widehat{v_j}$ is maximal on $\mathbb{B}(a,r)$ we deduce that $(dd^c h^*_{E,u})^n = 0$ on $\mathbb{B}(a,r)$. Hence $\operatorname{supp}(dd^c h^*_{E,u})^n \subset \overline{E}$.

Now assume that E is an arbitrary subset of Ω. Take an increasing sequence of subsets $\{E_j\}$ of E with $E_j \Subset \Omega$ for $j \geqslant 1$. Then $h^*_{E_j,u} \searrow \varphi \geqslant h^*_{E,u}$. By (i) $\varphi \in \mathcal{E}$. We show that $\varphi \leqslant h^*_{E,u}$ and hence, $h^*_{E_j,u} \searrow \varphi = h^*_{E,u}$. Indeed, by [3] for each $j \geqslant 1$ there exists a pluripolar set $F_j \subset E_j$ such that $h^*_{E_j,u} = h_{E_j,u} = u$ on $E_j \setminus F_j$. Thus $\varphi = u$ on $E \setminus F$ with $F = \bigcup\limits_{j=1}^{\infty} F_j$. It follows that $\varphi \leqslant h^*_{E\setminus F,u} = h^*_{E,u}$ because F is a pluripolar set and (ii). Theorem 4.2 in [7] implies that $(dd^c h^*_{E_j,u})^n \longrightarrow (dd^c h^*_{E,u})^n$ weakly. Since $\operatorname{supp}(dd^c h^*_{E_j,u})^n \subset \overline{E}$ for $j \geqslant 1$ it follows that $\operatorname{supp}(dd^c h^*_{E,u})^n \subset \overline{E}$. □

Proposition 3.2. *Let $\mathcal{E} \ni u_j \searrow u \in \mathcal{E}$ and $E \subset \Omega$. Then $h^*_{E,u_j} \searrow h^*_{E,u}$.*

Proof. Since the sequence $\{u_j\}$ decreases, so does the sequence $\{h^*_{E,u_j}\}$. Assume that $h^*_{E,u_j} \searrow h$ as $j \to \infty$. Obviously $h \geqslant h^*_{E,u}$ and $h \in \mathrm{PSH}^-(\Omega)$. On the other hand, since $h^*_{E,u_j} = h_{E,u_j} = u_j$ and $h^*_{E,u} = h_{E,u} = u$ on E outside a pluripolar set and by (ii) of Proposition 3.1 it follows that $h \leqslant h^*_{E,u}$. Hence $h = h^*_{E,u}$ and the conclusion follows. □

4. The weighted C_n-capacity

Definition 4.1. *As in* [8] *for each Borel set $E \subset \Omega$ and $u \in \mathcal{E}$ we define*

$$C_{n,u}(E) = C_{n,u}(E,\Omega) = \sup\Big\{ \int\limits_E (dd^c v)^n : v \in PSH \cap L^\infty(\Omega), \quad u \leqslant v \leqslant 0 \Big\}.$$

The Borel set function $E \mapsto C_{n,u}(E)$ is called C_n-capacity with the weight u or u-C_n-capacity.

Proposition 4.1. *Let $\mathcal{E} \ni u_j \searrow u \in \mathcal{E}$. Then $C_{n,u_j}(E) \nearrow C_{n,u}(E)$ as $j \to \infty$ for all Borel sets $E \Subset \Omega$.*

Proof. Since $\{u_j\} \searrow u$ then $C_{n,u_j}(E) \leqslant C_{n,u_{j+1}}(E) \leqslant C_{n,u}(E)$ for $j \geqslant 1$ and every Borel set $E \subset \Omega$. Hence $C_{n,u_j}(E) \nearrow \alpha \leqslant C_{n,u}(E)$ as $j \to \infty$. It remains to show that $\alpha \geqslant C_{n,u}(E)$. Given $\varphi \in \mathrm{PSH} \cap L^\infty(\Omega)$, $\; u \leqslant \varphi \leqslant 0$. By [3] for each $\varepsilon > 0$ there exists an open set $G \subset \Omega$ such that $C_n(G) < \varepsilon$ and $u\big|_{\Omega\setminus G}$ is continuous. Let $E \subset \Omega' \Subset \Omega$. By Dini's theorem $\{u_j\}$ uniformly converges to u on $\Omega' \setminus G$. Take j_o such that

