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THE UNIQUENESS OF VISCOSITY SOLUTIONS OF SECOND ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN A HILBERT SPACE OF TWO-DIMENSIONAL FUNCTIONS

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Abstract. In this paper we prove the uniqueness of viscosity solutions of the second order nonlinear partial differential equations associated with the Stokes and Euler operators in a Hilbert space of two-dimensional functions.

1. Introduction

Since the early 1980s, the concept of *viscosity solutions* introduced by M.G. Crandall and P.-L. Lions ([1], [3]) has been used in a large portion of research in a nonclassical theory of first-order nonlinear PDEs as well as in other types of PDEs. For convex Hamilton-Jacobi equations, the viscosity solution - characterized by a semiconcave stability condition, was first introduced by S.N. Kruzkov [6]. There is an enormous activity based on these studies. The primary virtues of this theory are that it allows merely nonsmooth functions to be solutions of nonlinear PDEs, this theory provides very general existence and uniqueness theorems, and it yields precise formulations of general boundary conditions. Let us mention here the names of M.G. Crandall, P.-L. Lions, L.C. Evans, H. Ishii, R. Jensen, V. Barbu, M. Bardi, G. Barles, Barron, L. Cappuzzo-Dolcetta, P. Dupuis, S. Lenhart, S. Osher, B. Perthame, P. Soravia, P.E. Souganidis, D. Tataru, Y. Tomita, N. Yamada,... and many others, whose contributions make great progress in nonlinear PDEs. The concept of viscosity solutions is motivated by the classical maximum principle which distinguishes it from other definitions of generalized solutions. The results of viscosity solutions were generalized to infinite dimensions by P.-L. Lions, H. Ishii, A. Swiech, D. Tataru... (see [4], [5], [7] and the references therein).

Let $\Omega \subset \mathbb{R}^2$ be the open and bounded set with smooth boundary. Set

$$\mathbb{H} = \text{the closure of } \{x \in \mathcal{D}(\Omega;\mathbb{R}^2),\ \text{div}\ x = 0\} \text{ in } L^2(\Omega;\mathbb{R}^2),$$

$$\mathbb{V} = \text{the closure of } \{x \in \mathcal{D}(\Omega;\mathbb{R}^2),\ \text{div}\ x = 0\} \text{ in } H_0^1(\Omega;\mathbb{R}^2)$$

and let $P_{\mathbb{H}}$ be the orthogonal projection in $L^2(\Omega;\mathbb{R}^2)$ onto $\mathbb{H}$. The operators $Ax = -P_{\mathbb{H}}\Delta x$ and $B(x,y) = P_{\mathbb{H}}[(x.\nabla)y]$ are called the Stokes and the Euler operators respectively. Let $\langle .,. \rangle$ and $|.|$ be the inner product and the norm in

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$\mathbb{H}$ space. This paper is concerned with fully nonlinear second order Hamilton-Jacobi-Bellman-Isaccs equations in the $\mathbb{H}$ space

$$(*)\qquad \begin{aligned} u(x) &+ \langle Ax + B(x,x), Du(x)\rangle \\ &+ F(x, Du(x), D^2u(x)) = 0,\ x \in \mathbb{H}. \end{aligned}$$

The assumptions about F will be given later in Section 3. The viscosity solution approach is adapted to those equations under consideration and the uniqueness of viscosity solutions is established.

We notice that, the equation $(*)$ was studied in [4] without Euler operator, and in [5] in case the function F does not depend on the second order partial derivatives of u.

This paper is presented in five sections. In Section 2, we give some preliminaries on the abstract spaces and the Stokes and Euler operators. Section 3 is devoted to the assumptions about F. In Section 4 and Section 5 we present the definition of a viscosity solution and we prove a general uniqueness result for (*).

2. Notation and preliminaries

2.1. **Abstract spaces and the Stokes operator.** We denote by $W^{m,p}(\Omega;\mathbb{R}^2)$ (or simply by $W^{m,p}(\Omega)$) the Sobolev space of order $0 \leqslant m \in \mathbb{R}$ and power $p \geq 1$ of functions with values in $\mathbb{R}^2$ (which can be seen as a product space $W^{m,p}(\Omega;\mathbb{R}^2) = [W^{m,p}(\Omega;\mathbb{R})]^2$). The norm of $x \in W^{m,p}(\Omega;\mathbb{R}^2)$ will be denote by $|x|_{m,p}$. We will use the notation L^p for $W^{0,p}$ and H^m for $W^{m,2}$. We will also be using the negative Sobolev spaces H^{-m}. The space $\mathbb{H}$ can be alternatively defined by

$$\mathbb{H} = \{x \in L^2(\Omega;\mathbb{R}^2),\ \text{div}\ x = 0 \text{ in } \mathcal{D}'(\Omega;\mathbb{R}^2), x.n = 0 \text{ in } H^{-\frac{1}{2}}(\partial\Omega)\}$$

where n is the outward normal to the boundary (see [5]). The Stokes operator A has the domain of definition

$$\mathcal{D}(A) = H^2(\Omega;\mathbb{R}^2) \cap \mathbb{V}.$$

It is well known that A is linear, positive definite, self-adjoint, A^{-1} is self-adjoint, injective and compact. For $\gamma \geq 0$ we denote by $\mathbb{V}_\gamma$ the domain of definition of $A^{\frac{\gamma}{2}}, \mathcal{D}(A^{\frac{\gamma}{2}})$, equipped with the norm

$$|x|_\gamma = |A^{\frac{\gamma}{2}}x|_{0,2}. \tag{2.1}$$

For $\gamma < 0$, the space $\mathbb{V}_\gamma$ is defined as the completion of $\mathbb{H}$ under the norm (2.1). If $\gamma > -\frac{1}{2}$ the norm of $\mathbb{V}_\gamma$ is equivalent to the norm of H^γ (see [5] or [8], [9]). Moreover, the space $\mathbb{V}_1$ coincides with $\mathbb{V}$. Identifying $\mathbb{H}$ with its dual, the space $\mathbb{V}_{-\gamma}$ is the dual of $\mathbb{V}_\gamma$ for $\gamma > 0$. We will also use the customary notation $\mathbb{V}'$ for the dual of $\mathbb{V}$ and the duality pairing between $\mathbb{V}'$ and $\mathbb{V}$ will be denoted by $\langle .,.\rangle$. The same symbol will also be used to denote the inner product in $\mathbb{H}$ if both entries are in $\mathbb{H}$.

