ACTA MATHEMATICA VIETNAMICA 167
Volume 31, Number 2, 2006, pp. 167-180

ON THE FUNCTIONAL EQUATION $P(f) = Q(g)$ IN NON-ARCHIMEDEAN FIELD

NGUYEN TRONG HOA

Abstract. In this paper, we study the existence of non-constant meromorphic solutions f and g of the functional equation $P(f) = Q(g)$, where $P(z)$ and $Q(z)$ are given nonlinear polynomials with coefficients in the non-Archimedean field $\mathbb{K}$.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field, complete for a non-trivial non-Archimedean absolute value, f be a non-constant meromorphic function and S be a subset of distinct elements in $\mathbb{K}$. Define

$$E_f(S) = \bigcup_{a \in S} \{(z, m) |\ z \text{ is zero of } f - a \text{ with multiplicity } m\}.$$

Two function f and g of the same type are said to *share S, counting multiplicity*, if $E_f(S) = E_g(S)$. A subset S is called an *unique range set* (a *URS* in short) for entire (or meromorphic) functions if for any two non-constant entire (or meromorphic) functions f and g such that $E_f(S) = E_g(S)$, one has f=g. Assume that S be a finite set, we set

$$P_S(z) = \prod_{a \in S} (x - a).$$

As a connection to the study of the uniqueness problem, Li and Yang ([3]) introduced the following definition.

Definition 1.1. A non-constant polynomial $P(z)$ is said to be an *unique polynomial* for entire (or meromorphic) functions if whenever $P(f) = P(g)$ for two non-constant entire (or meromorphic) functions f and g, it implies that $f = g$.

P(z) is said to be a *strong uniqueness polynomial* for entire (or meromorphic) functions if it satisfies the condition $P(f) = cP(g)$ for two non-constant entire (or meromorphic) functions f, g and some nonzero constant c, then it implies that $c = 1$ and $f = g$.

To demonstrate that the finite set S be a URS for entire (or meromorphic) functions, we prove that polynomial $P_S(z)$ is a strong uniqueness polynomial. If P is a strong uniqueness polynomial for entire (or meromorphic) functions, then the set of the zeros of P can be a URS.

Received November 08, 2005; in revised form April 12, 2006.
The author is partially supported by the National Basic Research Program of Vietnam.

Recently, H. H. Khoai and C. C. Yang ([1]) generalized the above studies by considering a pair of two nonlinear polynomials $P(z)$ and $Q(z)$ such that the only meromorphic solutions f, g satisfying $P(f) = Q(g)$ are constants. This problem is considered in the complex plane $\mathbb{C}$ by H. H. Khoai and C. C. Yang ([1]) as well as by C. C. Yang and P. Li ([2]).

In this paper, we find the conditions, such that functional equation $P(f) = Q(g)$ has no non-constant meromorphic solutions f, g in $\mathbb{K}$. To solve the functional equation, we study the hyperbolicity of the algebraic curve $\{P(x) - Q(y) = 0\}$ by estimating its genus. We shall do this by giving sufficiently many linear independent regular 1-forms of Wronskian type on that curve.

2. Main theorems

Definition 2.1. Let $P(z)$ be a nonlinear polynomial of degree n whose derivative is given by

$$P'(z) = c(z - \alpha_1)^{n_1} \dots (z - \alpha_k)^{n_k},$$

where $n_1 + \cdots + n_k = n - 1$ and $\alpha_1, \dots, \alpha_k$ are distinct zeros of P'. The number k is called *the derivative index of P*.

Polynomial $P(z)$ is said to satisfy *the condition separating the roots of P' (separation condition)* if $P(\alpha_i) \neq P(\alpha_j)$ for all $i \neq j = 1, 2, \dots, k$.

Let $P(x)$ and $Q(y)$ be two nonlinear polynomials of degrees n and m, respectively,

$$P(x) = a_n x^n + \dots + a_1 x + a_0, \quad Q(y) = b_m y^m + \dots + b_1 y + b_0. \tag{1}$$

Then, we have

$$P'(x) = n a_n (x - \alpha_1)^{n_1} \cdots (x - \alpha_k)^{n_k}, \tag{2}$$

$$Q'(y) = m b_m (y - \beta_1)^{m_1} \cdots (y - \beta_l)^{m_l}, \tag{3}$$

where $n_1 + \dots + n_k = n - 1$, $m_1 + \dots + m_l = m - 1$, $\alpha_1, \dots, \alpha_k$ are distinct zeros of P', and $\beta_1, \dots, \beta_l$ are distinct zeros of Q'. Define

$$\begin{aligned} \Delta &:= \{\alpha_i | \text{ there exist } \beta_j \text{ such that } P(\alpha_i) = Q(\beta_j)\}, \\ \Lambda &:= \{\beta_j | \text{ there exist } \alpha_i \text{ such that } P(\alpha_i) = Q(\beta_j)\}. \end{aligned} \tag{4}$$

Setting

$$I = \#\Delta, \; J = \#\Lambda. \tag{5}$$

Theorem 2.1. *Let $P(x), Q(y)$ be two nonlinear polynomials of degrees $n \geqslant m$, respectively, such that $P(x) - Q(y)$ has no linear factor. Suppose that k, l are the derivative indexes of P, Q, respectively. Then there exists no non-constant meromorphic functions f and g such that $P(f) = Q(g)$, if P and Q satisfy one of the following conditions*

(i) $k - I \geqslant n - m + 2$,

(ii) $l - J \geqslant 2$,

(iii) $k - I = 1$ *and* $n_1 \geqslant n - m + 2$, *where n_1 is the multiplicity of zero α_1 of P' such that $\alpha_1 \notin \Delta$,*

(iv) $l - J = 1$ *and* β_1 *is a unique zero of* Q' *such that* $\beta_1 \notin \Lambda$, *then* β_1 *is a multiple zero.*

Theorem 2.2. *Let* $P(x), Q(y)$ *be two nonlinear polynomials of degrees* n, m, *respectively,* $n \geqslant m$, *and* $P(x) - Q(y)$ *has no linear factor.* $\Lambda, \Delta, I, J, n_i, m_j$ *be defined as above. Rearrange* $\beta_j \in \Lambda$ *so that* $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_J$.