$$u_{j_o} < (1-\varepsilon)u \leqslant (1-\varepsilon)\varphi$$

on $\Omega' \setminus G$. We have

$$\begin{aligned}\alpha \geqslant C_{n,u_{j_o}}(E) &\geqslant \int_E (dd^c \max((1-\varepsilon)\varphi, u_{j_o}))^n \\ &\geqslant \int_{E\setminus G} (dd^c \max((1-\varepsilon)\varphi, u_{j_o}))^n = (1-\varepsilon)^n \int_{E\setminus G} (dd^c\varphi)^n \\ &= (1-\varepsilon)^n \Big[\int\limits_E (dd^c\varphi)^n - \int\limits_G (dd^c\varphi)^n\Big] \\ &\geqslant (1-\varepsilon)^n \Big[\int\limits_E (dd^c\varphi)^n - (\sup_G|\varphi|)^n C_n(G)\Big] \\ &\geqslant (1-\varepsilon)^n \int\limits_E (dd^c\varphi)^n - \varepsilon(\sup_G|\varphi|)^n.\end{aligned}$$

We tend $\varepsilon \searrow 0$ and have the inequality $\alpha \geqslant C_{n,u_{j_o}}(E) \geqslant \int_E (dd^c\varphi)^n$. Hence $\alpha \geqslant C_{n,u}(E)$ and the conclusion follows. $\square$

As a generalization of a result of Bedford-Taylor for the normal C_n-capacity in $\mathbb{C}^n$ (see Proposition 6.5 in [3]) we have the following.

Theorem 4.1. *Let $u \in \mathcal{E}$. Then*

$$C_{n,u}(E) = \int\limits_E (dd^c h^*_{E,u})^n$$

holds for all Borel sets $E \subset \Omega$. Moreover, if $u \in \mathcal{F}$ then

$$\int\limits_E (dd^c h^*_{E,u})^n = \inf\Big\{\int (dd^c v)^n : v \in \mathcal{F},\quad v \leqslant u \quad on \quad E \quad\Big\},$$

holds for all Borel sets $E \subset \Omega$.

Proof. Let $u \in \mathcal{F}$. Given $E \subset \Omega$ a Borel set and $v \in \mathcal{F}$, $v \leqslant u$ on E. Then $v \leqslant h^*_{E,u}$ and hence, $h^*_{E,u} \in \mathcal{F}$. Corollary 2.11 in [1] implies that

$$\int (dd^c h^*_{E,u})^n \leqslant \int (dd^c v)^n.$$

Take infimum over all $v \in \mathcal{F}$, $v \leqslant u$ on E we get

$$\int (dd^c h^*_{E,u})^n \leqslant \inf\Big\{\int (dd^c v)^n : v \in \mathcal{F},\quad v \leqslant u \quad \text{on} \quad E\Big\} := \alpha.$$

On the other hand, since $h^*_{E,u} \in \mathcal{F}$ and $h^*_{E,u} \leqslant u$ on E then $\int (dd^c h^*_{E,u})^n \geqslant \alpha$. Therefore, the second equality is proved.

To prove the first equality we consider the partial case when $u \in \mathcal{E}_0 \cap C(\overline{\Omega})$. Let E be a compact set of Ω. Then $u \leqslant h^*_{E,u} \leqslant 0$ and hence, $h^*_{E,u} \in \mathcal{F} \cap L^\infty(\Omega)$.