We recall below Sobolev imbeddings and other inequalities that we will need in the remainder of the paper.

-*Sobolev imbedding type inequalities:* If $m \geq 0, mp \leqslant 2$ and $p \leqslant q \leqslant \frac{2p}{2-mp}$ then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, i.e.,

$$|x|_{0,q} \leqslant C|x|_{m,p}, \quad x \in W^{m,p}(\Omega).$$

(noting that when $mp = 2$ the imbedding holds for all $p \leqslant q < +\infty$). Combining the above with the equivalence of norm of $\mathbb{V}_\gamma$ and H^γ, we find that for $\gamma \in (0, 1]$ and $q \in [2, \frac{2}{1-\gamma}]$ ($q \in [2, +\infty)$ if $\gamma = 1$) $\mathbb{V}_\gamma \hookrightarrow L^q(\Omega)$, i.e.,

$$|x|_{0,q} \leqslant C|x|_\gamma, \quad x \in \mathbb{V}_\gamma. \tag{2.2}$$

-*Interpolation inequality:* If an operator S generates an analytic semigroup, then there exists a constant C such that, for every $z \in \mathcal{D}(S)$ and $0 \leqslant \gamma \leqslant 1$,

$$|S^\gamma z| \leqslant C|Sz|^\gamma |z|^{1-\gamma}. \tag{2.3}$$

Let $\gamma \in (0, 1], \alpha \in (0, \gamma)$. For every $\sigma > 0$, these exists $C_\sigma > 0$ such that

$$|A^\alpha z| \leqslant \sigma|A^\gamma z| + C_\sigma|z|, \quad \forall z \in \mathcal{D}(A^\gamma). \tag{2.4}$$

2.2. **The Euler operator.** We define the trilinear form $b(.,.,.) : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ as

$$b(x, y, z) = \int_\Omega z(\xi)(x(\xi).\nabla_\xi)y(\xi)d\xi$$

and the bilinear operator $B(.,.) : \mathbb{V} \times \mathbb{V} \to \mathbb{V}'$ as

$$\langle B(x, y), z\rangle = b(x, y, z), \quad z \in \mathbb{V}.$$

This is just another way to introduce the operator B that we have already used in Section 1. By the incompressibility condition ($\text{div}x = 0$) we have

$$b(x, y, y) = 0, \quad b(x, y, z) = -b(x, z, y). \tag{2.5}$$

2.3. **Preliminaries on the operators and spaces of operators.** Throughout this subsection E will denote a real separable Hilbert space endowed with the inner product $\langle .,.\rangle$ and the norm $|.|$. We denote by $L(E)$ the Banach space of continuous linear operators $T : E \to E$ with the operator norm $\|.\|$, and we set

$$\sum(E) = \{T \in L(E),\ T - \text{self-adjoint}\}.$$

For any Hilbert spaces E and $\tilde{E}$, we denote by $UC(E, \tilde{E})$, $BUC(E, \tilde{E})$ the Banach space of all functions $\varphi : E \to \tilde{E}$ which are, respectively, uniformly continuous, uniformly continuous and bounded on E with the usual norm

$$\|\varphi\| = \sup_{x\in E}|\varphi(x)|_{\tilde{E}}.$$

We say that a function $\rho : [0, +\infty) \to [0, +\infty)$ is a *modulus* if ρ is continuous, nondecreasing, subadditive, and $\rho(0) = 0$. Subadditivity in particular implies that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\rho(r) \leqslant \varepsilon + C_\varepsilon r, \quad \text{for every } r \geq 0.$$

Moreover, a function $\omega : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a *local modulus* if ω is continuous, nondecreasing in both variables, subadditive in the first variable, and $\omega(0, r) = 0$, for every $r \geq 0$.

Let $f : M \subset E \to R$ be a functional on the subset M of E. Then function f is called *weakly sequentially lower (upper) semicontinuous* iff, for each $x \in M$, and each sequence (x_n) in $M, x_n \rightharpoonup x$ (the weakly convergence) implies

$$\liminf_{n\to\infty} f(x_n) \geq f(x)$$

$$\text{(respectively: } \limsup_{n\to\infty} f(x_n) \leqslant f(x)).$$

Function f is called *weakly sequentially continuous* iff it is both weakly sequentially lower semicontinuous and weakly sequentially upper semicontinuous.

Proposition 2.1 (see [10]). *Suppose that the functional $f : E \to R$ has the following two properties*

(i) *f is weakly sequentially upper semicontinuous;*

(ii) *$f(x) \to -\infty$ as $|x| \to +\infty$.*

Then f has a global maxima.

Next, we will give an important property of partial sup-convolution.

Let E be a separable infinite dimensional Hilbert space which is written as a product $E = Z \times W$ where Z, W are Hilbert spaces and Z is finite dimensional, $u : E \to \mathbb{R}$ is a functional. We define the *partial sup-convolution* of u by

$$\hat{u}(z,\omega) = \sup_{\hat{w}\in W}\Big(u(z,\hat{\omega}) - \frac{\alpha}{2}|\hat{\omega}-\omega|^2\Big), \quad \alpha > 0.$$

Proposition 2.2 (see [2]). *Let u be a weakly sequentially upper semicontinuous functional and satisfy*

$$u(z,\omega) \leqslant a_R + \frac{K}{2}|\omega|^2, \ \textit{for } z \in Z, |z| \leqslant R,$$

where $a_R \geq 0$ for $R > 0$ and $K \geq 0$. Then $\hat{u}$ is also weakly sequentially upper semicontinuous.

