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ON COLORINGS OF SPLIT GRAPHS

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Abstract. A graph $G = (V, E)$ is called a split graph if there exists a partition $V = I \cup K$ such that the subgraphs of G induced by I and K are empty and complete graphs, respectively. In this paper, we determine chromatic polynomials for split graphs and characterize chromatically unique split graphs. Some sufficient conditions for split graphs to be Class one are also proved. In particular, we prove that the conjecture posed by Hilton and Zhao is true for split graphs.

1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$, $E(G)$ (or V, E in short), and $\overline{G}$ will denote its vertex-set, its edge-set and its complementary graph, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). Further, for $W \subseteq V(G)$ the set $W \cap N_G(S)$ is denoted by $N_W(S)$. If $S = \{v\}$, then $N(S)$ and $N_W(S)$ are denoted shortly by $N(v)$ and $N_W(v)$, respectively. For a vertex $v \in V(G)$, the degree of v (resp., the degree of v with respect to W), denoted by $\deg(v)$, (resp., $\deg_W(v)$), is $|N_G(v)|$ (resp., $|N_W(v)|$). The subgraph of G induced by $W \subseteq V(G)$ is denoted by $G[W]$. The empty and complete graphs of order n are denoted by O_n and K_n, respectively. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

A graph $G = (V, E)$ is called a *split graph* if there exists a partition $V = I \cup K$ such that $G[I]$ and $G[K]$ are empty and complete graphs, respectively. We will denote such a graph by $S(I \cup K, E)$. The notion of split graphs was introduced in 1977 by Földes and Hammer [10]. A role that split graphs play in graph theory is clarified in [10] and in [5], [18], [19]. These graphs have been paid attention also because they have connection with packing and knapsack problems [9], with the matroid theory [11], with Boolean functions [20], with the analysis of parallel processes in computer programming [12] and with the task allocation in distributed systems [13]. Many generalizations of split graphs have been made.

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The newest one is the notion of bisplit graphs introduced by Brandstädt et al. [4].

Both vertex colorings (or simply colorings), and edge colorings of graphs have applications in many problems. Because of this, they are important topics in the graph theory. In this paper we consider colorings and edge colorings of split graphs. We recall now some definitions. Let G be a graph. For a positive integer λ, a *λ-coloring* of G is a mapping $f : V(G) \to \{1, 2, \ldots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. Two λ-colorings f and g are considered different if and only if there exists $u \in V(G)$ such that $f(u) \neq g(u)$. Let $P(G, \lambda)$ (or simply $P(G)$ if there is no danger of confusion), denote the number of distinct λ-colorings of G. It is well-known that for any graph G, $P(G, \lambda)$ is a polynomial in λ, called the *chromatic polynomial* of G. The notion of chromatic polynomials was first introduced by Birkhoff [2] in 1912 as a quantitative approach to tackle the four-color problem. The smallest positive integer λ such that $P(G, \lambda) > 0$ is called the *chromatic number* of G and is denoted by $\chi(G)$. We say that a graph G is *n-chromatic* if $n = \chi(G)$ and is *r-critical* if $\chi(G) = r$ and $\chi(H) < \chi(G)$ for every subgraph H of G with $H \neq G$. Cycles C_n with n odd are examples of 3-critical graphs. Two graphs G and H are called *chromatically equivalent* or in short *χ-equivalent*, and we write in notation $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is called *chromatically unique* or in short *χ-unique* if $G' \cong G$ (i.e., G' is isomorphic to G), for any graph G' such that $G' \sim G$. For examples, all cycles are χ-unique [16]. A connected χ-unique graph G is called *weakly χ-unique* if the graph $G \cup O_1$ is not χ-unique; otherwise, G is called *strongly χ-unique*. The notion of χ-unique graphs was first introduced and studied by Chao and Whitehead [7] in 1978. The readers can see the surveys [16] and [17] for more informations about χ-unique graphs.

