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SHARP WEIGHTED INEQUALITIES FOR MULTILINEAR INTEGRAL OPERATORS

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ABSTRACT. A sharp inequality for some multilinear operator related to a certain integral operator is obtained. The operator includes the Littlewood-Paley operator, the Marcinkiewicz operator and the Bochner-Riesz operator. As an application, we derive the weighted L^p $(p > 1)$ norm inequalities and $L \log L$ type estimate for the multilinear operator.

1. PRELIMINARIES AND RESULTS

In this paper, we will study some multilinear integral operators which are defined as follows. Suppose m_j $(j = 1, \cdots, l)$ are the positive integers, $m_1 + \cdots + m_l = m$ and A_j $(j = 1, \cdots, l)$ are the functions on R^n. Let $F_t(x, y)$ be defined on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x, y) f(y) dy$$

and

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x - y|^m} F_t(x, y) f(y) dy$$

for every bounded and compactly supported function f, where

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x - y)^\alpha.$$

Let H be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H. Then, the multilinear operator related to F_t is defined by

$$T^A(f)(x) = ||F_t^A(f)(x)||,$$

where F_t satisfies the following condition: for any fixed $\varepsilon > 0$,

$$||F_t(x, y)|| \leq C|x - y|^{-n}$$

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and

$$||F_t(y,x)-F_t(z,x)|| \le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon}$$

if $2|y-z| \le |x-z|$. We put $T(f)(x) = ||F_t(f)(x)||$ and assume that T is bounded on $L^p(R^n)$ for $1 < p < \infty$ and weak (L^1, L^1)-bounded.

Note that for $m = 0$, T^A is just the multilinear commutator of T and A (see [8], [9], [15]). For $m > 0$, T^A is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been studied intensively (see [1]-[5]). In [7], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [13], the authors obtained a sharp estimate for the multilinear commutator.

The purpose of this paper is to prove a sharp inequality for the multilinear operator T^A. By using the sharp estimate, we obtain the weighted L^p $(p > 1)$ norm inequalities and $L\log L$ type estimate for the multilinear operators. In Section 3, we will give some applications.

First, let us introduce some notations. Throughout this paper, Q denotes a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)-f_Q|dy,$$

where $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known (see [6]) that

$$f^{\#}(x) = \sup_{x\in Q}\inf_{c\in C} \frac{1}{|Q|}\int_Q |f(y)-c|dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^{\infty}(R^n)$ and $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. For $0 < r < \infty$, we put

$$f_r^{\#}(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator. For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ for $k \ge 2$.

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ. The Φ-average for a function f is defined by

$$||f||_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \le 1\right\}$$

and the maximal function associated to Φ by

$$M_{\Phi}(f)(x) = \sup_{x\in Q} ||f||_{\Phi,Q}.$$

The Young functions to be used in this paper are

$$\Phi(t) = \exp(t^r) - 1 \quad \text{and} \quad \Psi(t) = t\log^r(t+e).$$

The corresponding Φ-average and maximal functions are denoted by $||\cdot||_{\exp L^r,Q}$, $M_{\exp L^r}$ and $||\cdot||_{L(\log L)^r,Q}$, $M_{L(\log L)^r}$. For any $r>0$ and $m\in N$, we have the following inequality (see [13]):

$$M(f)\le M_{L(\log L)^r}(f),\quad M_{L(\log L)^m}(f)\approx M^{m+1}(f).$$

Set, for $r\ge 1$,

$$||b||_{Osc_{\exp L^r}}=\sup_Q ||b-b_Q||_{\exp L^r,Q},$$

the space $Osc_{\exp L^r}$ is defined by

$$Osc_{\exp L^r}=\{b\in L^1_{\log}(R^n):||b||_{osc_{\exp L^r}}<\infty\}.$$

It has been known (see [11]) that

$$||b-b_{2Q}||_{\exp L^r,2^kQ}\le Ck||b||_{Osc_{\exp L^r}}.$$

It is obvious that $Osc_{\exp L^r}$ coincides with the BMO space if $r=1$. And $Osc_{\exp L^r}\subset BMO$ if $r>1$. For $1\le p<\infty$, we denote the Muckenhoupt weights (see [6]) by A_p.

Now we state our main results as following.