Let $A : \mathcal{D}(A) \subset E \to E$ be a linear operator. We will call that

A is *monotone* iff $\langle Av, v\rangle \geq 0, \forall v \in \mathcal{D}(A)$;

A is *maximal monotone* iff A is monotone and $R(I + A) = E$.

Proposition 2.3 (see [11]). *Let $A : \mathcal{D}(A) \subset E \to E$ be a linear symmetric monotone operator. Then A is maximal monotone iff it is self-adjoint.*

Proposition 2.4 (see [11]). *Let $A : \mathcal{D}(A) \subset E \to E$ be maximal monotone on E. Then, it follows from $Av_n \rightharpoonup b$ and $v_n \to v$ as $n \to \infty$ that $Av = b$.*

3. The assumptions on F

Let $\mathbb{H}_1 \subset \mathbb{H}_2 \subset \cdots$ be finite dimensional subspaces of $\mathbb{H}$ generated by eigenvectors of A such that $\overline{\cup_{N=1}^{\infty}\mathbb{H}_N} = \mathbb{H}$. Given any $N \in \mathbb{N}$, denote by P_N the orthogonal projection in $\mathbb{H}$ onto $\mathbb{H}_N$. Let $Q_N = I - P_N$ and let $\mathbb{H}_N^{\perp} = Q_N\mathbb{H}$. We then have an orthogonal decomposition $\mathbb{H} = \mathbb{H}_N \times \mathbb{H}_N^{\perp}$ and we will denote by x_N an element of $\mathbb{H}_N$ and by $x_N^{\perp}$ an element of $\mathbb{H}_N^{\perp}$. For $x \in \mathbb{H}$, we will write $x = (P_N x, Q_N x)$. We make the following assumptions about F.

Hypothesis F :

(F_0) There exists $\beta \in (0, \frac{1}{2})$ such that the function $F : \mathbb{V}_\beta \times \mathbb{V}_\beta \times \sum(\mathbb{H}) \to \mathbb{R}$ is continuous (in the topology of $\mathbb{V}_\beta \times \mathbb{V}_\beta \times \sum(\mathbb{H})$);

(F_1) $F(x,p,S_1) \leqslant F(x,p,S_2), \quad \forall x,p \in \mathbb{V}_\beta, \forall S_1 \geq S_2$, where $S_1 \geq S_2$ iff $S_1 - S_2$ is monotone;

(F_2) There exists a modulus ρ such that

$$|F(x,p,S_1) - F(x,q,S_2)| \leqslant \rho\big((1+|x|_\beta)|p-q|_\beta + (1+|x|_\beta^2)\|S_1 - S_2\|\big),$$

$\forall x,p,q \in \mathbb{V}_\beta$ and $\forall S_1, S_2 \in \sum(\mathbb{H})$;

(F_3) There exists a modulus ω such that, $\forall \varepsilon > 0, \forall N \geq 1, \eta = 1-\beta, \forall x,y \in \mathbb{V}_\beta$ and $X, Y \in \sum(\mathbb{H}_N)$ such that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{2}{\varepsilon}\begin{pmatrix} P_N A^{-\eta} P_N & -P_N A^{-\eta} P_N \\ -P_N A^{-\eta} P_N & P_N A^{-\eta} P_N \end{pmatrix} \tag{3.1}$$

we have

$$F\Big(x, \frac{A^{-\eta}(x-y)}{\varepsilon}, X\Big) - F\Big(y, \frac{A^{-\eta}(x-y)}{\varepsilon}, Y\Big) \geq -\omega\Big(|x-y|_\beta\big(1 + \frac{|x-y|_\beta}{\varepsilon}\big)\Big);$$

(F_4) For every $R < +\infty, |\lambda| \leqslant R, x, p \in \mathbb{V}_\beta$,

$$\sup\big\{|F(x,p,S_N + \lambda Q_N) - F(x,p,S_N)| : S_N = P_N S P_N, \|S\| \leqslant R\big\} \to 0 \text{ as } N \to \infty.$$

Remark. By the properties of moduli, condition (F_2) guarantees the existence of a constant C such that for all $x, p \in \mathbb{V}_\beta$, for all $S \in \sum(\mathbb{H})$,

$$|F(x,p,S)| \leqslant C\Big(1 + (1+|x|_\beta)|p|_\beta + (1+|x|_\beta^2)\|S\|\Big) + |F(x,0,0)|. \tag{3.2}$$

4. Viscosity solutions

The definition of a viscosity solution that we propose here has its predecessors in [4] and [5].

Definition 4.1. A function $\psi : \mathbb{H} \to \mathbb{R}$ is a *test function* for (*) if

$$\psi(x) = \varphi(x) + \delta|x|^2,$$

where

(1) $\delta > 0$;
(2) $\varphi \in C^2(\mathbb{H})$ and is weakly sequentially lower semicontinuous;
(3) $D\varphi \in UC(\mathbb{H}, \mathbb{H}) \cap UC(\mathcal{D}(A^{\frac{1}{2}-\varepsilon}), \mathbb{V})$, for some $\varepsilon = \varepsilon(\varphi) > 0$;
(4) $D^2\varphi \in BUC(\mathbb{H}, \sum(\mathbb{H}))$.