An edge coloring of a graph G can be defined similarly. Namely, an *edge λ-coloring* of a graph G is a mapping $f : E(G) \to \{1, 2, \ldots \lambda\}$ such that two adjacent edges have distinct images. The *chromatic index* of G, denoted by $\chi'(G)$, is the smallest positive integer λ such that G has an edge λ-coloring. In 1964, Vizing [24] proved that $\chi'(G)$ is equal to either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. A graph G is said to be *Class one* (resp., *Class two*), if $\chi'(G) = \Delta(G)$ (resp., $\Delta(G) + 1$). We say that G is *Class-two-critical* if it is connected, Class two and $G - e$ is Class one for every edge $e \in E(G)$. For examples, all cycles C_n with n even are Class one; all cycles C_n with n odd are Class two and moreover they are Class-two-critical.

The *core* of a graph G, denoted by G_Δ, is the subgraph of G induced by the set of all vertices of degree $\Delta(G)$. We say that G is *overfull* if $|E(G)| > \left\lfloor \frac{|V(G)|}{2} \right\rfloor \Delta(G)$. It is easy to see that if G is overfull, then G is Class two. Vizing [25] proved that, if G_Δ has at most two vertices, then G is Class one. Let P^* be a graph obtained from the Petersen graph by removing one vertex. Then it is not difficult to see that $\chi(P^*) = 3$ and $\chi'(P^*) = 4$. Thus, P^* is Class two, but it is not overfull. In [15] Hilton and Zhao posed the following conjecture:

Conjecture 1. *Let G be a connected graph such that $\Delta(G_\Delta) \leqslant 2$ and $G \neq P^*$. Then G is Class two if and only if G is overfull.*

In this paper, we shall determine chromatic polynomials for split graphs and characterize chromatically unique split graphs (Section 2). Namely, we shall prove that a connected split graph G is χ-unique if and only if G is isomorphic to a connected split graph $G' = S(I' \cup K', E')$ with $|I'| = 1$, and a disconnected split graph G is χ-unique if and only if $G \cong H \cup O_k$, where $k \geqslant 1$ and $H = S(I' \cup K', E')$ is a connected split graph such that $|I'| = 1$ but $|N(I')| > 1$ if $|K'| > 1$. Some sufficient conditions for split graphs to be Class one will be given in Section 3. In particular, we shall prove that Conjecture 1 is true for split graphs. The reader can see other particular cases where Conjecture 1 is true in [6].

2. Vertex colorings

First of all, we prove the following theorem for chromatic polynomials of split graphs.

Theorem 2.1. *Let $G = S(I \cup K, E)$ be a split graph with $I = \{u_1, u_2, \ldots, u_m\}$, $deg(u_i) = t_i$ for $i = 1, 2, \ldots, m$ and $|K| = n$. Then*

$$P(G, \lambda) = \lambda(\lambda - 1)\ldots(\lambda - n + 1)(\lambda - t_1)\ldots(\lambda - t_m).$$

Proof. Let $K = \{v_1, v_2, \ldots, v_n\}$ and f be a coloring of G using λ colors $1, 2, \ldots, \lambda$. It is clear that $f(v_1) \in \{1, 2, \ldots, \lambda\}$,
$f(v_2) \in \{1, 2, \ldots, \lambda\} \setminus \{f(v_1)\}, \ldots, f(v_n) \in \{1, 2, \ldots, \lambda\} \setminus \{f(v_1), \ldots, f(v_{n-1})\}$ and $f(u_i) \in \{1, 2, \ldots, \lambda\} \setminus f(N(u_i))$ for $i = 1, 2, \ldots, m$. Therefore, the number of distinct λ-colorings of G is

$$P(G, \lambda) = \lambda(\lambda - 1)\ldots(\lambda - n + 1)(\lambda - t_1)\ldots(\lambda - t_m).$$

□

Using the above result we now characterize χ-unique split graphs. We need the following lemmas.