From the definition of $C_{n,u}(E)$ it follows that $C_{n,u}(E) \geqslant \int_E (dd^c h^*_{E,u})^n$. Thus it remains to show that

$$\int (dd^c \varphi)^n \leqslant \int_E (dd^c h^*_{E,u})^n$$

for all $\varphi \in \mathrm{PSH} \cap L^\infty(\Omega)$, $u \leqslant \varphi \leqslant 0$. By Choquet's lemma [5] we can find an increasing sequence $\{u_j\} \subset \mathcal{F}$ which converges to $h_{E,u}$. Let $u \leqslant \varphi \leqslant 0$, $\varphi \in \mathrm{PSH} \cap L^\infty(\Omega)$ be given. It is easy to see that $E \subset \{u_j \leqslant \varphi\}$ for all $j \geqslant 1$. Moreover, $\{u_{j+1} \leqslant \varphi\} \subset \{u_j \leqslant \varphi\}$ and hence, $\chi_{\{u_j \leqslant \varphi\}} \searrow \chi_{\{h_{E,u} \leqslant \varphi\}}$ as $j \to \infty$. We have

$$\int\limits_E (dd^c\varphi)^n \leqslant \int\limits_{\{u_j \leqslant \varphi\}} (dd^c\varphi)^n = \int\limits_\Omega \chi_{\{u_j \leqslant \varphi\}} (dd^c\varphi)^n.$$

Applying the monotone convergence theorem it follows that

$$\int\limits_E (dd^c\varphi)^n \leqslant \int_\Omega \chi_{\{h_{E,u} \leqslant \varphi\}} (dd^c\varphi)^n = \int\limits_{\{h_{E,u} \leqslant \varphi\}} (dd^c\varphi)^n = \int\limits_{\{h^*_{E,u} \leqslant \varphi\}} (dd^c\varphi)^n$$

because the set $\{h^*_{E,u} \leqslant \varphi\}$ is different from the set $\{h_{E,u} \leqslant \varphi\}$ a pluripolar set. However, $\varphi = \max(\varphi, u) \in \mathcal{F}$ and the Corollary 2.11 in [1] implies that

$$\int\limits_{\{h^*_{E,u} \leqslant \varphi\}} (dd^c\varphi)^n \leqslant \int\limits_{\{h^*_{E,u} \leqslant \varphi\}} (dd^c h^*_{E,u})^n \leqslant \int (dd^c h^*_{E,u})^n.$$

Hence, the equality $C_{n,u}(E) = \int (dd^c h^*_{E,u})^n$ holds for the case E is compact.

Now assume that $E \subset \Omega$ is an open set. Let $\{E_j\}_{j \geqslant 1}$ be an exhaustion increasing sequence of compact sets of E. Because $\int\limits_{E_j} (dd^c\varphi)^n \nearrow \int\limits_E (dd^c\varphi)^n$ for $\varphi \in \mathrm{PSH} \cap L^\infty(\Omega)$ it is easy to see that $\sup\limits_j C_{n,u}(E_j) = C_{n,u}(E)$. It follows that

$$C_{n,u}(E) = \lim_{j\to\infty} C_{n,u}(E_j) = \lim_{j\to\infty} \int (dd^c h^*_{E_j,u})^n = \int (dd^c h^*_{E,u})^n.$$

Here the last equality follows from $\mathcal{E}_0 \ni h^*_{E_j,u} \searrow h^*_{E,u} \in \mathcal{F}$ and Proposition 5.1 in [7]. Thus we have the first equality for $E \subset \Omega$ which is either compact or open. To prove this equality for all Borel subsets $E \subset \Omega$ we consider the Borel set function $C^*_{n,u}$ defined by

$$C^*_{n,u}(E) = \inf\Big\{ C_{n,u}(G) : \quad E \subset G, \quad G \text{ is open}\Big\}.$$

From the definition of $C^*_{n,u}$ and since the first equality holds for open sets G of Ω and $\mathcal{F} \ni h^*_{E,u} \geqslant h_{G,u} \in \mathcal{F}$ it follows that

$$C^*_{n,u}(E) \geqslant \int (dd^c h^*_{E,u})^n$$

holds for all Borel sets $E \subset \Omega$. To prove the opposite inequality $C^*_{n,u}(E) \leqslant \int (dd^c h^*_{E,u})^n$ we take $\varphi_j \in \mathrm{PSH}^-(\Omega)$, $\varphi_j \leqslant u$ on E such that