Definition 4.2. A weakly sequentially upper (lower) semicontinuous function $u : \mathbb{H} \to \mathbb{R}$ is a *viscosity subsolution* (respectively: *viscosity supersolution*) of (*) if for every test function ψ, whenever $u - \psi$ has a local maximum (respectively: $u + \psi$ has a local minimum) at x then $x \in \mathbb{V}$ and

$$u(x) + \langle Ax + B(x, x), D\psi(x)\rangle + F(x, D\psi(x), D^2\psi(x)) \leqslant 0$$

$$(\text{resp.: } u(x) + \langle Ax + B(x, x), -D\psi(x)\rangle + F(x, -D\psi(x), -D^2\psi(x)) \geq 0),$$

where $\langle Ax, y\rangle := \langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}y\rangle,\ \forall x, y \in \mathbb{V}$.

A function u is a *viscosity solution* of (*) if it is both a viscosity subsolution and a viscosity supersolution.

5. The uniqueness of viscosity solutions

We denote by $\mathcal{K}$ the class of functions $u : \mathbb{H} \to \mathbb{R}$ such that u is weakly sequentially continuous, bounded and Lipschitz continuous in $|.|_{-\eta}$ norm on bounded subsets of $\mathbb{H}$.

Theorem 5.1. *Let Hypothesis F hold. Let $u, -v \leqslant M$ for some constant M, u be a viscosity subsolution of (*) and v be a viscosity supersolution of (*). If u and $-v$ are Lipschitz continuous in $|.|_{-\eta}$ norm on bounded subsets of $\mathbb{H}$ then $u \leqslant v$ on $\mathbb{H}$. Moreover, if (*) has a viscosity solution $u \in \mathcal{K}$ then it is unique.*

First, we need some prepairing.

Let E be a Hilbert space, $u : E \to [-\infty, +\infty]$ be a functional. If $\hat{x} \in E$ and $(p, X) \in E \times \sum(E)$ we say that $(p, X) \in J^{2,+}u(\hat{x})$ provided that

$$u(x) \leqslant u(\hat{x}) + \langle p, x - \hat{x}\rangle + \frac{1}{2}\langle X(x - \hat{x}), x - \hat{x}\rangle + o(|x - \hat{x}|^2)$$

as $x \to \hat{x}$. The closure of $J^{2,+}u(x)$, $\bar{J}^{2,+}u(x)$, is defined as follows:

$$\bar{J}^{2,+}u(x) = \Big\{(p, X) \in E \times \sum(E) : \exists (x_n, p_n, X_n) \in E \times E \times \sum(E) :$$
$$(p_n, X_n) \in J^{2,+}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X)\Big\}.$$

We are interested in the situation where $E = E_1 \times E_2$ is the product of two spaces and $u(x_1, x_2) = u_1(x_1) + u_2(x_2)$. The proposition below is a straightforward corollary from Theorem 3.2 in [3].

Proposition 5.1. *Let $u_i, i = 1, 2$ be upper semicontinuous on $\mathbb{R}^N$ and φ be twice continuously differentiable on $\mathbb{R}^{2N}$. Suppose that*

$$u_1(x_1) + u_2(x_2) - \varphi(x_1, x_2)$$

has a local maximum at $(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^{2N}$. Then, for every $\alpha > 0$ there are $X_1, X_2 \in \sum(\mathbb{R}^N)$ such that

$$(D_{x_i}\varphi(\hat{x}), X_i) \in \bar{J}^{2,+}u_i(\hat{x}_i), \quad i = 1, 2,$$

and the block diagonal matrix with entries X_i satisfies

$$-\Big(\frac{1}{\alpha} + \|\phi\|\Big)I \leqslant \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leqslant \phi + \alpha\phi^2,$$

where $\phi = D^2\varphi(\hat{x}) \in \sum(\mathbb{R}^{2N})$.

The norm of the symmetric matrix ϕ used above is

$$\|\phi\| = \sup\big\{|\lambda| : \lambda \text{ is an eigenvalue of } \phi\big\} = \sup\big\{|\langle\phi\xi, \xi\rangle| : |\xi| \leqslant 1\big\}.$$

Remark 5.1. Proposition 5.1 is also true if we take the finite dimensional Hilbert space $\mathbb{H}_N$ instead of $\mathbb{R}^N$. Then, for

$$u_1(x_1) = \tilde{u}_1(x_N), \quad u_2(x_2) = -\tilde{v}_1(y_N)$$

and

$$\varphi(x_N, y_N) = \frac{1}{2\varepsilon}\langle P_N A^{-\eta} P_N(x_N - y_N), x_N - y_N\rangle, \quad x_N, y_N \in \mathbb{H}_N$$

we have

$$D_{x_N}\varphi(\hat{x}_N, \hat{y}_N) = -D_{y_N}\varphi(\hat{x}_N, \hat{y}_N) = \frac{1}{\varepsilon}P_N A^{-\eta} P_N(\hat{x}_N - \hat{y}_N)$$

and

$$\phi = D^2\varphi(\hat{x}_N, \hat{y}_N) = \frac{1}{\varepsilon}\begin{pmatrix} P_N A^{-\eta} P_N & -P_N A^{-\eta} P_N \\ -P_N A^{-\eta} P_N & P_N A^{-\eta} P_N \end{pmatrix}.$$

Thus

$$\phi^2 = \frac{2}{\varepsilon^2}\begin{pmatrix} (P_N A^{-\eta} P_N)^2 & -(P_N A^{-\eta} P_N)^2 \\ -(P_N A^{-\eta} P_N)^2 & (P_N A^{-\eta} P_N)^2 \end{pmatrix}.$$