Lemma 2.1. *Let $G = S(I \cup K, E)$ be a split graph with $|K| = n$ and $k = \max\{\deg(u) \mid u \in I\}$. Then*

(i) *G is n-chromatic if and only if $k < n$;*
(ii) *G is $(n + 1)$-chromatic if and only if $k = n$.*

Proof. Let $I = \{u_1, u_2, \ldots, u_m\}$ and $\deg(u_i) = t_i$ for $i = 1, 2, \ldots, m$. By Theorem 2.1, we have

$$P(G, \lambda) = \lambda(\lambda - 1)\ldots(\lambda - n + 1)(\lambda - t_1)\ldots(\lambda - t_m).$$

Since $n \geqslant t_i$ for any $i = 1, 2, \ldots, m$ and $\chi(G)$ is the smallest positive integer λ such that $P(G, \lambda) > 0$, it is not difficult to see now that $\chi(G) = n$ if $n \neq t_i$ for every $i = 1, 2, \ldots, m$ and $\chi(G) = n + 1$ if $n = t_i$ for some $i = 1, 2, \ldots, m$. □

TABLE 1. The graphs $G_n^m(t_1,\ldots,t_m)$ and $H_n^m(t_1,\ldots,t_m)$

The graph $G=(V,E)$	The vertex-set $V=I\cup K$	The edge-set $E=E_1\cup E_2\cup\ldots\cup E_{m+1}$
$G_n^m(t_1,\ldots,t_m)$ $(2\leqslant m,$ $1\leqslant t_1\leqslant t_2\leqslant$ $\ldots\leqslant t_m<n)$	$I=\{u_1,\ldots,u_m\}$, $K=\{v_1,\ldots,v_n\}$.	$E_1=\{u_1v_1,u_1v_2,\ldots,u_1v_{t_1}\}$, $E_2=\{u_2v_1,u_2v_2,\ldots,u_2v_{t_2}\}$, ... $E_m=\{u_mv_1,u_mv_2,\ldots,u_mv_{t_m}\}$, $E_{m+1}=\{v_iv_j\mid i\neq j;i,j=1,\ldots,n\}$.
$H_n^m(t_1,\ldots,t_m)$ $(2\leqslant m,$ $1\leqslant t_1\leqslant t_2\leqslant$ $\ldots\leqslant t_m<n)$	$I=\{u_1,\ldots,u_m\}$, $K=\{v_1,\ldots,v_n\}$.	$E_1=\{u_1v_1,u_1v_2,\ldots,u_1v_{t_1}\}$, $E_2=\{u_2v_1,u_2v_2,\ldots,u_2v_{t_2}\}$, ... $E_{m-1}=\{u_{m-1}v_1,\ldots,u_{m-1}v_{t_{m-1}}\}$, $E_m=\{u_mv_2,u_mv_3,\ldots,u_mv_{(t_m+1)}\}$, $E_{m+1}=\{v_iv_j\mid i\neq j;i,j=1,\ldots,n\}$.

The results of the following lemma were proved in [22]. So we omit their proofs here.

Lemma 2.2 ([22])**.** *Let G and H be two χ-equivalent graphs. Then*

(i) $|V(G)|=|V(H)|$;
(ii) $|E(G)|=|E(H)|$;
(iii) $\chi(G)=\chi(H)$;
(iv) *G is connected if and only if H is connected;*
(v) *G is 2-connected if and only if H is 2-connected.*

In Table 1 we define the graphs $G_n^m(t_1,\ldots,t_m)$ and $H_n^m(t_1,\ldots,t_m)$. The conditions that m,n and $t_1,t_2,\ldots,t_m$ must be satisfied for the corresponding graphs are indicated in the parentheses under their name in Column 1. The subsets I and K of the vertex-set V for each of these graphs are indicated in Column 2. Finally, in Column 3, we present the edges of the corresponding graphs. It is clear by definition that $G_n^m(t_1,\ldots,t_m)$ and $H_n^m(t_1,\ldots,t_m)$ are split graphs.

Lemma 2.3. (i) $G_n^m(t_1,\ldots,t_m)\sim H_n^m(t_1,\ldots,t_m)$;
(ii) $G_n^m(t_1,\ldots,t_m)\not\cong H_n^m(t_1,\ldots,t_m)$.