Theorem 1.1. *Let $r_j\ge 1$ and $D^\alpha A_j\in Osc_{\exp L^{r_j}}$ for all α with $|\alpha|=m_j$ and $j=1,\cdots,l$. Let $1/r=1/r_1+\cdots 1/r_l$. Then for any $0<p<1$, there exists a constant $C>0$ such that for any $f\in C_0^\infty(R^n)$ and $x\in R^n$,*

$$(T^A(f))^\#_p(x)\le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}(f)(x).$$

Theorem 1.2. *Let $r_j\ge 1$ and $D^\alpha A_j\in Osc_{\exp L^{r_j}}$ for all α with $|\alpha|=m_j$ and $j=1,\cdots,l$.*

(1) *If $1<p<\infty$ and $w\in A_p$, then*

$$||T^A(f)||_{L^p(w)}\le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}}\right)||f||_{L^p(w)};$$

(2) *Let $w\in A_1$, $1/r=1/r_1+\cdots+1/r_l$ and $\Phi(t)=t\log^{1/r}(t+e)$. Then there exists a constant $C>0$ such that for all $\lambda>0$,*

$$\begin{aligned}&w(\{x\in R^n:|T^A(f)(x)|>\lambda\})\\ \le& C\int_{R^n}\Phi\left(\lambda^{-1}\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}}\right)|f(x)|\right)w(x)dx.\end{aligned}$$

2. Proof of Theorems 1 and 2

First, we state some lemmas.

Lemma 2.1 ([3]). *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and for some $q > n$. Then*

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2.2 ([6, p. 485]). *Let $0 < p < q < \infty$ and $f \ge 0$. For $1/r = 1/p - 1/q$ we put*

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q},$$

$$N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p} / ||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p} ||f||_{WL^q}.$$

Lemma 2.3 ([13]). *Let $r_j \ge 1$ for $j = 1, \cdots, m$, and let $1/r = 1/r_1 + \cdots + 1/r_m$. Then*

$$\frac{1}{|Q|} \int_Q |f_1(x) \cdots f_m(x) g(x)| dx \le ||f||_{\exp L^{r_1},Q} \cdots ||f||_{\exp L^{r_m},Q} ||g||_{L(\log L)^{1/r},Q}.$$

Proof of Theorem 1.1.

It suffices to prove for $f \in C_0^\infty(R^n)$ and for some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|} \int_Q |T^A(f)(x) - C_0|^p dx \right)^{1/p}$$
$$\le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and

$$\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} (D^\alpha A_j)_{\tilde{Q}} x^\alpha.$$

Then $R_{m_j+1}(A_j;x,y)=R_{m_j+1}(\tilde{A}_j;x,y)$ and $D^\alpha\tilde{A}_j=D^\alpha A_j-(D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha|=m_j$. We write, for $f_1=f\chi_{\tilde{Q}}$ and $f_2=f\chi_{R^n\setminus\tilde{Q}}$,

$$F_t^A(f)(x)=\int\limits_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)f(y)dy$$
$$=\int\limits_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)f_2(y)dy$$
$$+\int\limits_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)f_1(y)dy$$
$$-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int\limits_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)F_t(x,y)f_1(y)dy$$
$$-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int\limits_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)F_t(x,y)f_1(y)dy$$
$$+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int\limits_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}F_t(x,y)f_1(y)dy,$$

Then

$$\left|T^A(f)(x)-T^{\tilde{A}}(f_2)(x_0)\right|=\left|||F_t^A(f)(x)||-||F_t^{\tilde{A}}(f_2)(x_0)||\right|$$
$$\le||F_t^A(f)(x)-F_t^{\tilde{A}}(f_2)(x_0)||\le\left\|\int\limits_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)f_1(y)dy\right\|$$
$$+\left\|\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int\limits_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)F_t(x,y)f_1(y)dy\right\|$$
$$+\left\|\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int\limits_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)F_t(x,y)f_1(y)dy\right\|$$
$$+\left\|\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int\limits_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}F_t(x,y)f_1(y)dy\right\|$$
$$+\left|T^{\tilde{A}}(f_2)(x)-T^{\tilde{A}}(f_2)(x_0)\right|$$
$$:=I_1(x)+I_2(x)+I_3(x)+I_4(x)+I_5(x).$$