We notice that, if $0<\lambda_1<\lambda_2<\cdots$ are eigenvalues of operator A then we obtain

$$\phi=\frac{1}{\varepsilon}\begin{pmatrix}\lambda_1^{-\eta} & & 0 & -\lambda_1^{-\eta} & & 0\\ & \ddots & & & \ddots & \\ 0 & & \lambda_N^{-\eta} & 0 & & -\lambda_N^{-\eta}\\ -\lambda_1^{-\eta} & & 0 & \lambda_1^{-\eta} & & 0\\ & \ddots & & & \ddots & \\ 0 & & -\lambda_N^{-\eta} & 0 & & \lambda_N^{-\eta}\end{pmatrix},$$

$$\phi^2=\frac{2}{\varepsilon^2}\begin{pmatrix}\lambda_1^{-2\eta} & & 0 & -\lambda_1^{-2\eta} & & 0\\ & \ddots & & & \ddots & \\ 0 & & \lambda_N^{-2\eta} & 0 & & -\lambda_N^{-2\eta}\\ -\lambda_1^{-2\eta} & & 0 & \lambda_1^{-2\eta} & & 0\\ & \ddots & & & \ddots & \\ 0 & & -\lambda_N^{-2\eta} & 0 & & \lambda_N^{-2\eta}\end{pmatrix}.$$

Therefore, if we choose $\alpha=\frac{\varepsilon}{2\lambda}$ with $\lambda=\sup_j\{\lambda_j^{-\eta}\}$ then it follows from Proposition 5.2 that there exists $X_N, Y_N\in\sum(\mathbb{H}_N)$ such that

$$\big(\frac{1}{\varepsilon}P_NA^{-\eta}P_N(\hat{x}_N-\hat{y}_N),X_N\big)\in\bar{J}^{2,+}\tilde{u}_1(\hat{x}_N),$$
$$\big(-\frac{1}{\varepsilon}P_NA^{-\eta}P_N(\hat{x}_N-\hat{y}_N),-Y_N\big)\in\bar{J}^{2,+}(-\tilde{v}_1)(\hat{y}_N)$$

and X_N, Y_N satisfy

$$\begin{pmatrix}X_N & 0\\ 0 & -Y_N\end{pmatrix}\leqslant\phi+\alpha\phi^2\leqslant\frac{2}{\varepsilon}\begin{pmatrix}P_NA^{-\eta}P_N & -P_NA^{-\eta}P_N\\ -P_NA^{-\eta}P_N & P_NA^{-\eta}P_N\end{pmatrix}.$$

Remark 5.2. Let $(p,X)\in\bar{J}^{2,+}u(\bar{x})$. Then by the definition of $\bar{J}^{2,+}u(\bar{x})$, there exists $(x_n,p_n,X_n)\in\mathbb{R}^N\times\mathbb{R}^N\times\sum(\mathbb{R}^N)$, $(p_n,X_n)\in J^{2,+}u(x_n)$ and

$$(x_n,u(x_n),p_n,X_n)\to(\bar{x},u(\bar{x}),p,X)\text{ as }n\to\infty.$$

Since $(p_n,X_n)\in J^{2,+}u(x_n)$, as $x\to x_n$ we have

$$u(x)\leqslant u(x_n)+\langle p,x-x_n\rangle+\frac{1}{2}\langle X_n(x-x_n),x-x_n\rangle+o(|x-x_n|^2).$$

Setting

$$\varphi_n(x)=u(x_n)+\langle p_n,x-x_n\rangle+\frac{1}{2}\langle(X_n+\frac{1}{n}I)(x-x_n),x-x_n\rangle+o(|x-x_n|^2),$$

we obtain $\varphi_n\in C^2(\mathbb{R}^N), D\varphi_n=p_n, D^2\varphi_n=X_n+\frac{1}{n}I$. Moreover, $u-\varphi_n$ has a local unique, strict maxima at x_n. Global strict maximum at x_n of $u-\varphi_n$ may be achieved by first restricting u to some compact neighborhood K of x_n and then extending the restriction to $\mathbb{R}^N$ by $u(x) = \inf_{\mathbb{R}^N} u(x)$ if $x \notin K$ (still writing u). The compactness of K guarantees that u is still upper semicontinuous on $\mathbb{R}^N$.

Proof of Theorem 5.1.
Step 1: For $\epsilon, \delta > 0$, we consider the function

$$\Phi(x, y; \varepsilon, \delta) := u(x) - v(y) - \frac{|x-y|^2_{-\eta}}{2\varepsilon} - \delta(|x|^2 + |y|^2).$$

We will prove that Φ has a global maximum at $(\bar{x}, \bar{y})$ ($\bar{x}, \bar{y}$ depend on ε and δ) satisfying

$$\limsup_{\delta \searrow 0} \limsup_{\varepsilon \searrow 0} \delta(|\bar{x}|^2 + |\bar{y}|^2) = 0 \tag{5.1}$$

and

$$\limsup_{\varepsilon \searrow 0} \Big(\frac{|\bar{x}-\bar{y}|^2_{-\eta}}{2\varepsilon}\Big) = 0 \quad \text{for fixed } \delta > 0. \tag{5.2}$$

Since $u, -v$ are bounded from above, weakly sequentially upper semicontinuous, and A^{-1} is compact, the function Φ is also weakly sequentially upper semicontinuous in $\mathbb{H} \times \mathbb{H}$. Therefore, by Proposition 2.1, it has a global maximum at $(\bar{x}, \bar{y})$.