If $J \geqslant 2$, P *satisfies the separation condition and* $P(\alpha_{i_t}) = Q(\beta_t)$ *for* $i_t \in \{1, 2, \ldots, I\}$, $t = 1, 2$, *then there exists no pair of non-constant meromorphic functions* f *and* g *such that* $P(f) = Q(g)$ *if one of the following conditions is satisfied*

(i) $m_2 \geqslant 2$, $m_1 \geqslant n_{i_1}$ *and* $m_2 \geqslant n_{i_2}$, *or*

(ii) $n_{i_1} > m_1$, $m_2 \geqslant n_{i_2}, m_2 > 2$ *and* $\frac{m_1+1}{m_1} \geqslant \frac{n_{i_1}-m_1}{m_2-2}$, *or*

(iii) $m_1 \geqslant n_{i_1}$, $n_{i_2} > m_2 \geqslant 2, m_1 > 2$ *and* $\frac{m_2+1}{m_2} \geqslant \frac{n_{i_2}-m_2}{m_1-2}$, *or*

(iv) $n_{i_1} > m_1$, $n_{i_2} > m_2 > 2$, $\frac{m_1+1}{m_1} \geqslant \frac{n_{i_1}-m_1}{m_2-2}$ *and* $\frac{m_2+1}{m_2} \geqslant \frac{n_{i_2}-m_2}{m_1-2}$.

If $J = 1$ *and* $\beta_1 \in \Lambda$ *with multiplicities* m_1, *then there exists no non-constant meromorphic functions* f *and* g *such that* $P(f) = Q(g)$ *if*

$$\sum_{t|\alpha_t \in \Delta} n_t - (n - m + 2) \geqslant m_1 \geqslant \max_{t|\alpha_t \in \Delta}\{n_t\}.$$

Corollary 2.1. *If the hypotheses of Theorem 2.2 are satisfied, then there exists no pair of non-constant meromorphic functions* f *and* g *such that* $P(f) = Q(g)$ *provided* $J \geqslant 2$ *and* $m_1 + m_2 - 3 \geqslant \max\{n_{i_1}, n_{i_2}\}$.

In the case $P \equiv Q$, we obtain the following result.

Theorem 2.3. *Assume that* $P(z)$ *is nonlinear polynomial of degrees* n *and* P *satisfies the separation condition. Suppose that* $\alpha_1, \ldots, \alpha_k$ *are distinct zeros of* P' *with multiplicities* $n_1, \ldots, n_k$, *respectively. Rearrange* α_i *so that* $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k$. *Then there exists no non-constant meromorphic functions* $f \neq g$ *such that* $P(f) = P(g)$ *if and only if* $k \geqslant 3$ *or* $k = 2$ *and* $\min\{n_1, n_2\} \geqslant 2$.

3. Proofs of the main theorems

Suppose that $H(X, Y, Z)$ is a homogeneous polynomial of degree n and

$$C := \{(X : Y : Z) \in \mathbb{P}^2(\mathbb{K}) |\ H(X, Y, Z) = 0\}.$$

Put

$$W_1 = W(X, Y) = \begin{vmatrix} X & Y \\ dX & dY \end{vmatrix}, \quad W_2 = W(Y, Z) = \begin{vmatrix} Y & Z \\ dY & dZ \end{vmatrix},$$

$$W_3 = W(X, Z) = \begin{vmatrix} X & Z \\ dX & dZ \end{vmatrix}.$$

Assume that $R(X, Y, Z)$ and $S(X, Y, Z)$ are two homogeneous polynomials in $\mathbb{P}^2(\mathbb{K})$. Let

$$\omega_i = \frac{R(X, Y, Z)}{S(X, Y, Z)} W_i,$$

with $i = 1, 2, 3$. If $R(X, Y, Z)$ and $S(X, Y, Z)$ such that $\deg S = \deg R + 2$ then ω_i is a well-defined rational 1-form on $\mathbb{P}^2(\mathbb{K})$.

Definition 3.1. Let C be an algebraic curve in $\mathbb{P}^2(\mathbb{K})$. An 1-form ω on C is said to be *regular* if it is the pull-back of a rational 1-form on $\mathbb{P}^2(\mathbb{K})$ such that the pole set of ω does not intersect C. A well-defined rational regular 1-form on C is said to be an *1-form of Wronskian type.*

Notice that to solve the functional equation $P(f) = Q(g)$, is the same as to find meromorphic functions f, g on $\mathbb{K}$ such that $(f(z), g(z))$ in curve $\{P(x) - Q(y) = 0\}$. On the other hand, if C is hyperbolic on $\mathbb{K}$ and f, g be meromorphic functions such that $(f(z), g(z)) \in C$, for all $z \in \mathbb{K}$, then f and g are constant (see [6]). Therefore, to show that this equation has no non-constant solution, we shall prove the hyperbolicity of $\{P(x) - Q(y) = 0\}$. By Picard-Berkovich's theorem in the p-adic case, a curve C in $\mathbb{K}$ is hyperbolic if and only if the genus of the curve C is at least 1.

It is well-known that the genus g of a algebraic curve C is equal to the dimension of the space of regular 1-forms on C. Therefore, to compute the genus, we have to construct a basis of the space of regular 1-forms on C.