$\varphi_j \nearrow h^*_{E,u}$ q.e - C_n-capacity. Let $\lambda_j \nearrow 1$ and put $G_j = \{\varphi_j < \lambda_j u\}$. Then $\{G_j\}$ is a decreasing sequence of open neighbourhoods of E and

$$\varphi_j/\lambda_j \leqslant h_{G_j,u} \leqslant h^*_{E,u}$$

for $j \geqslant 1$. Hence $h_{G_j,u} \nearrow h^*_{E,u}$. Note that $h_{G_j,u} \in \mathcal{F}$ for $j \geqslant 1$. Hence Proposition 5.1 in [7] implies that

$$\lim_{j\to\infty} \int (dd^c h_{G_j,u})^n = \int (dd^c h^*_{E,u})^n.$$

Therefore,

$$C^*_{n,u}(E) \leqslant \lim_{j\to\infty} C_{n,u}(G_j) = \int (dd^c h^*_{E,u})^n.$$

This proves that

$$C^*_{n,u}(E) = \int (dd^c h^*_{E,u})^n \tag{4.1}$$

holds for every Borel set $E \subset \Omega$.

Using (4.1) we prove that $C^*_{n,u}$ is a generalized capacity. Hence the Choquet's theorem (see [5]) implies that

$$C_{n,u}(E) = C^*_{n,u}(E) = \int (dd^c h^*_{E,u})^n \tag{4.2}$$

holds for all Borel sets $E \subset \Omega$.

Obviously, if $E \subset F \subset \Omega$ then $C^*_{n,u}(E) \leqslant C^*_{n,u}(F)$. At the same time, from (4.1) we notice that $C^*_{n,u}(K_j) \searrow C^*_{n,u}(K)$ for every sequence of compact sets $\{K_j\}$, $K_j \searrow K$. On the other hand, using (i) of Proposition 3.1 and repeating the same arguments as in the proof of Theorem 3.1.8 in [5] we get that $C^*_{n,u}(E_j) \nearrow C^*_{n,u}(E)$ for every sequence of subsets $E_j \nearrow E$. Thus $C^*_{n,u}$ is a generalized capacity.

Now we complete the proof of Theorem 4.1 as follows. Let $u \in \mathcal{E}$. By [7] there exists a decreasing sequence $\{u_j\} \in \mathcal{E}_0 \cap C(\overline{\Omega})$ such that $u_j \searrow u$ as $j \to \infty$. Since (4.2) holds for every u_j and using Proposition 4.2 and Proposition 3.3 and Proposition 5.1 in [7] we have

$$C_{n,u}(E) = \lim_j C_{n,u_j}(E) = \lim_j \int (dd^c h^*_{E,u_j})^n = \int (dd^c h^*_{E,u})^n$$

holds for all $E \subset \Omega$. Theorem 4.1 is completely proved. $\square$

Remark 1. In [8] the authors proved a weaker form of Theorem 4.1. Namely they proved that if $u \in \mathcal{E}$ then for every Borel set $E \Subset \Omega$

$$C_{n,u}(\mathring{E}) \leqslant \int (dd^c h^*_{E,u})^n \leqslant C_{n,u}(\overline{E}).$$

Corollary 4.1. *Let E be a subset of Ω. Then the following are equivalent*

(i) *E is pluripolar.*
(ii) *$C^*_{n,u}(E) = 0$ for all $u \in \mathcal{E}$.*
(iii) *There exists $u \in \mathcal{E}, u \neq 0$ such that $C^*_{n,u}(E) = 0$.*