Setting

$$m_1(\varepsilon, \delta) := \sup_{x,y \in \mathbb{H}} \Phi(x, y; \varepsilon, \delta),$$

$$m_2(\delta) := \sup_{x,y \in \mathbb{H}} \{u(x) - v(y) - \delta(|x|^2 + |y|^2)\},$$

we have that

$$m = \lim_{\delta \searrow 0} m_2(\delta) \quad \text{and} \quad m_2(\delta) = \lim_{\varepsilon \searrow 0} m_1(\varepsilon, \delta).$$

Now

$$m_1(\varepsilon, \delta) = \Phi(\bar{x}, \bar{y}; \varepsilon, \delta) = u(\bar{x}) - v(\bar{y}) - \frac{|\bar{x}-\bar{y}|^2_{-\eta}}{2\varepsilon} - \delta(|\bar{x}|^2 + |\bar{y}|^2)$$

and, for fixed δ,

$$\begin{aligned} m_1(\varepsilon, \delta) + \frac{|\bar{x}-\bar{y}|^2_{-\eta}}{4\varepsilon} &= u(\bar{x}) - v(\bar{y}) - \frac{|\bar{x}-\bar{y}|^2_{-\eta}}{4\varepsilon} - \delta(|\bar{x}|^2 + |\bar{y}|^2) \\ &\leqslant m_1(2\varepsilon, \delta). \end{aligned}$$

Thus

$$\frac{|\bar{x}-\bar{y}|^2_{-\eta}}{4\varepsilon} \leqslant m_1(2\varepsilon, \delta) - m_1(\varepsilon, \delta).$$

This gives (5.2). Similarly,

$$\begin{aligned} m_1(\varepsilon, \delta) + \frac{\delta}{2}(|\bar{x}|^2 + |\bar{y}|^2) &= u(\bar{x}) - v(\bar{y}) - \frac{|\bar{x}-\bar{y}|^2_{-\eta}}{2\varepsilon} - \frac{\delta}{2}(|\bar{x}|^2 + |\bar{y}|^2) \\ &\leqslant m_1(\varepsilon, \frac{\delta}{2}) \end{aligned}$$

which gives

$$\frac{\delta}{2}(|\bar{x}|^2 + |\bar{y}|^2) \leqslant m_1(\varepsilon, \frac{\delta}{2}) - m_1(\varepsilon, \delta).$$

From this we obtain (5.1).

Step 2: Next we will prove that $\bar{x}, \bar{y} \in \mathbb{V}$.

We now fix $N \in \mathbb{N}$. Then

$$\begin{aligned}|x-y|^2_{-\eta} &= \langle A^{-\eta}(x-y), x-y\rangle \\ &= \langle P_N A^{-\eta} P_N(x-y), x-y\rangle + |A^{\frac{-\eta}{2}} Q_N(x-y)|^2,\end{aligned}$$

and we have

$$\begin{aligned}|A^{\frac{-\eta}{2}} Q_N(x-y)|^2 \leqslant\ & 2\langle Q_N A^{-\eta} Q_N(\bar{x}-\bar{y}), x-y\rangle \\ &- \langle Q_N A^{-\eta} Q_N(\bar{x}-\bar{y}), \bar{x}-\bar{y}\rangle \\ &+ 2|A^{\frac{-\eta}{2}} Q_N(x-\bar{x})|^2 + 2|A^{\frac{-\eta}{2}} Q_N(y-\bar{y})|^2\end{aligned}$$

with equality if and only if $x = \bar{x}, y = \bar{y}$. Therefore, if we define

$$\begin{aligned}u_1(x) = u(x) &- \frac{\langle x, Q_N A^{-\eta} Q_N(\bar{x}-\bar{y})\rangle}{\varepsilon} + \frac{\langle Q_N A^{-\eta} Q_N(\bar{x}-\bar{y}), \bar{x}-\bar{y}\rangle}{2\varepsilon} \\ &- \frac{|A^{-\frac{\eta}{2}} Q_N(x-\bar{x})|^2}{\varepsilon} - \delta|x|^2\end{aligned}$$

and

$$v_1(y) = v(y) - \frac{\langle y, Q_N A^{-\eta} Q_N(\bar{x}-\bar{y})\rangle}{\varepsilon} + \frac{|A^{-\frac{\eta}{2}} Q_N(y-\bar{y})|^2}{\varepsilon} + \delta|y|^2,$$

it follows that the function

$$\tilde{\Phi}(x,y) := u_1(x) - v_1(y) - \frac{\langle P_N A^{-\eta} P_N(x-y), x-y\rangle}{2\varepsilon}$$

always satisfies $\tilde{\Phi} \leqslant \Phi$ and attains a strict global maximum at $\bar{x}, \bar{y}$. Moreover,

$$\tilde{\Phi}(\bar{x}, \bar{y}) = \Phi(\bar{x}, \bar{y}).$$

We now define, for $x_N, y_N \in \mathbb{H}_N$, the functions

$$\tilde{u}_1(x_N) := \sup_{x_N^\perp \in \mathbb{H}_N^\perp} u_1(x_N, x_N^\perp), \quad \tilde{v}_1(y_N) := \inf_{y_N^\perp \in \mathbb{H}_N^\perp} v_1(y_N, y_N^\perp).$$

Since the assumptions about $u, -v$ and the weakly sequentially continuity of the inner product, using Proposition 2.2 with $\omega = 0$ we see that $\tilde{u}_1$ and $-\tilde{v}_1$ are upper semicontinuous on $\mathbb{H}_N$. Moreover, by the definition of u_1, v_1 and by the form of $\tilde{\Phi}$, it follows that

$$\tilde{u}_1(P_N\bar{x}) = u_1(\bar{x}), \qquad \tilde{v}_1(P_N\bar{y}) = v_1(\bar{y}). \tag{5.3}$$

Now define the map $\Phi_N : \mathbb{H}_N \times \mathbb{H}_N \to \mathbb{R}$ as

$$\begin{aligned}\Phi_N(x_N, y_N) &:= \tilde{u}_1(x_N) - \tilde{v}_1(y_N) - \frac{\langle P_N A^{-\eta} P_N(x_N - y_N), x_N - y_N)\rangle}{2\varepsilon} \\ &= \sup_{x_N^\perp, y_N^\perp \in \mathbb{H}_N^\perp} \tilde{\Phi}\big((x_N, x_N^\perp), (y_N, y_N^\perp)\big).\end{aligned}$$