Proof. Assertion (i) follows immediately from Theorem 2.1 and the definitions of $G_n^m(t_1,\dots,t_m)$ and $H_n^m(t_1,\dots,t_m)$. In order to prove Assertion (ii), we set

$$A_i = \{u \in V(G_n^m(t_1,\dots,t_m)) \mid \deg(u) = i\},$$
$$B_i = \{u \in V(H_n^m(t_1,\dots,t_m)) \mid \deg(u) = i\}.$$

It is not difficult to see that

$$A_{m+n-1} = \{v_1, v_2, \dots, v_{t_1}\} \text{ and } B_{m+n-1} = \{v_2, v_3, \dots, v_{t_1}\}.$$

So, $|A_{m+n-1}| = |B_{m+n-1}| + 1$. It follows that $G_n^m(t_1,\dots,t_m) \not\cong H_n^m(t_1,\dots,t_m)$ because otherwise $|A_i| = |B_i|$ for every $i = 1, 2, \dots, m+n-1$. □

We shall apply the following known results on r-critical (see Chapter 8 in [3]), and strongly χ-unique (see [16] and [23]), graphs to prove Theorems 2.2 and 2.3.

Lemma 2.4 ([3]). (i) *Every r-chromatic graph contains an r-critical subgraph.*
(ii) *If H is an r-critical graph and H is not complete, then $|V(H)| \geqslant r+2$.*

Lemma 2.5 ([16]). *Let G be a connected χ-unique graph. Then G is strongly χ-unique if and only if G is 2-connected.*

Lemma 2.6 ([23]). *Let G be a disconnected graph. Then G is χ-unique if and only if $G \cong H \cup O_k$, where $k \geqslant 1$ and H is a strongly χ-unique graph.*

Now we characterize χ-unique split graphs.

Theorem 2.2. *A connected split graph $G = S(I \cup K, E)$ is χ-unique if and only if G is isomorphic to a connected split graph $G' = S(I' \cup K', E')$ with $|I'| = 1$.*

Proof. First we prove the necessity. Suppose that $G = S(I \cup K, E)$ is a connected χ-unique split graph with $|I| = m$, $|K| = n$. If $m \geqslant 3$ or $m = 2$ but $\deg(u) < n$ for any $u \in I$, then by Lemma 2.3, it is not difficult to see G is not χ-unique, a contradiction. So $m = 1$ or $m = 2$ but there exists $u \in I$ such that $\deg(u) = n$. It follows that G is isomorphic to a connected split graph $G' = S(I' \cup K', E')$ with $|I'| = 1$.

Now we prove the sufficiency. Suppose $G' = S(I' \cup K', E')$ is a connected split graph with $I' = \{u\}$ and $|K'| = n$. If $\deg(u) = n$ then G' is the complete graph K_{n+1}. Therefore, G' is χ-unique. So we may assume that $1 \leqslant \deg(u) < n$. Let R be a graph such that $P(R,\lambda) = P(G',\lambda)$. By Lemmas 2.1 and 2.2, R is n-chromatic of order $n+1$. By Assertion (i) of Lemma 2.4, R has an n-critical subgraph H. It is clear that $|V(H)| \geqslant n$. If $|V(H)| > n$, then H is not complete because H is n-chromatic. By Assertion (ii) of Lemma 2.4, $|V(H)| \geqslant n+2$, contradicting the fact that $|V(H)| \leqslant |V(R)| = n+1$. Hence we must have $|V(H)| = n$. Now if H is not complete then it is not difficult to see that $\chi(H) < n$, a contradiction. It follows that H is complete and therefore $R = S(I \cup K, E)$ with $I = \{u^*\}, K = V(H)$. Since $P(R,\lambda) = P(G',\lambda)$, by Theorem 2.1 it is not difficult to see that $\deg_R(u^*) = \deg_{G'}(u)$. It follows that $R \cong G'$. Thus, G' is χ-unique. □