Let P and Q be two nonlinear polynomials of degrees n and m, respectively, in $\mathbb{K}$, defined by (1). Without loss of generality, we can assume that $n \geqslant m$. We set

$$H(x, y) := P(x) - Q(y).$$

$$F(X, Y, Z) := Z^n \Big\{ P(\frac{X}{Z}) - Q(\frac{Y}{Z}) \Big\}. \tag{6}$$

$$C := \{(X : Y : Z) \in \mathbb{P}^2(\mathbb{K}) \mid F(X, Y, Z) = 0\}. \tag{7}$$

Define

$$P'(X, Z) := Z^{n-1} P'(\frac{X}{Z}) \ , \ Q'(Y, Z) := Z^{m-1} Q'(\frac{Y}{Z}),$$

then

$$\begin{aligned}
\frac{\partial F}{\partial X} &= P'(X, Z), \\
\frac{\partial F}{\partial Y} &= -Z^{n-m} Q'(Y, Z), \\
\frac{\partial F}{\partial Z} &= \sum_{i=0}^{n-1} (n - i) a_i X^i Z^{n-1-i} - \sum_{j=0}^{m'} (n - j) b_j Y^j Z^{n-1-j},
\end{aligned}$$

where

$$m' = \begin{cases} n - 1 & \text{if} \quad n = m \\ m & \text{if} \quad n > m. \end{cases}$$

Then, by Euler's theorem, for all points $(X : Y : Z) \in C$, we have

$$\frac{\partial F}{\partial X} X + \frac{\partial F}{\partial Y} Y + \frac{\partial F}{\partial Z} Z = nF(X, Y, Z) = 0. \tag{8}$$

The equation of the tangent space of C the point $(X : Y : Z) \in C$ is defined by

$$\frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial Y}dY + \frac{\partial F}{\partial Z}dZ = 0. \tag{9}$$

From (8) and (9), we obtain

$$\frac{\partial F}{\partial X} = \frac{\begin{vmatrix} Y & Z \\ dY & dZ \end{vmatrix}}{\begin{vmatrix} X & Y \\ dX & dY \end{vmatrix}} \frac{\partial F}{\partial Z}, \quad \frac{\partial F}{\partial Y} = \frac{\begin{vmatrix} Z & X \\ dZ & dX \end{vmatrix}}{\begin{vmatrix} X & Y \\ dX & dY \end{vmatrix}} \frac{\partial F}{\partial Z}.$$

Hence,

$$\frac{W(Y,Z)}{\dfrac{\partial F}{\partial X}} = \frac{W(Z,X)}{\dfrac{\partial F}{\partial Y}} = \frac{W(X,Y)}{\dfrac{\partial F}{\partial Z}}. \tag{10}$$

Setting

$$\eta := \frac{W(Y,Z)}{\dfrac{\partial F}{\partial X}} = \frac{W(Z,X)}{\dfrac{\partial F}{\partial Y}} = \frac{W(X,Y)}{\dfrac{\partial F}{\partial Z}},$$

we obtain

$$\begin{aligned} \eta &= \frac{W(Y,Z)}{P'(X,Z)} = \frac{W(X,Z)}{Z^{n-m}Q'(Y,Z)} \\ &= \frac{W(X,Y)}{\sum_{i=0}^{n-1}(n-i)a_i X^i Z^{n-1-i} - \sum_{j=0}^{m'}(n-j)b_j Y^j Z^{n-1-j}}. \end{aligned} \tag{11}$$

In order to prove the main results, we need the following lemmas.

Lemma 3.1. *Let P, Q be two nonlinear polynomials of degrees n, m, respectively, where, $n \geqslant m$, and C be a projective curve defined by (7). If $P(\alpha_i) \neq Q(\beta_j)$ for all zeros α_i of P' and β_j of Q', then we have the following assertions*

(i) *If $n = m$ or $n = m+1$, then C is non-singular in $\mathbb{P}^2(\mathbb{K})$.*

(ii) *If $n - m \geqslant 2$, then the point $(0 : 1 : 0)$ be a unique singular point of C in $\mathbb{P}^2(\mathbb{K})$.*

Proof. By the hypothesis of the Lemma, $P(\alpha_i) \neq Q(\beta_j)$ for all zeros α_i of P' and β_j of Q', we conclude that C is non-singular in $\mathbb{P}^2(\mathbb{K}) \setminus [Z = 0]$. Now we consider the singularity of C in $[Z = 0]$. Assume that $(X : Y : 0)$ is a singular point of C. We have

$$\frac{\partial F}{\partial X}(X, Y, 0) = \frac{\partial F}{\partial Y}(X, Y, 0) = \frac{\partial F}{\partial Z}(X, Y, 0) = 0.$$

If $n = m$ or $n = m + 1$, then the above system has no root in $\mathbb{P}^2(\mathbb{K})$.

If $n - m \geqslant 2$, then the system has a unique root $(0 : 1 : 0)$ in $\mathbb{P}^2(\mathbb{K})$.

Thus, if $n = m$ or $n = m + 1$ then C is a smooth curve. If $n - m \geqslant 2$ then C is singular with a unique singular point at $(0 : 1 : 0)$. $\square$

Remark 3.1. (i) We also require that the 1-form, defined by (11), is non trivial when it restricts to a component of $\mathbb{K}$. This is equivalent to the condition that the nominators are not identically zero when they restrict to a component of C i.e., the Wronskians $W(X,Y), W(X,Z), W(Y,Z)$ are not identically zero. It means that the homogeneous polynomial defining C has no linear factors of the forms $aX-bY,\ aY-bZ,$ or $aX-bZ$, with $a,b\in\mathbb{K}$ if $P\neq Q$. Indeed, on the contrary, suppose that $aX-bZ$ is a factor of curve C defined by (7). Without loss of generality, we can take $a\neq 0$. Since $aX-bZ$ is a factor of $F(X,Y,Z)$, we have

$$0=F(\frac{b}{a}Z,Y,Z)=Z^n\{P(\frac{\frac{b}{a}Z}{Z})-Q(\frac{Y}{Z})\}=Z^n\{P(\frac{b}{a})-Q(\frac{Y}{Z})\},$$

which gives $P(\frac{b}{a})\equiv Q(\frac{Y}{Z})$ for all Y,Z, a contradiction.

(ii) Assume that $P(\alpha_i)\neq Q(\beta_j)$ for all zeros α_i of P' and β_j of Q' and $m>n$. If $m=n+1$ then C is non-singular in $\mathbb{P}^2(\mathbb{K})$. If $m-n\geqslant 2$ then the point $(1:0:0)$ is an unique singular point of C in $\mathbb{P}^2(\mathbb{K})$.