Proof. (i) $\Longrightarrow$ (ii) By the definition of $C_{n,u}$ it is easy to see that the set function $E \mapsto C^*_{n,u}(E)$ is subadditive. Thus it is enough to consider the case $E \Subset \Omega$. By Theorem 4.1 we have

$$C^*_{n,u}(E) = \int_\Omega (dd^c h^*_{E,u})^n.$$

On the other hand, if E is pluripolar then $h^*_{E,u} \equiv 0$ on Ω and the desired conclusion follows.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i) Without loss of generality we may assume that $E \Subset \Omega$. Then $h^*_{E,u} \in \mathcal{F}$ and

$$\int (dd^c h^*_{E,u})^n = C^*_{n,u}(E) = 0.$$

Since $h^*_{E,u} \in \mathcal{F}$ it follows that $h^*_{E,u} = 0$. By [3] there exists $a \in \Omega$ such that $h^*_{E,u}(a) = h_{E,u}(a) = 0$. Therefore, for each $j \geqslant 1$ we can find $v_j \in \mathcal{F}$ such that $v_j \leqslant u$ on E and $v_j(a) > -2^{-j}$. Put $v = \sum\limits_{j=1}^{\infty} v_j$. Then v is plurisubharmonic on Ω with $v(a) > -1$ and

$$v(z) = \sum_{j=1}^{\infty} v_j(z) \leqslant \sum_{j=1}^{\infty} u(z) = -\infty$$

for $z \in E$ because $u(z) < 0$ for all $z \in E$. $\square$

5. A capacity characterization of $C_{n,u}$.

In this section we give a capacity characterization of the set function $E \mapsto C_{n,u}(E)$. Namely we prove the following.

Theorem 5.1. *Let $u \in \mathcal{F}$. Then the following are equivalent:*

(i) *$C_{n,u}$ is a generalized capacity on Ω.*
(ii) *$C_{n,u} \geqslant (dd^c u)^n$.*
(iii) *$u \in \mathcal{F}^a$.*
(iv) *$C_{n,u} << C_n$.*
(v) *Every psh function v on Ω is q.e $C_{n,u}$-continuous, i.e $\forall\ \varepsilon > 0\ \exists\ G$ open $\subset \Omega$, $C_{n,u}(G) < \varepsilon$ such that $v\big|_{\Omega\setminus G}$ is continuous.*

Proof. (i) $\Longrightarrow$ (ii). Since $C_{n,u}$ and $(dd^c u)^n$ are generalized capacities then by the Choquet's theorem (see [5]) it suffices to show that for each compact set $E \Subset \Omega$, $C_{n,u}(E) \geqslant \int_E (dd^c u)^n$. For each $j \geqslant 1$, put

$$E_j = \Big\{ z \in \Omega : \operatorname{dist}(z, E_j) < \frac{1}{j} \Big\}.$$

Then E_j are open neighbourhoods of E, $E_j \searrow E$. By the hypothesis and $h_{E_j,u} = u$ on E_j and because E_j are open we have

$$\begin{aligned} C_{n,u}(E) &= \lim_j C_{n,u}(E_j) = \lim_j \int (dd^c h_{E_j,u})^n \\ &= \lim_j \int\limits_{\overline{E_j}} (dd^c h_{E_j,u})^n \geqslant \lim_j \int\limits_{E_j} (dd^c h_{E_j,u})^n \\ &= \lim_j \int\limits_{E_j} (dd^c u)^n = \int\limits_{E} (dd^c u)^n. \end{aligned}$$

(ii) $\Longrightarrow$ (iii). Since $\int\limits_E (dd^c u)^n \leqslant C_{n,u}(E) = \int (dd^c h^*_{E,u})^n = 0$ for every pluripolar set E we have $u \in \mathcal{F}^a$.

(iii) $\Longrightarrow$ (iv). It suffices to show that $\lim\limits_j C_{n,u}(E_j) = 0$ for all decreasing sequences of regular compact sets E_j in Ω with $\lim\limits_j C_n(E_j) = 0$. Without loss of generality we may assume that $u \leqslant -1$ on Ω. Note that $h_{E_j,\Omega} \nearrow 0$ q.e - C_n-capacity. Then it follows that

$$\begin{aligned} C_{n,u}(E_j) &= \int (dd^c h_{E_j,u})^n = \int\limits_{E_j} (dd^c h_{E_j,u})^n \\ &\leqslant \int\limits_{E_j} (-h_{E_j,\Omega})(dd^c h_{E_j,u})^n \\ &\leqslant \int (-h_{E_j,\Omega})(dd^c h_{E_j,u})^n \\ &\leqslant \int (-h_{E_j,\Omega})(dd^c u)^n \end{aligned}$$