It is not difficult to check that Φ_N attains a strict global maximum over $\mathbb{H}_N \times \mathbb{H}_N$ at $(\bar{x}_N, \bar{y}_N) = (P_N\bar{x}, P_N\bar{y})$. By Remarks 5.1 and 5.2, for every $n \in \mathbb{N}$, there exist $x_N^n, y_N^n \in \mathbb{H}_N$ such that

$$x_N^n \to \bar{x}_N, \quad y_N^n \to \bar{y}_N, \quad \tilde{u}_1(x_N^n) \to \tilde{u}_1(\bar{x}_N), \quad \tilde{v}_1(y_N^n) \to \tilde{v}_1(\bar{y}_N) \tag{5.4}$$

as $n \to \infty$, and there are functions $\varphi_n, \psi_n \in C^2(\mathbb{H}_N)$ such that $\tilde{u}_1 - \varphi_n$ and $-\tilde{v}_1 + \psi_n$ have unique, strict, global maxima at x_N^n and y_N^n respectively, and

$$\begin{aligned} D\varphi_n(x_N^n) &\to \frac{1}{\varepsilon} P_N A^{-\eta} P_N(\bar{x}_N - \bar{y}_N), \\ D\psi_n(y_N^n) &\to \frac{1}{\varepsilon} P_N A^{-\eta} P_N(\bar{x}_N - \bar{y}_N), \end{aligned} \tag{5.5}$$

$$D^2\varphi_n(x_N^n) \to X_N, \quad D^2\psi_n(y_N^n) \to Y_N, \tag{5.6}$$

where X_N, Y_N satisfy (3.1).

Consider finally the map $\Phi_N^n : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ defined as

$$\Phi_N^n(x, y) := u_1(x) - v_1(y) - \varphi_n(P_N x) + \psi_n(P_N y). \tag{5.7}$$

This map has the variables split and, by the definition of u_1 and v_1, attains its global maximum at some point $(\hat{x}^n, \hat{y}^n)$. This point depends also on N but we will drop this dependence since N is now fixed. Setting now

$$\bar{\varphi}_{N,n}(x) := \frac{\langle x, Q_N A^{-\eta} Q_N(\bar{x} - \bar{y})\rangle}{\varepsilon} + \frac{|A^{-\frac{\eta}{2}} Q_N(x - \bar{x})|^2}{\varepsilon} + \varphi_n(P_N x),$$

we easily see that $\psi(x) = \bar{\varphi}_{N,n}(x) + \delta|x|^2$ is a test function of (*). From (5.7) it follows that $u(x) - \psi(x)$ has a maximum at $\hat{x}^n$. Therefore, by the definition of viscosity subsolution, $\hat{x}^n \in \mathbb{V}$ and

$$\begin{aligned} u(\hat{x}^n) + \langle A\hat{x}^n, D\psi(\hat{x}^n)\rangle + \langle B(\hat{x}^n, \hat{x}^n), D\psi(\hat{x}^n)\rangle \\ + F\big(\hat{x}^n, D\psi(\hat{x}^n), D^2\psi(\hat{x}^n)\big) \leqslant 0 \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} D\psi(\hat{x}^n) &= D\varphi_n(P_N\hat{x}^n) + \frac{A^{-\eta}Q_N(\bar{x} - \bar{y})}{\varepsilon} + \frac{2A^{-\eta}Q_N(\hat{x}^n - \bar{x})}{\varepsilon} + 2\delta\hat{x}^n, \\ D^2\psi(\hat{x}^n) &= D^2\varphi_n(P_N\hat{x}^n) + \frac{2A^{-\eta}Q_N}{\varepsilon} + 2\delta I. \end{aligned}$$

We now write

$$\hat{x}^n = (P_N\hat{x}^n, Q_N\hat{x}^n), \quad \hat{y}^n = (P_N\hat{y}^n, Q_N\hat{y}^n).$$

Then, for every $x_N^\perp, y_N^\perp \in \mathbb{H}_N$ we have

$$\begin{aligned} &\tilde{u}_1(P_N\hat{x}^n) - \tilde{v}_1(P_N\hat{y}^n) - \varphi_n(P_N\hat{x}^n) + \psi_n(P_N\hat{y}^n) \\ &\quad \geq u_1(P_N\hat{x}^n, Q_N\hat{x}^n) - v_1(P_N\hat{y}^n, Q_N\hat{y}^n) - \varphi_n(P_N\hat{x}^n) + \psi_n(P_N\hat{y}^n) \\ &\quad \geq u_1(x_N^n, x_N^\perp) - v_1(y_N^n, y_N^\perp) - \varphi_n(x_N^n) + \psi_n(y_N^n). \end{aligned}$$

Therefore taking suprema over $x_N^\perp$ and $y_N^\perp$ in the above inequality we obtain

$$\begin{aligned}&\tilde{u}_1(P_N\hat{x}^n) - \tilde{v}_1(P_N\hat{y}^n) - \varphi_n(P_N\hat{x}^n) + \psi_n(P_N\hat{y}^n)\\ &\quad \geq u_1(P_N\hat{x}^n, Q_N\hat{x}^n) - v_1(P_N\hat{y}^n, Q_N\hat{y}^n) - \varphi_n(P_N\hat{x}^n) + \psi_n(P_N\hat{y}^n)\\ &\quad \geq \tilde{u}_1(x_N^n) - \tilde{v}_1(y_N^n) - \varphi_n(x_N^n) + \psi_n(y_N^n).\end{aligned}$$

This implies that

$$P_N\hat{x}^n = x_N^n, \quad P_N\hat{y}^n = y_N^n, \quad u_1(\hat{x}^n) = \tilde{u}_1(x_N^n), \quad v_1(\hat{y}^n) = \tilde{v}_1(y_N^n),$$

which, together with (5.4) and (5.3), yields

$$u_1(\hat{x}^n) \to u_1(\bar{x}), \quad v_1(\hat{y}^n) \to v_1(\bar{y}) \quad \text{as } n \to \infty. \tag{5.9}$$