Next, we recall the following notations. Let C be a curve on $\mathbb{K}$ defined by a homogeneous polynomial $F(X,Y,Z)=0$ and let ρ be a point of C. A holomorphic map

$$\phi=(\phi_1,\phi_2,\phi_3):\triangle_\epsilon=\{t\in\mathbb{K}||t|<\epsilon\}\Longrightarrow C,$$

with $\phi(0)=\rho$, is referred to a *holomorphic parameterization* of C at ρ. Local holomorphic parameterization always exists for sufficiently small ϵ. If ϕ is a local holomorphic parameterization of C at ρ, then the Laurent expansion of $F\circ\phi(t)$ at ρ has the form

$$F\circ\phi(t)=\sum_{i=p}^{q}c_it^i,\quad c_p\neq 0.$$

The order of F at ρ (it is also the order of $F\circ\phi(t)$ at $t=0$) is defined by p and denoted by

$$p:=\mathrm{ord}_{\rho,\phi}F=\mathrm{ord}_{t=0}F(\phi(t)).$$

Assume that $\varphi(x,y)$ is an analytic function in x,y and is singular at (a,b). The Puiseux expansion of $\varphi(x,y)$ at $\rho:=(a,b)$ is given by

$$[x=a+a_\alpha t^\alpha+\text{higher terms},\quad y=b+b_\beta t^\beta+\text{higher terms}],$$

where $\alpha,\beta\in\mathbb{N}^*$ and $a_\alpha,b_\beta\neq 0$. The α (respectively, β) is the order (also the multiplicity number) of x at ρ (respectively, the order of y at ρ) for F and is denoted by

$$\alpha:=\mathrm{ord}_{\rho,\varphi}(x)\quad(\text{respectively, }\beta:=\mathrm{ord}_{\rho,\varphi}(y)).$$

Denote by $\alpha_1,\ldots,\alpha_k$ zeros of P' with multiplicities $n_1,\ldots,n_k$, and by $\beta_1,\ldots,\beta_l$ zeros of Q' with multiplicities $m_1,\ldots,m_l$, respectively, then singularities of C in $\mathbb{P}^2(\mathbb{K})\backslash[Z=0]$ are $(\alpha_i:\beta_j:1)$, which satisfy $P(\alpha_i)=Q(\beta_j)$. Let

$$\begin{aligned}&\Gamma:=\{(\alpha_i:\beta_j:1)\mid(\alpha_i:\beta_j:1)\text{ is a singular point of }C\},\\ &\Delta:=\{\alpha_i\mid(\alpha_i:\beta_j:1)\text{ is a singular point of }C\},\\ &\Lambda:=\{\beta_j\mid(\alpha_i:\beta_j:1)\text{ is a singular point of }C\}.\end{aligned}\tag{12}$$

Setting $I = \#\Delta$, $J = \#\Lambda$, we have $k \geqslant I$, $l \geqslant J$. Without loss of generality, we can take

(13) $$\Delta = \{\alpha_1, \dots, \alpha_I\}, \ \ \Lambda = \{\beta_1, \dots, \beta_J\} \ \text{and} \ m_1 \geqslant m_2 \geqslant \dots \geqslant m_J.$$

Lemma 3.2. *Suppose that $\Delta, \Lambda, \alpha_i, \beta_j, n_i, m_j$ be defined as above. Then, the 1-forms*

$$\theta := \frac{W(X,Z)}{\prod_{j|\beta_j \notin \Lambda}(Y - \beta_j Z)^{m_j}},$$
$$\sigma := \frac{Z^{n-m} W(Y,Z)}{\prod_{i|\alpha_i \notin \Delta}(X - \alpha_i Z)^{n_i}},$$

are regular on C.

Proof. By the hypotheses of the lemma, θ is regular on C because no point of the set $\{(\alpha_i : \beta_j : 1) |\ \beta_j \notin \Lambda, i = 1, 2, \dots, k\}$ is in C.

Note that

$$\begin{aligned}\sigma &= \frac{Z^{n-m}\prod_{i|\alpha_i \in \Delta}(X - \alpha_i Z)^{n_i}}{\prod_{j=1}^{k}(X - \alpha_j Z)^{n_j}} W(Y,Z)\\ &= \frac{pZ^{n-m}\prod_{i|\alpha_i \in \Delta}(X - \alpha_i Z)^{n_i}}{P'(X,Z)} W(Y,Z)\\ &= \frac{p\prod_{i|\alpha_i \in \Delta}(X - \alpha_i Z)^{n_i}}{Q'(Y,Z)} W(X,Z),\end{aligned}$$

where, $p = na_n \neq 0$. Because $Q'(Y,Z)\ |_{X=0,Y=1,Z=0} = mb_m \neq 0$ and no point of the set $\{(\alpha_i : \beta_j : 1) |\ \alpha_i \notin \Delta, j = 1, 2, \dots, l\}$ is in C, σ is regular on C. □

Proposition 3.1. *Suppose that $n \geqslant m$, $P(x) - Q(y)$ has no linear factor and $k, l, \Delta, \Lambda, I, J, n_i, m_j$ be defined as above. Then the curve C is hyperbolic if one of following conditions is satisfied*

(i) $\sum_{i|\alpha_i \notin \Delta} n_i \geqslant n - m + 2$,

(ii) $\sum_{j|\beta_j \notin \Lambda} m_j \geqslant 2$.

Proof. Set

$$\vartheta := Z^{\sum_{j|\beta_j \notin \Lambda} m_j - 2}\theta.$$

By Lemma 3.2, ϑ is a well-defined regular 1-form of Wronskian type on C if $\sum_{j|\beta_j \notin \Lambda} m_j \geqslant 2$. Hence $g_C \geqslant 1$, that is, C is hyperbolic, if $\sum_{j|\beta_j \notin \Lambda} m_j \geqslant 2$.