Theorem 2.3. *A disconnected split graph $G = S(I \cup K, E)$ is χ-unique if and only if $G \cong H \cup O_k$, where $k \geqslant 1$ and $H = S(I' \cup K', E')$ is a connected split graph such that $|I'| = 1$ but $|N(I')| > 1$ if $|K'| > 1$.*

Proof. First we prove the necessity. Let $G = S(I \cup K, E)$ be a disconnected χ-unique split graph. Then by Lemma 2.6, $G \cong G' \cup O_k$, where $k \geqslant 1$ and G' is a strongly χ-unique split graph. By Theorem 2.2, G' is isomorphic to a split graph $H = S(I' \cup K', E')$ with $|I'| = 1$. By Lemma 2.5, H is 2-connected. So $|N(I')| > 1$ if $|K'| > 1$.

Now we prove the sufficiency. Let $G \cong H \cup O_k$, where $k \geqslant 1$ and $H = S(I' \cup K', E')$ is a connected split graph such that $|I'| = 1$ but $|N(I')| > 1$ if $|K'| > 1$. If $|K'| = 1$, then by Lemma 2.2 it is not difficult to see that the graph H is strongly χ-unique. If $|K'| > 1$, then H is 2-connected because $|N(I')| > 1$. By Lemma 2.5, H is strongly χ-unique. Therefore, by Lemma 2.6, G is χ-unique. □

3. Edge colorings

In this section we consider the problem of determining when a split graph is Class one. Without loss of generality we may assume that all split graphs considered in this section are graphs without isolated vertices. We need the following Lemmas 3.1–3.4 to prove our results.

Lemma 3.1 ([21])**.** *If G is a graph of order $2n + 1$ and $\Delta(G) = 2n$, then G is Class one if and only if $|E(\overline{G})| \geqslant n$.*

Lemma 3.2 ([8])**.** *Let G be a split graph. If $\Delta(G)$ is odd, then G is Class one.*

Lemma 3.3 ([14])**.** *Let G be a connected Class two graph with $\Delta(G_\Delta) \leqslant 2$. Then*

1. *G is Class-two-critical;*
2. *$\delta(G_\Delta) = 2$;*
3. *$\delta(G) = \Delta(G) - 1$, unless G is an odd cycle;*
4. *$N(V(G_\Delta)) = V(G)$.*

Lemma 3.4. *Let $G = S(I \cup K, E)$ be a split graph with $\Delta(G) = |V(G)| - 1$. Then G is Class two if and only if G is overfull.*

Proof. Suppose that $|I| = m, |K| = n$ and G is Class two. By Lemma 3.2, $\Delta(G)$ is even and therefore $|V(G)|$ is odd. By Lemma 3.1, we have

$$\begin{aligned} |E(G)| &= |K_{m+n}| - |E(\overline{G})| \\ &> \frac{(n+m)(n+m-1)}{2} - \frac{n+m-1}{2} \\ &= \frac{n+m-1}{2}(n+m-1) \\ &= \left\lfloor \frac{n+m}{2} \right\rfloor \Delta(G) \\ &= \left\lfloor \frac{|V(G)|}{2} \right\rfloor \Delta(G). \end{aligned}$$

Thus, G is overfull. The converse is already well-known to be true. □

Now we prove some sufficient conditions for split graphs to be Class one.

Theorem 3.1. *Let $G = S(I \cup K, E)$ be a split graph and $G_1 = G - E(G[K])$. If $\Delta(G_1) = \deg(v)$ for some vertex $v \in K$, then G is Class one.*

Proof. If the order of $G_2 = G[K]$ is even then G_2 has an edge coloring f_2 using $\Delta(G_2)$ colors $1, 2, \ldots, \Delta(G_2)$. The graph G_1 is bipartite. So G_1 has an edge coloring f_1 using $\Delta(G_1)$ colors $\Delta(G_2) + 1, \ldots, \Delta(G_2) + \Delta(G_1)$. Since $\Delta(G_1) = \deg(v)$ for some vertex $v \in K$, it is clear that the mapping

$$f : E \to \{1, 2, \ldots, \Delta(G_2), \Delta(G_2) + 1, \ldots, \Delta(G_2) + \Delta(G_1)\}$$

such that $f(e) = f_1(e)$ if $e \in E(G_1)$ and $f(e) = f_2(e)$ if $e \in E(G_2)$ is an edge coloring of G. Since $\Delta(G) = \Delta(G_1) + \Delta(G_2)$, it follows that $\chi'(G) = \Delta(G)$ and G is Class one.