Thus

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int\limits_Q \left|T^A(f)(x)-T^{\tilde{A}}(f_2)(x_0)\right|^p dx\right)^{1/p}\\
\le&\left(\frac{C}{|Q|}\int\limits_Q I_1(x)^p dx\right)^{1/p}+\left(\frac{C}{|Q|}\int\limits_Q I_2(x)^p dx\right)^{1/p}\\
&\quad+\left(\frac{C}{|Q|}\int\limits_Q I_3(x)^p dx\right)+\left(\frac{C}{|Q|}\int\limits_Q I_4(x)^p dx\right)\\
&\quad+\left(\frac{C}{|Q|}\int\limits_Q I_5(x)^p dx\right)^{1/p}\\
:=&I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 2.1, we get

$$R_{m_j}(\tilde{A}_j;x,y)\le C|x-y|^{m_j}\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}}.$$

Thus, by Lemma 2.2 and the weak type (1,1) of T, we obtain

$$\begin{aligned}
I_1 &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}}\right)\left(\frac{1}{|Q|}\int\limits_Q |T(f_1)(x)|^p dx\right)^{1/p}\\
&= C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}}\right)|Q|^{-1}\frac{||T(f_1)\chi_Q||_{L^p}}{|Q|^{1/p-1}}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}}\right)|Q|^{-1}||T(f_1)||_{WL^1}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}}\right)|Q|^{-1}||f_1||_{L^1}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}}\right)M_{L(\log L)^{1/r}}(f)(\tilde{x}).
\end{aligned}$$

For I_2 and I_3, note that $||\chi_Q||_{\exp L^{r_2},Q} \le C$. Similarly as in the estimation of I_1, using Lemma 2.3 instead of Lemma 2.2, we get

$$\begin{aligned}
I_2 &\le C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{Osc_{\exp L^{r_2}}} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 f_1)(x)|^p dx \right)^{1/p} \\
&\le C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{Osc_{\exp L^{r_2}}} \sum_{|\alpha_1|=m_1} |Q|^{-1} ||T(D^{\alpha_1}\tilde{A}_1 f_1)(x)\chi_Q||_{WL^1} \\
&\le C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{Osc_{\exp L^{r_2}}} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) f_1(x)| dx \\
&\le C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{Osc_{\exp L^{r_2}}} ||\chi_Q||_{\exp L^{r_2},Q} \\
&\quad \times \sum_{|\alpha_1|=m_1} ||D^{\alpha_1}A_1 - (D^{\alpha_1}A_1)_{\tilde{Q}}||_{\exp L^{r_1},\tilde{Q}} ||f||_{L(\log L)^{1/r},\tilde{Q}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(\tilde{x})
\end{aligned}$$

and

$$I_3 \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

Similarly, for I_4, by using Lemma 2.3 we get

$$\begin{aligned}
I_4 &\le C \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1)(x)|^p dx \right)^{1/p} \\
&\le C \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} |Q|^{-1} ||T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1)\chi_Q||_{WL^1} \\
&\le C \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{|Q|} \int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) D^{\alpha_2}\tilde{A}_2(x) f_1(x)| dx \\
&\le C \sum_{|\alpha_1|=m_1} ||D^{\alpha_1}A_1 - (D^{\alpha_1}A_1)_{\tilde{Q}}||_{\exp L^{r_1},\tilde{Q}} \\
&\quad \times \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2 - (D^{\alpha_2}A_2)_{\tilde{Q}}||_{\exp L^{r_2},\tilde{Q}} ||f||_{L(\log L)^{1/r},\tilde{Q}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
&F_t^{\tilde{A}}(f_2)(x) - F_t^{\tilde{A}}(f_2)(x_0)\\
=&\int_{R^n} \left(\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(x_0,y)}{|x_0-y|^m}\right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j;x,y) f_2(y)dy\\
&+\int_{R^n} \left(R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x_0,y)\right) \frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m} F_t(x_0,y) f_2(y)dy\\
&+\int_{R^n} \left(R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x_0,y)\right) \frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m} F_t(x_0,y) f_2(y)dy\\
&-\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m} F_t(x,y)\right.\\
&\qquad\qquad \left. - \frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} F_t(x_0,y)\right] D^{\alpha_1}\tilde{A}_1(y) f_2(y)dy\\
&-\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m} F_t(x,y)\right.\\
&\qquad\qquad \left. - \frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} F_t(x_0,y)\right] D^{\alpha_2}\tilde{A}_2(y) f_2(y)dy\\
&+\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} F_t(x,y)\right.\\
&\qquad\qquad \left. - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} F_t(x_0,y)\right] D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y) f_2(y)dy\\
=&I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.
\end{aligned}$$