Setting

$$\varsigma := Z^{\sum_{i|\alpha_i \notin \Delta} n_i - (n-m+2)}\sigma,$$

and arguing similarly as above, we can that the curve C is hyperbolic if

$$\sum_{i|\alpha_i \notin \Delta} n_i \geqslant n - m + 2.$$

This completes the proof. □

Assume that $(\alpha_i : \beta_j : 1)$ is a singular point of C. Then

$$P(x) - P(\alpha_i) = \sum_{t=n_i+1}^{n} a_t (x-\alpha_i)^t,$$

$$Q(y) - Q(\beta_j) = \sum_{t=m_j+1}^{m} b_t (y-\beta_j)^t,$$

when $a_{n_i+1} \neq 0, b_{m_j+1} \neq 0, P(\alpha_i) = Q(\beta_j)$. Therefore, we have

$$\begin{aligned} F(X,Y,Z) &= Z^n \{P(\frac{X}{Z}) - Q(\frac{Y}{Z})\} \\ &= Z^n \left\{ \{P(\frac{X}{Z}) - P(\alpha_i)\} - \{Q(\frac{Y}{Z}) - Q(\beta_j)\} \right\} \\ &= \sum_{t=n_i+1}^{n} a_t (X-\alpha_i Z)^t - Z^{n-m} \sum_{t=m_j+1}^{m} b_t (Y-\beta_j Z)^t. \end{aligned}$$

Using the Puiseux expansion of $F(X,Y,Z)$ at $\rho_{ij} = (\alpha_i : \beta_j : 1)$, we have

$$(n_i+1)\mathrm{ord}_{\rho_{ij},F}(X-\alpha_i Z) = (m_j+1)\mathrm{ord}_{\rho_{ij},F}(Y-\beta_j Z). \tag{14}$$

Suppose that $\rho_1 = (\alpha_{i_1} : \beta_{j_1} : 1)$ and $\rho_2 = (\alpha_{i_2} : \beta_{j_2} : 1)$ are two distinct finite singular points of C. Setting

$$L_{12} := \begin{cases} (X-\alpha_{i_1}Z) - \frac{\alpha_{i_2}-\alpha_{i_1}}{\beta_{j_2}-\beta_{j_1}}(Y-\beta_{j_1}Z) & \text{if } \beta_{j_1} \neq \beta_{j_2} \\ (Y-\beta_{j_2}Z) - \frac{\beta_{j_2}-\beta_{j_1}}{\alpha_{i_2}-\alpha_{i_1}}(X-\alpha_{i_2}Z) & \text{if } \alpha_{i_1} \neq \alpha_{i_2}, \end{cases}$$

we conclude that $L_{12}(\alpha_{i_1}, \beta_{j_1}, 1) = L_{12}(\alpha_{i_2}, \beta_{j_2}, 1) = 0$ and

$$\mathrm{ord}_{\rho_t,F} L_{12} \geqslant \min\{\mathrm{ord}_{\rho_t,F}(X-\alpha_{i_t}Z), \mathrm{ord}_{\rho_t,F}(Y-\beta_{j_t}Z)\}.$$

Hence, by (14) we have

$$\mathrm{ord}_{\rho_t,F} L_{12} \geqslant \begin{cases} \mathrm{ord}_{\rho_t,F}(X-\alpha_{i_t}Z) & \text{if } m_{j_t} < n_{i_t} \\ \mathrm{ord}_{\rho_t,F}(Y-\beta_{j_t}Z) & \text{if } m_{j_t} \geqslant n_{i_t} \end{cases} \tag{15}$$

for $t = 1, 2$.

We have the following proposition.

Proposition 3.2. *Let P, Q be nonlinear polynomials such that $P(x) - Q(y)$ has no linear factor. Let C be a projective curve defined by (7), $\Gamma = \{(\alpha_{i_j} : \beta_j : 1)\}$ be the set of all finite singular points of C, and let $\Lambda = \{\beta_1, \ldots, \beta_J\}$ (defined by (12)), where $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_J$. In addition, assume that $(\alpha_{i_1} : \beta_1 : 1)$, $(\alpha_{i_2} : \beta_2 : 1) \in \Gamma$, and P satisfies the separation condition. Then the curve C is hyperbolic if $J \geqslant 2$ and one of following conditions is satisfied*

(i) $m_2 \geqslant 2$, $m_1 \geqslant n_{i_1}$ *and* $m_2 \geqslant n_{i_2}$, *or*

(ii) $n_{i_1} > m_1$, $m_2 \geqslant n_{i_2}, m_2 > 2$ *and* $\frac{m_1+1}{m_1} \geqslant \frac{n_{i_1}-m_1}{m_2-2}$, *or*

(iii) $m_1 \geqslant n_{i_1}$, $n_{i_2} > m_2 \geqslant 2, m_1 > 2$ *and* $\frac{m_2+1}{m_2} \geqslant \frac{n_{i_2}-m_2}{m_1-2}$, *or*

(iv) $n_{i_1} > m_1$, $n_{i_2} > m_2 > 2$, $\frac{m_1+1}{m_1} \geqslant \frac{n_{i_1}-m_1}{m_2-2}$ *and* $\frac{m_2+1}{m_2} \geqslant \frac{n_{i_2}-m_2}{m_1-2}$.

Proof. By the hypotheses, if $\rho_1 = (\alpha_{i_1} : \beta_1 : 1) \neq \rho_2 = (\alpha_{i_2} : \beta_2 : 1)$ then $\beta_1 \neq \beta_2$. Indeed, assume to the contrary that $\beta_1 = \beta_2$. Since $\rho_1 \neq \rho_2$, we obtain $\alpha_{i_1} \neq \alpha_{i_2}$. Hence $P(\alpha_{i_1}) = Q(\beta_1) = Q(\beta_2) = P(\alpha_{i_2})$, which is a contradiction. Let

$$L := (X - \alpha_{i_1} Z) - \frac{\alpha_{i_2} - \alpha_{i_1}}{\beta_2 - \beta_1}(Y - \beta_1 Z).$$

By (14) and (15) we get

$$\text{ord}_{\rho_t,F} L \geqslant \begin{cases} \text{ord}_{\rho_t,F}(X - \alpha_{i_t} Z) & \text{if } m_t < n_{i_t} \\ \text{ord}_{\rho_t,F}(Y - \beta_t Z) & \text{if } m_t \geqslant n_{i_t} \end{cases} \tag{16}$$

for $t = 1, 2$. The rational 1-form

$$\omega := \frac{L^{m_1+m_2-2}}{(Y - \beta_1 Z)^{m_1}(Y - \beta_2 Z)^{m_2}} W(X, Z),$$

is well-defined (since $m_1 \geqslant m_2 \geqslant 1$). We claim that ω is regular. To prove this we need only to check the regularity at $\rho_t = (\alpha_{i_t} : \beta_t : 1)$, for $t = 1, 2$. The ω is regular at ρ_t if the 1-forms

$$\chi_t := \frac{L^{m_1+m_2-2}}{(Y - \beta_t Z)^{m_t}} W(X, Z),$$

are regular at ρ_t with $t = 1, 2$.