where the last inequality follows from Corollary 2.11 in [1] and $h_{E_j,u} \in \mathcal{F}$. Hence,

$$\overline{\lim_j} C_{n,u}(E_j) \leqslant \overline{\lim_j} \int (-h_{E_j,\Omega})(dd^c u)^n = \int (-h)(dd^c u)^n$$

where $h_{E_j,\Omega} \nearrow h$ with $h = 0$ q.e - C_n- capacity. Since $u \in \mathcal{F}^a$ it follows that $\lim\limits_j C_{n,u}(E_j) = 0$.

(iv) $\Longrightarrow$ (v). It is a consequence of $C_{n,u} << C_n$, and the quasi-continuity in C_n-capacity of psh functions has been proved in [3].

(v) $\Longrightarrow$ (i). From the definition of $C_{n,u}$ it follows that if $E_1 \subset E_2 \subset \Omega$ then $C_{n,u}(E_1) \leqslant C_{n,u}(E_2)$. On the other hand, if $E_j \nearrow E$, $E_j \Subset \Omega$, $E \Subset \Omega$ then the proof of Proposition 3.1 (iii) implies that $h^*_{E_j,u} \searrow h^*_{E,u}$. Note that $h^*_{E_j,u}$ and $h^*_{E,u}$ belong to $\mathcal{F}$ then Proposition 5.1 in [7] implies that

$$\lim_j \int (dd^c h^*_{E_j,u})^n = \int (dd^c h^*_{E,u})^n.$$

But by Theorem 4.1 we have

$$C_{n,u}(E_j) = \int (dd^c h^*_{E_j,u})^n$$

and

$$C_{n,u}(E) = \int (dd^c h^*_{E,u})^n.$$

Hence, $\lim_j C_{n,u}(E_j) = C_{n,u}(E)$. It remains to show that if $\{K_j\}$ is a sequence of compact sets in Ω, $K_j \searrow K$ then $C_{n,u}(K_j) \searrow C_{n,u}(K)$. Let $\varphi = \lim_j h^*_{K_j,u}$. Then $\varphi \leqslant h^*_{K,u}$ and $\varphi = \max(\varphi, h^*_{K_1,u})$. Hence, $\varphi \in \mathcal{F}$ because $h^*_{K_1,u} \in \mathcal{F}$. We prove that $\varphi \geqslant h^*_{K,u}$. Fix $v \in \mathcal{F}$, $v \leqslant u$ on K. Put $E_j = \{v < u + \frac{1}{j}\}$. By the hypothesis for each $j \geqslant 1$ take an open subset G_j of Ω such that $C_{n,u}(G_j) < \frac{1}{j}$ and $u\big|_{\Omega\setminus G_j}$, $v\big|_{\Omega\setminus G_j}$ are continuous. Moreover, we may assume that $G_j \supset G_{j+1}$. Let $F_j = \{z \in \Omega \setminus G_j : v(z) < u(z) + \frac{1}{j}\} \cup G_j$. Then F_j is an open neighbourhood of K for $j \geqslant 1$. For each $j \geqslant 1$ choose $K_{s_j} \subset F_j$. Then we have

$$\begin{aligned}\varphi &\geqslant h^*_{K_{s_j},u} \geqslant h_{F_j,u} \\ &\geqslant h_{\{z\in\Omega\setminus G_j : v(z)<u(z)+\frac{1}{j}\},u} + h_{G_j,u} \\ &\geqslant v - \frac{1}{j} + h_{G_j,u} \quad \text{for} \quad j \geqslant 1.\end{aligned}$$