By Lemma 2.1, we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned}
&|R_{m_j}(\tilde{A}_j;x,y)|\\
&\le C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} (||D^{\alpha_j}A||_{Osc_{\exp L^{r_j}}} + |(D^{\alpha_j}A)_{\tilde{Q}(x,y)} - (D^{\alpha_j}A)_{\tilde{Q}}|)\\
&\le Ck|x-y|^{m_j} \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A||_{Osc_{\exp L^{r_j}}}.
\end{aligned}$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, by the condition of F_t we obtain

$$||I_5^{(1)}|| \le C \int_{R^n} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}} + \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{m+n+\varepsilon}} \right) \prod_{j=1}^{2} |R_{m_j}(\tilde{A}_j; x, y)||f_2(y)|dy$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right)$$

$$\times \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} k^2 \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}} \right) |f(y)|dy$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |f(y)|dy$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) M(f)(\tilde{x}).$$

For $I_5^{(2)}$ and $I_5^{(3)}$, by the formula

$$R_{m_j}(\tilde{A}; x, y) - R_{m_j}(\tilde{A}; x_0, y) = \sum_{|\beta|<m_j} \frac{1}{\beta!} R_{m_j-|\beta|}(D^{\beta}\tilde{A}; x, x_0)(x-y)^{\beta}$$

(see [3]) and by Lemma 2.1, we have

$$|R_{m_j}(\tilde{A}; x, y) - R_{m_j}(\tilde{A}; x_0, y)|$$
$$\le C \sum_{|\beta|<m_j} \sum_{|\alpha|=m_j} |x-x_0|^{m_j-|\beta|} |x-y|^{|\beta|} ||D^{\alpha} A||_{Osc_{\exp L^{r_j}}},$$

Thus

$$||I_5^{(2)}|| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} k \frac{|x-x_0|}{|x_0-y|^{n+1}} |f(y)|dy$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) M(f)(\tilde{x})$$

and

$$||I_5^{(3)}|| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) M(f)(\tilde{x}).$$

For $I_5^{(4)}$ and $I_5^{(5)}$, similarly as in the estimation of $I_5^{(1)}$, $I_5^{(2)}$ and I_2, we get

$$\left\|I_5^{(4)}\right\| \le C \sum_{|\alpha_1|=m_1} \int_{R^n} \left\{ \left\| \frac{(x-y)^{\alpha_1} F_t(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1} F_t(x_0,y)}{|x_0-y|^m} \right\| \times |R_{m_2}(\tilde{A}_2; x,y)||D^{\alpha_1}\tilde{A}_1(y)| \right\} |f_2(y)|dy$$

$$+ C \sum_{|\alpha_1|=m_1} \int_{R^n} \left\{ \left| R_{m_2}(\tilde{A}_2; x,y) - R_{m_2}(\tilde{A}_2; x_0,y) \right| \times \frac{|(x_0-y)^{\alpha_1}| \, \|F_t(x_0,y)\|}{|x_0-y|^m} |D^{\alpha_1}\tilde{A}_1(y)| \right\} |f_2(y)|dy$$

$$\le C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{Osc_{\exp L^{r_2}}} \times \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k(2^{-k}+2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(y)||f(y)|dy$$

$$\le C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{Osc_{\exp L^{r_2}}} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k(2^{-k}+2^{-\varepsilon k}) \times ||D^{\alpha_1}A_1 - (D^{\alpha_1}A_1)_{\tilde{Q}}||_{\exp L^{r_1}, 2^k\tilde{Q}} ||f||_{L(\log L)^{1/r}, 2^k\tilde{Q}}$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(\tilde{x})$$

and

$$||I_5^{(5)}|| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

For $I_5^{(6)}$, using Lemma 2.3 we obtain

$$||I_5^{(6)}|| \le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \left\| \frac{(x-y)^{\alpha_1+\alpha_2} F_t(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2} F_t(x_0,y)}{|x_0-y|^m} \right\| \times |D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f_2(y)|dy$$

$$\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^{\infty} (2^{-k}+2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|dy$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

Thus

$$||I_5|| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

This completes the proof of Theorem 1.1. □

By Theorem 1.1 and the boundedness of $M_{L(\log L)^{1/r}}$, we can obtain Theorem 1.2.

3. Applications

In this section, we shall apply Theorems 1.1 and 1.2 to several particular operators such as the Littlewood-Paley operator, the Marcinkiewicz operator and the Bochner-Riesz operator.