First of all, we check the regularity of χ_1 at ρ_1. If $m_1 \geqslant n_{i_1}$, by (16) we have

$$\text{ord}_{\rho_1,F} \frac{L^{m_1+m_2-2}}{(Y - \beta_1 Z)^{m_1}} \geqslant (m_2 - 2)\text{ord}_{\rho_1,F}(Y - \beta_1 Z). \tag{17}$$

If $n_{i_1} > m_1$, by (16), we obtain

$$\begin{aligned} \text{ord}_{\rho_1,F} \frac{L^{m_1+m_2-2}}{(Y - \beta_1 Z)^{m_1}} &= (m_1 + m_2 - 2)\text{ord}_{\rho_1,F}(X - \alpha_{i_1} Z) - m_1 \text{ord}_{\rho_1,F}(Y - \beta_1 Z) \\ &= \frac{(m_1 + 1)(m_2 - 2) - m_1(n_{i_1} - m_1)}{n_{i_1} + 1} \text{ord}_{\rho_1,F}(Y - \beta_1 Z). \end{aligned} \tag{18}$$

From (17) and (18) it follows that

$$\text{ord}_{\rho_1,F} \frac{L^{m_1+m_2-2}}{(Y - \beta_1 Z)^{m_1}} \geqslant \begin{cases} (m_2 - 2)\text{ord}_{\rho_1,F}(Y - \beta_1 Z) & \text{if } \ m_1 \geqslant n_{i_1} \\ \frac{(m_1+1)(m_2-2)-m_1(n_{i_1}-m_1)}{n_{i_1}+1} \text{ord}_{\rho_1,F}(Y - \beta_1 Z) & \text{if } \ m_1 < n_{i_1}. \end{cases}$$

Thus, χ_1 is regular at ρ_1 if one of following conditions is satisfied

(i) $m_1 \geqslant n_{i_1}$ and $m_2 \geqslant 2$, or

(ii) $n_{i_1} > m_1 \geqslant m_2 > 2$ and $\frac{m_1+1}{m_1} \geqslant \frac{n_{i_1}-m_1}{m_2-2}$.

The regularity of χ_2 at ρ_2 can be checked similarly. Thus, ω is regular on C if one of conditions of the proposition is satisfied. □

In the case $J = \#\Lambda = 1$, we obtain following result.

Proposition 3.3. *Let P, Q be two nonlinear polynomials such that $P(x) - Q(y)$ has no linear factor, C be a projective curve defined by (7). Assume that $\Gamma = \{(\alpha_i : \beta_1 : 1)\}$ is the set of all finite singular points of C, where $\alpha_1, \alpha_2, \ldots, \alpha_I$ are* *zeros of P' with multiplicities $n_1, n_2, \ldots, n_I$, respectively; β_1 is zero of Q' with multiplicities m_1. Then the curve C is hyperbolic if*

$$\sum_{i=1}^{I} n_i - (n-m+2) \geqslant m_1 \geqslant \max_{I\geqslant i\geqslant 1}\{n_i\}.$$

Proof. Let

$$\varsigma := \frac{Z^{\sum_{i=1}^{I} n_i-(2+m_1)}(Y-\beta_1 Z)^{m_1}}{\prod_{i=1}^{I}(X-\alpha_i Z)^{n_i}} W(Y,Z).$$

Then

$$\begin{aligned}
\varsigma &= \frac{Z^{\sum_{i=1}^{I} n_i-(2+m_1)}(Y-\beta_1 Z)^{m_1}\prod_{j|\alpha_j\notin\Delta}(X-\alpha_j Z)^{n_j}}{\prod_{i=1}^{k}(X-\alpha_i Z)^{n_i}} W(Y,Z)\\
&= \frac{pZ^{\sum_{i=1}^{I} n_i-(2+m_1)}(Y-\beta_1 Z)^{m_1}\prod_{j|\alpha_j\notin\Delta}(X-\alpha_j Z)^{n_j}}{P'(X,Z)} W(Y,Z)\\
&= \frac{pZ^{\sum_{i=1}^{I} n_i-(n-m+2+m_1)}Z^{n-m}(Y-\beta_1 Z)^{m_1}\prod_{j|\alpha_j\notin\Delta}(X-\alpha_j Z)^{n_j}}{Z^{n-m}Q'(Y,Z)} W(X,Z)\\
&= \frac{pZ^{\sum_{i=1}^{I} n_i-(n-m+2+m_1)}\prod_{j|\alpha_j\notin\Delta}(X-\alpha_j Z)^{n_j}}{\prod_{i=2}^{l}(Y-\beta_i Z)^{m_i}} W(X,Z),
\end{aligned}$$

where $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_I\}$, $p = na_n \neq 0$, is regular in $C \cap [Z=0]$ if

$$\sum_{i=1}^{I} n_i - (n-m+2+m_1) \geqslant 0.$$

By (14),

$$(n_i+1)\mathrm{ord}_{\rho_i,F}(X-\alpha_i Z) = (m_1+1)\mathrm{ord}_{\rho_i,F}(Y-\beta_1 Z),$$

We have

$$\begin{aligned}
&m_1\mathrm{ord}_{\rho_i,F}(Y-\beta_1 Z) - n_i\mathrm{ord}_{\rho_i,F}(X-\alpha_i Z)\\
&= \mathrm{ord}_{\rho_i,F}(X-\alpha_i Z) - \mathrm{ord}_{\rho_i,F}(Y-\beta_1 Z),
\end{aligned}$$

for all $\alpha_i \in \Delta$ and $\rho_i := (\alpha_i : \beta_1 : 1)$. Hence, ς is regular at point ρ_i if

$$\mathrm{ord}_{\rho_i,F}(X-\alpha_i Z) - \mathrm{ord}_{\rho_i,F}(Y-\beta_1 Z) \geqslant 0,$$

that is, $m_1 \geqslant n_i$ for all i such that $\alpha_i \in \Delta$. Therefore, ς is regular in C if

$$\sum_{i=1}^{I} n_i - (n-m+2) \geqslant m_1 \geqslant \max_{I\geqslant i\geqslant 1}\{n_i\}.$$