Notice that $h_{G_j,u} \leqslant h_{G_{j+1},u}$. Let $\psi = (\lim_j h_{G_j,u})^*$. Then it is easy to see that $\mathcal{F} \ni \psi \geqslant h_{G_j,u}$ for $j \geqslant 1$. By [7] we have

$$\int (dd^c\psi)^n \leqslant \int (dd^c h_{G_j,u})^n = C_{n,u}(G_j) \longrightarrow 0$$

as $j \to \infty$. Therefore, $\int (dd^c\psi)^n = 0$ and hence, by [7] we have $\psi = 0$. Thus $h_{G_j,u} \longrightarrow 0$ a.e $d\lambda$. From the inequality

$$\varphi \geqslant v - \frac{1}{j} + h_{G_j,u}$$

it follows that $\varphi \geqslant v$. Hence, $\varphi \geqslant h^*_{K,u}$. However, since $(dd^c h^*_{K_j,u})^n \longrightarrow (dd^c h^*_{K,u})^n$ weakly, it follows that

$$\begin{aligned}\lim_j C_{n,u}(K_j) &= \lim_j \int (dd^c h^*_{K_j,u})^n \\ &\leqslant \lim_j \int_{K_1} (dd^c h^*_{K_j,u})^n \leqslant \int_{K_1} (dd^c h^*_{K,u})^n \\ &= \int (dd^c h^*_{K,u})^n = C_{n,u}(K) \leqslant \lim_j C_{n,u}(K_j).\end{aligned}$$

Hence, $\lim_j C_{n,u}(K_j) = C_{n,u}(K)$ and $C_{n,u}$ is a generalized capacity. $\square$

Let Ω be a hyperconvex domain in $\mathbb{C}^n$. As in [9] we define the Borel set function $\mathcal{P}_{n,u}$ on Ω given by

$$\mathcal{P}_{n,u}(E) := \int_\Omega (-h^*_{E,u})(z)d\lambda(z)$$

for all Borel sets $E \subset \Omega$, $d\lambda$ denotes the Lebesgue measure in $\mathbb{C}^n$.

In the case $u \equiv -1$ on Ω we write $\mathcal{P}_n = \mathcal{P}_{n,u}$. Sadullaev in [9] proved that if Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ then $C_n << \mathcal{P}_n << C_n$. For the case $C_{n,u}$ and $\mathcal{P}_{n,u}$ we have

Proposition 5.1. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}^a$. Then*

$$\mathcal{P}_{n,u} << C_{n,u} << \mathcal{P}_{n,u}.$$

Proof. Without loss of generality we may assume that $u \leqslant -1$ on Ω. This yields that $h^*_{E,u} \leqslant h^*_E$ for $E \subset \Omega$. Assume that $u \in \mathcal{F}^a$. Then by Theorem 5.1 $C_{n,u} << C_n$. By the above mentioned result of Sadullaev we have $C_{n,u} << \mathcal{P}_n << \mathcal{P}_{n,u}$. The last relation follows from the inequality

$$\mathcal{P}_n(E) = -\int h^*_E d\lambda \leqslant -\int_\Omega h^*_{E,u} d\lambda = \mathcal{P}_{n,u}(E).$$

Hence, $C_{n,u} << \mathcal{P}_{n,u}$. On the other hand, take a strictly psh exhaustion function ρ of Ω with $-1 \leqslant \rho < 0$. It is easy to see that

$$\mathcal{P}_{n,u} << \mathcal{P}_{n,u,\rho} << \mathcal{P}_{n,u}$$

where $\mathcal{P}_{n,u,\rho}(E) := -\int h^*_{E,u}(dd^c\rho)^n$.

Since $h^*_{E,u} \in \mathcal{F}$ integrating by parts we have

$$\begin{aligned}\mathcal{P}_{n,u,\rho}(E) &= -\int h^*_{E,u}(dd^c\rho)^n = -\int \rho(dd^c h^*_{E,u}) \wedge (dd^c\rho)^{n-1}\\ &\leqslant \int (dd^c h^*_{E,u}) \wedge (dd^c\rho)^{n-1} \leqslant \cdots\\ &\leqslant \int (dd^c h^*_{E,u})^n = C_{n,u}(E),\end{aligned}$$

which holds for every Borel set $E \Subset \Omega$.

Thus $\mathcal{P}_{n,u} << C_{n,u} << \mathcal{P}_{n,u}$ and Proposition 5.1 is completely proved. ☐

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