Application 1: Littlewood-Paley operator.

Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

(i) $\int_{R^n} \psi(x)dx = 0$,

(ii) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(iii) $|\psi(x+y) - \psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ whenever $2|y| < |x|$;

The Littlewood-Paley multilinear operator is defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x-y|^m} \psi_t(x-y) f(y)dy$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f) = \psi_t * f$. Define

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

which is the Littlewood-Paley operator (see [14]).

Let H be the space $H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t \right)^{1/2} < \infty \right\}$. Then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H. It is clear that

$$g_\psi(f)(x) = ||F_t(f)(x)|| \quad \text{and} \quad g_\psi^A(f)(x) = ||F_t^A(f)(x)||.$$

We know that g_ψ is bounded on $L^p(R^n)$ for all $1 < p < \infty$ and weak (L^1, L^1)-bounded. Thus

$$(g_\psi^A(f))_p^{\#}(x) \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(x)$$

for any $f \in C_0^\infty(R^n)$ and $0 < p < 1$,

$$||g_\psi^A(f)||_{L^p(w)} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) ||f||_{L^p(w)}$$

for any $w \in A_p$ and $1 < p < \infty$,

$$w(\{x \in R^n : g_\psi^A(f)(x) > \lambda\})$$
$$\le C \int_{R^n} \Phi \left(\lambda^{-1} \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) |f(x)| \right) w(x)dx$$

for any $w \in A_1$ and all $\lambda > 0$.

Application 2: Marcinkiewicz operator.

Let $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x - y|^\gamma$. The Marcinkiewicz multilinear operator is defined by

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x - y|^m} \frac{\Omega(x - y)}{|x - y|^{n-1}} f(y)dy.$$

Set $F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy$. Define $\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}$, which is the Marcinkiewicz operator (see [15]).

Let H be the space $H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}$. Then, it is clear that

$$\mu_\Omega(f)(x) = ||F_t(f)(x)|| \quad \text{and} \quad \mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||.$$

We know that μ_Ω is bounded on $L^p(R^n)$ for all $1 < p < \infty$ and weak (L^1, L^1)-bounded. Thus

$$(\mu_\Omega^A(f))_p^\#(x) \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) M_{L(\log L)^{1/r}}(f)(x)$$

for any $f \in C_0^\infty(R^n)$ and $0 < p < 1$,

$$||\mu_\Omega^A(f)||_{L^p(w)} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}} \right) ||f||_{L^p(w)}$$

for any $w \in A_p$ and $1 < p < \infty$,

$$w(\{x \in R^n : \mu_\Omega^A(f)(x) > \lambda\}) \leq C \int_{R^n} \Phi\left(\lambda^{-1} \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}}\right) |f(x)|\right) w(x)dx$$

for any $w \in A_1$ and all $\lambda > 0$.

Application 3: Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$. Put

$$B_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x-y|^m} B_t^\delta(x-y) f(y)dy,$$

where $B_t^\delta(z) = t^{-n} B^\delta(z/t)$ for $t > 0$. The maximal Bochner-Riesz multilinear commutator is defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0} |B_{\delta,t}^A(f)(x)|;$$

We also put $B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$, which is the Bochner-Riesz operator (see [9], [10]).

Let H be the space $H = \left\{h : ||h|| = \sup_{t>0} |h(t)| < \infty\right\}$. It is clear that

$$B_*^\delta(f)(x) = ||B_t^\delta(f)(x)|| \text{ and } B_{\delta,*}^A(f)(x) = ||B_{\delta,t}^A(f)(x)||.$$

We know that $B_{\delta,*}$ is bounded on $L^p(R^n)$ for all $1 < p < \infty$ and weak (L^1, L^1)-bounded. Thus

$$(B_{\delta,*}^A(f))_p^\#(x) \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}}\right) M_{L(\log L)^{1/r}}(f)(x)$$

for any $f \in C_0^\infty(R^n)$ and $0 < p < 1$,

$$||B_{\delta,*}^A(f)||_{L^p(w)} \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}}\right) ||f||_{L^p(w)}$$

for any $w \in A_p$ and $1 < p < \infty$,

$$w(\{x \in R^n : B_{\delta,*}^A(f)(x) > \lambda\}) \leq C \int_{R^n} \Phi\left(\lambda^{-1} \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{\exp L^{r_j}}}\right) |f(x)|\right) w(x)dx$$

for any $w \in A_1$ and all $\lambda > 0$.

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