This completes the proof. □

Remark 3.2. If $m_1 + m_2 - 3 \geqslant \max\{n_{i_1}, n_{i_2}\}$, then

$$\frac{(m_1+1)(m_2-2)-m_1(n_{i_1}-m_1)}{n_{i_1}+1} = \frac{(m_1+m_2-2)(m_1+1)}{n_{i_1}+1} - m_1 \geqslant 1,$$

$$\frac{(m_2+1)(m_1-2)-m_2(n_{i_2}-m_2)}{n_{i_2}+1} = \frac{(m_1+m_2-2)(m_2+1)}{n_{i_2}+1} - m_2 \geqslant 1.$$

Thus, we have $\omega = \dfrac{L^{m_1+m_2-2}}{(Y-\beta_1 Z)^{m_1}(Y-\beta_2 Z)^{m_2}} W(X,Z)$ is regular on C.

Lemma 3.3. *If $k = I = J = l = 1$, then there exist non-constant meromorphic functions f, g such that $P(f) = Q(g)$.*

Proof. If $k = I = J = l = 1$, then can rewrite the equation $P(f) = Q(g)$ in the form $(f-\alpha)^n = (bg-\beta)^m$, where $b \neq 0$. Assume that h is a non-constant meromorphic function. Set

$$f = \alpha + h^m, \ g = \frac{1}{b}h^n + \frac{\beta}{b}.$$

Then f and g are non-constant meromorphic solutions of equation $P(f) = Q(g)$. □

Proof of Theorem 2.1. From Proposition 3.1, if $\sum_{j|\beta_j \notin \Lambda} m_j - 2 \geqslant 0$, i.e., $p = \sum_{j|\beta_j \notin \Lambda} m_j \geqslant 2$, then the functional equation $P(f) = Q(g)$ has no solution in the set of non-constant meromorphic functions. As $m_j \geqslant 1$, we conclude that if $l - J \geqslant 2$ then $p \geqslant 2$. If $l - J = 1$, then there only exists a unique zero β_1 with multiplicity m_1 of Q' such that $P(\alpha) \neq Q(\beta_1)$, with all zeros α of P'. Since $m_1 \geqslant 2$, we have $p = m_1 \geqslant 2$. Therefore, (ii) and (iv) are valid.

Note that $\sum_{i|\alpha_i \notin \Delta} n_i \geqslant k - I$. Therefore, if $k - I \geqslant n - m + 2$ then the curve C is hyperbolic. If $k - I = 1$ and $n_1 \geqslant n - m + 2$, then

$$\sum_{i|\alpha_i \notin \Delta} n_i = n_1 \geqslant n - m + 2.$$

Thus, we obtain (i) and (iii). This completes the proof. □

Proof of Theorem 2.2 and Corollary 2.1. Theorem 2.2 can be derived from Propositions 3.2 and 3.3. Corollary 2.1 follows from Theorem 2.2 and Remark 3.2. □

Proof of Theorem 2.3. Let

$$H^*(x,y) := \frac{P(x)-P(y)}{x-y}.$$

$$F^*(X,Y,Z) := Z^{n-1}H^*(\frac{X}{Z}, \frac{Y}{Z}).$$

$$C^* := \{(X,Y,Z) \in \mathbb{P}^2(\mathbb{K}) \ |F^*(X,Y,Z) = 0\}.$$

By Remark 3.1, $F^*(X,Y,Z)$ has no factor of the forms $aX - bY,\ aX - bZ,\ aY - bZ$. Assume that $F^*(X,Y,Z)$ has no factor of the form $aX + bY + cZ$. Then, the curve

C^* is only singular in $\mathbb{P}^2(\mathbb{K})$ at $\rho_i = (\alpha_i : \alpha_i : 1)$, with $\{\alpha_i \mid i = 1, 2, \dots, k\}$ being the set of distinct zeros of P'. We have

$$\begin{aligned}\frac{\partial F^*}{\partial X} &= \frac{P'(X,Z) - F^*(X,Y,Z)}{X - Y},\\ \frac{\partial F^*}{\partial Y} &= \frac{-P'(Y,Z) + F^*(X,Y,Z)}{X - Y},\\ \frac{\partial F^*}{\partial Z} &= \sum_{i=1}^{n-1}(n-i)a_i Z^{n-i-1}\sum_{t=0}^{i-1} X^{i-t-1}Y^t.\end{aligned}$$

Note that if $(X : Y : Z) \in C^*$ then $F^*(X,Y,Z) = 0$. From (11) we obtain

$$\begin{aligned}\eta &= \frac{(X-Y)W(Y,Z)}{P'(X,Z)} = \frac{(X-Y)W(X,Z)}{P'(Y,Z)}\\ &= \frac{(X-Y)W(X,Y)}{\sum_{i=1}^{n-1}(n-i)a_i(X^i - Y^i)Z^{n-1-i}}.\end{aligned}$$

Let

$$\theta := na_n(X-Y)^{n-4}\eta = \frac{(X-Y)^{n-3}W(X,Z)}{(Y-\alpha_1 Z)^{n_1}\dots(Y-\alpha_k Z)^{n_k}}.$$

Since $\mathrm{ord}_{\rho_i,F^*}(X-Y) \geqslant \mathrm{ord}_{\rho_i,F^*}(X-\alpha_i Z) = \mathrm{ord}_{\rho_i,F^*}(Y-\alpha_i Z)$,

$$\begin{aligned}\mathrm{ord}_{\rho_i,F^*}\frac{(X-Y)^{n-3}}{\prod_{t=1}^k (Y-\alpha_t Z)^{n_t}} &= (n-3)\mathrm{ord}_{\rho_i,F^*}(X-Y) - n_i\mathrm{ord}_{\rho_i,F^*}(Y-\alpha_i Z)\\ &\geqslant (n-n_i-3)\mathrm{ord}_{\rho_i,F^*}(Y-\alpha_i Z)\\ &= (\sum_{i\neq t=1}^{k} n_t - 2)\mathrm{ord}_{\rho_i,F^*}(Y-\alpha_i Z).\end{aligned}$$

This implies that if $\sum_{i\neq t=1}^k n_t \geqslant 2$ then θ is regular at ρ_i, with $i = 1, 2, \dots, k$. Since $n_1 \geqslant n_2 \geqslant \dots \geqslant n_k \geqslant 1$, we conclude that if $k \geqslant 3$ or $k = 2$ and $\min\{n_1, n_2\} \geqslant 2$ then the curve C^* is hyperbolic.