If the order of $G_2 = G[K]$ is odd then G_2 has an edge coloring f_2 using $\Delta(G_2)+1$ colors $1, 2, \ldots, \Delta(G_2)+1$. The graph G_1 has an edge coloring f_1 using $\Delta(G_1)$ colors $\Delta(G_2) + 2, \ldots, \Delta(G_1) + \Delta(G_2) + 1$. Let f be the edge coloring of G such that

(i) $f(e) = f_2(e)$ if $e \in E(G_2)$,

(ii) $f(e) = f_1(e)$ if $e \in E(G_1)$ and $f_1(e) \neq \Delta(G_1) + \Delta(G_2) + 1$,

(iii) For $e = uv$ with $u \in I, v \in K$ and $f_1(e) = \Delta(G_1) + \Delta(G_2) + 1$, $f(e)$ is the color from $\{1, 2, \ldots, \Delta(G_2) + 1\}$ not used in the star with the star center v (i.e., in the subgraph induced by the edges of G incident with v). Thus, we obtain an edge coloring f for G which used only $\Delta(G_1) + \Delta(G_2) = \Delta(G)$ colors, i.e., G is Class one. □

Theorem 3.2. *Let $G = S(I \cup K, E)$ be a split graph and $A = V(G_\Delta)$. If $|N_I(S)| \geqslant |S|$ for every $S \subseteq A$, then G is Class one.*

Proof. If $\Delta(G)$ is odd then by Lemma 3.2, G is Class one. So we may assume that $\Delta(G)$ is even. It is not difficult to see that $A \subseteq K$. Since $|N_I(S)| \geqslant |S|$ for every $S \subseteq A$, by the Hall's theorem on matching G contains a complete matching M from A to I. Let $A = \{v_1, \ldots, v_k\}$ and $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$, where $u_1, u_2, \ldots, u_k \in I$. Consider the graph $G' = G - M$. It is clear that $\Delta(G') = \Delta(G) - 1$. Since $\Delta(G)$ is even, $\Delta(G')$ is odd and G' has an edge coloring f_1 using $\Delta(G) - 1$ colors $1, 2, \ldots, \Delta(G) - 1$. Let f be the edge coloring of G such that

(i) $f(e) = f_1(e)$ if $e \in E(G')$,

(ii) $f(e) = \Delta(G)$ if $e \in M$.

Then f is an edge $\Delta(G)$-coloring for G. Thus, G is Class one. □

Theorem 3.3. *Let $G = S(I \cup K, E)$ be a connected split graph with deg$(u) \leqslant 2$ for each $u \in I$ and $G \ncong K_3$. Then G is Class one.*

Proof. Let $|I| = m$ and $|K| = n$. If $\deg_I(v) \geqslant 2$ for some $v \in K$, then by Theorem 3.1 G is Class one. So we may assume that $\deg_I(v) \leqslant 1$ for any $v \in K$. It is clear that $\Delta(G) = n$. If $n = 1$ or 2 then it is clear that G is Class one. So we may assume that $n > 2$ and n is even because otherwise G is Class one by Lemma 3.2. Since $\Delta(G) = n$ and $\deg(u) \leqslant 2$ for each $u \in I$, we may assume that

$$\begin{aligned} I &= \{u_1, u_2, \ldots, u_m\}, \\ V(G_\Delta) &= K = \{v_1, v_2, \ldots, v_n\}, \\ N(u_i) &= \{v_i\} \text{ for } i = 1, 2, \ldots, p, \\ N(u_i) &= \{v_{2i-p-1}, v_{2i-p}\} \text{ for } i = p+1, \ldots, m. \end{aligned}$$