Now we consider the cases $k = 1$ and $k = 2$, $\min\{n_1, n_2\} < 2$.

If $k = 1$ then $P(x) = a(x-\alpha)^n + b$, with $a, b \in \mathbb{K}$, $a \neq 0$. Let $1 \neq \epsilon \in \mathbb{K}$ such that $\epsilon^n = 1$ and h is any non-constant meromorphic function. We set $f = h + \alpha$, $g = \epsilon h + \alpha$. Then $P(f) = Q(g)$.

In the case $k = 2$ and $\min\{n_1, n_2\} < 2$, we have $n_1 = n_2 = 1$ or $n_1 \geqslant 2$, $n_2 = 1$. If $n_1 = n_2 = 1$, then $n = 3$ and $P = ax^3 + bx^2 + cx + d$ with $a \neq 0$, $b^2 - 3ac \neq 0$. From the equation $P(f) = P(g)$ and the fact $f \neq g$ we have

$$a(f+g)^2 + b(f+g) + c = afg.$$

Let $f = u + v$, $g = u - v$. We observe that

$$(u - \frac{i}{\sqrt{3}}v + \frac{b}{3a})(u + \frac{i}{\sqrt{3}}v + \frac{b}{3a}) = \frac{b^2 - 3ac}{9a^2},$$

with $i^2 = -1$. Assume that h is any non-constant meromorphic function. Setting

$$u - \frac{i}{\sqrt{3}}v + \frac{b}{3a} = \frac{b^2 - 3ac}{9a^2}h, \quad u + \frac{i}{\sqrt{3}}v + \frac{b}{3a} = \frac{1}{h},$$

we see that

$$f = \left\{\frac{b^2 - 3ac}{9a^2}\right\}\left\{\frac{i - \sqrt{3}}{2i}\right\}h + \left\{\frac{i + \sqrt{3}}{2i}\right\}\frac{1}{h} - \frac{b}{3a}$$

and

$$g = \left\{\frac{b^2 - 3ac}{9a^2}\right\}\left\{\frac{i + \sqrt{3}}{2i}\right\}h + \left\{\frac{i - \sqrt{3}}{2i}\right\}\frac{1}{h} - \frac{b}{3a}$$

constitute a solution of the equation $P(f) = P(g)$.

If $k = 2, n_1 \geqslant 2$ and $n_2 = 1$, by Proposition 1 ([9]) the curve C^* has only one singular point $\rho_1 = (\alpha_1 : \alpha_1 : 1)$ with multiplicity n_1. Assume that F^* is reducible at ρ_1, i.e., $F^* = HG$ where H is a proper irreducible factor of F^*. Let n_H, n_G be the multiplicity of ρ_1 in $H = 0$ and $G = 0$, respectively. Then we have $n_H + n_G = n_1$ and $\deg H + \deg G = \deg F^* = n - 1 = n_1 + 1$. Since $\deg H \geqslant n_H$, $\deg G \geqslant n_G$, by Bezout's theorem we obtain $n_H n_G = (\deg H)(\deg G)$. Then we have $n_H = \deg H$, $n_G = \deg G$ and $n_1 = n_1 + 1$, a contradiction. Therefore, F^* is irreducible and curve C^* has genus zero. This completes the proof. $\square$

Acknowledgments. I am grateful to the referee, the editor and Dr. Ta Thi Hoai An for several useful suggestions and comments to improve the quality of this paper. The author is partially supported by the National Basic Research Program of Vietnam

References

[1] H. H. Khoai and C. C. Yang, *On the functional equation $P(f) = Q(g)$*, Value Distribution Theory, Kluwer Academic Publishers, 2004, pp. 201 - 207.

[2] C. C. Yang and P. Li, *Some further results on the functional equation $P(f) = Q(g)$*, Value Distribution Theory and its Trends, ACAA, Kluwer Academic Publishers, 2004.

[3] C. C. Yang and P. Li, *On the unique range sets of meromorphic functions, Proc. Amer. Math. Soc.* **124** (1996), 177-195.

[4] A. Escassut and C. C. Yang, *The functional equation $P(f) = Q(g)$ in a p-adic field,* J. Number Theory, **105** (2004), 344-360.

[5] F. Gross and C. C. Yang, *On preimages and range sets of meromorphic functions,* Proc. Japan Acad., **58** (1982), 17-20.

[6] P. C. Hu and C. C. Yang, *Meromorphic functions over non-Archimedean field,* Kluwer Academic Publishers, 2000.

[7] H. Fujimoto, *On uniqueness of meromorphic functions sharing finite sets,* Amer. J. Math. **122** (2000), 1175-1203.

[8] B. Shiffman, *Uniqueness of entire and meromorphic functions sharing finite sets,* Complex Variables **4** (2001), 433-450.

[9] J. T-Y. Wang, *Uniqueness polynomials and bi-unique range sets for rational functions and non-Achimedean meromorphic functions,* Acta Arith. **104** (2002), 183-200.

[10] T. T. H. An, J. T-Y. Wang and P-M. Wong, *Strong uniqueness polynomials: The complex case,* Complex Var. Theory Appl. **49** (2004), 25-54.

[11] F. Gross, *Factorization of meromorphic functions and some open problems,* Complex analytic (Proc. Conf., Univ. Kentucki, Lexington, Ky.,1976), 51-67,Lecture Notes in Math., Vol. **599**, Springer, Berlin, 1977.

[12] A. Boutabaa and A. Escassut, *Applications of the p-adic Nevanlinna theory to functional equations,* Annale de I' Institut Fourier **50** (2000), 751-766.

Daklak Pedagogical College
Buon Ma Thuot, Vietnam
E-mail address: `ntronghoa@@math.ac.vn`