Let $L = [l_{i,j}]$ with $i, j \in \{1, 2, \ldots, n\}$ be the latin square of order n such that $l_{i,j} \equiv i + j (\bmod n)$ where numbers modulo n are $1, 2, \ldots, n$. We will use the latin square L to obtain an edge n-coloring f for G. Since $n > 2$, it is not difficult to see that $l_{2i-1,2i-1} \neq l_{2i,2i}$ for $i = 1, 2, \ldots, m$. Let f be the edge coloring of G such that

$$\begin{aligned} f(v_i v_j) &= l_{i,j} \text{ for } 1 \leqslant i \neq j \leqslant n, \\ f(v_i u_i) &= l_{i,i} \text{ for } i = 1, 2, \ldots, p, \\ f(v_{2i-p-1} u_i) &= l_{2i-p-1,2i-p-1} \text{ for } i = p+1, \ldots, m, \\ f(v_{2i-p} u_i) &= l_{2i-p,2i-p} \text{ for } i = p+1, \ldots, m. \end{aligned}$$

Then f is an edge $\Delta(G)$-coloring for G. Thus, G is Class one. □

The following theorem shows that Conjecture 1 posed by Hilton and Zhao in [15] and already mentioned in Section 1 is true for split graphs.

Theorem 3.4. *Let $G = S(I \cup K, E)$ be a connected split graph with $\Delta(G_\Delta) \leqslant 2$. Then G is Class two if and only if G is overfull.*

Proof. It is already well-known that if G is overfull then G is Class two. We prove now the converse. Suppose that G is Class two. If $G \cong K_3$ then G is overfull. So we may assume that $G \ncong K_3$. Let $A = V(G_\Delta)$. It is not difficult to see that $A \subseteq K$ and $|A| \leqslant 3$. If $|A| \leqslant 2$, then by using Theorems 3.2 and 3.3 it is not difficult to show that G is Class one. So we may assume that $|A| = 3$. By Lemma 3.3, $N_I(A) = I$. If $|N_I(v)| \geqslant 3$ for some $v \in A$, then $|N_I(S)| \geqslant |S|$ for every $S \subseteq A$. By Theorem 3.2, G is Class one, a contradiction. So $|N_I(v)| \leqslant 2$ for every $v \in A$. If $|I| \geqslant 3$, then it is not difficult to see that $|N_I(S)| \geqslant |S|$ for every $S \subseteq A$. Again by Theorem 3.2, G is Class one, a contradiction again. So $|I| \leqslant 2$. First assume that $|N_I(v)| = 2$ for some $v \in A$. Then $|I| = 2$ and $\Delta(G) = |V(G)| - 1$, by Lemma 3.4, G is overfull. Now we assume that $|N_I(v)| = 1$ for every $v \in A$. If $|I| = 2$, say $I = \{u_1, u_2\}$, then without loss of generality we may assume that $A = \{v_1, v_2, v_3\}$, $N(u_1) = \{v_1\}$, $N(u_2) = \{v_2, v_3\}$. Consider the graph $G' = G - u_1 v_1$. By Lemma 3.3, G is Class-two-critical. So G' is Class one. Let f_1 be an edge coloring of G' using $\Delta(G') = \Delta(G)$ colors $1, 2, \ldots, \Delta(G)$. Let f be the edge coloring of G such that

(i) $f(e) = f_1(e)$ if $e \in E(G')$;

(ii) $f(u_1v_1)$ is a color in $\{1, 2, \ldots, \Delta(G)\}$ which is not used to color any edges incident to v_1.

Then f is an edge $\Delta(G)$-coloring for G, i.e., G is Class one, a contradiction. So $|I| = 1$. Then $\Delta(G) = |V(G)| - 1$. Again by Lemma 3.4, G is overfull. □

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