Finally, since

$$u_1(\hat{x}^n) = \tilde{u}_1(P_N\hat{x}^n), \quad v_1(\hat{y}^n) = \tilde{v}_1(P_N\hat{y}^n)$$

and

$$u_1(\bar{x}) = \tilde{u}_1(P_N\bar{x}), \quad v_1(\bar{y}) = \tilde{v}_1(P_N\bar{y}),$$

formula (5.9), together with the weakly sequentially upper semicontinuity of u_1 and $-v_1$, implies

$$\hat{x}^n \to \bar{x}, \quad \hat{y}^n \to \bar{y} \quad \text{as } n \to \infty. \tag{5.10}$$

Therefore, since

$$\langle A\hat{x}^n, D\psi(\hat{x}^n)\rangle = \langle A^{\frac{1}{2}}\hat{x}^n, A^{\frac{1}{2}}D\bar{\varphi}_{N,n}(\hat{x}^n)\rangle + 2\delta|A^{\frac{1}{2}}\hat{x}^n|^2,$$

using (5.10), (5.9), (3.2), (F_0) and (2.4), it follows from (5.8) that $|A^{\frac{1}{2}}\hat{x}^n|$ are bounded independently of n. Then there exists a subsequence of $A^{\frac{1}{2}}\hat{x}^n$ (still written as $A^{\frac{1}{2}}\hat{x}^n$) converges weakly to b. Thanks to (5.10), since $A^{\frac{1}{2}}$ is maximal monotone (by Proposition 2.3) and Proposition 2.4, we have $b = A^{\frac{1}{2}}\bar{x}$. Thus $\bar{x} \in \mathbb{V}$ and

$$A^{\frac{1}{2}}\hat{x}^n \rightharpoonup A^{\frac{1}{2}}\bar{x} \quad \text{as } n \to \infty. \tag{5.11}$$

Similarly, we also get $\bar{y} \in \mathbb{V}$.

Step 3: We now would like to pass to the limit as $n \to \infty$ in (5.8) keeping ε, δ, N fixed.

Since $A^{-\frac{\eta}{2}}$ is compact we conclude that, as $n \to \infty$,

$$A^{\frac{1-\eta}{2}}\hat{x}^n = A^{-\frac{\eta}{2}}(A^{\frac{1}{2}}\hat{x}^n) \to A^{-\frac{\eta}{2}}(A^{\frac{1}{2}}\bar{x}) = A^{\frac{1-\eta}{2}}\bar{x}. \tag{5.12}$$

Using (5.4), (5.5), (5.6), (5.10) we have, as $n \to \infty$,

$$\begin{aligned}&D\psi(\hat{x}^n) \to \frac{1}{\varepsilon}A^{-\eta}(\bar{x} - \bar{y}) + 2\delta\bar{x},\\ &D^2\psi(\hat{x}^n) \to X_N + \frac{2A^{-\eta}Q_N}{\varepsilon} + 2\delta I \leqslant X_N + \frac{2\|A^{-\eta}\|Q_N}{\varepsilon} + 2\delta I,\end{aligned}$$

which together with (5.11), (5.12) and the weakly semicontinuity of the norm implies that

$$\liminf_{n\to\infty}\langle A\hat{x}^n, D\psi(\hat{x}^n)\rangle \geq \left\langle A^{\frac{1-\eta}{2}}\bar{x}, \frac{A^{\frac{1-\eta}{2}}(\bar{x} - \bar{y})}{\varepsilon}\right\rangle + 2\delta|A^{\frac{1}{2}}\bar{x}|^2.$$

Therefore, using the last inequality, (2.5) and (F_1), letting $n \to \infty$ in (5.8) we get

$$
\begin{aligned}
&u(\bar{x}) + \frac{1}{\varepsilon}\big\langle A^{\frac{1-\eta}{2}}\bar{x}, A^{\frac{1-\eta}{2}}(\bar{x}-\bar{y})\big\rangle + \frac{1}{\varepsilon}\langle B(\bar{x},\bar{x}), A^{-\eta}(\bar{x}-\bar{y})\rangle \\
&\quad +2\delta|\bar{x}|_1^2 + F\Big(\bar{x}, \frac{1}{\varepsilon}A^{-\eta}(\bar{x}-\bar{y}) + 2\delta\bar{x}, \\
&\qquad X_N + \frac{2}{\varepsilon}\|A^{-\eta}\|Q_N + 2\delta I\Big) \leqslant 0.
\end{aligned} \tag{5.13}
$$

We now eliminate the terms with δ and N. Using (F_2) we have

$$
\begin{aligned}
&F\Big(\bar{x}, \frac{1}{\varepsilon}A^{-\eta}(\bar{x}-\bar{y}), X_N + \frac{2}{\varepsilon}\|A^{-\eta}\|Q_N\Big) - \rho\big(d\delta(1+|\bar{x}|_\beta^2)\big) \\
&\quad \leqslant F\Big(\bar{x}, \frac{1}{\varepsilon}A^{-\eta}(\bar{x}-\bar{y}) + 2\delta\bar{x}, X_N + \frac{2}{\varepsilon}\|A^{-\eta}\|Q_N + 2\delta I\Big)
\end{aligned} \tag{5.14}
$$

for some constant $d > 0$. Now, given $\tau > 0$, let K_τ be such that

$$\rho(s) \leqslant \tau + K_\tau s.$$

Applying (2.4) with $\alpha = \frac{\beta}{2}$ and $\gamma = \frac{1}{2}$ we obtain

$$\rho\big(d\delta(1+|\bar{x}|_\beta^2)\big) \leqslant \delta|\bar{x}|_1^2 + \delta C_\tau|\bar{x}|^2 + \tau + K_\tau d\delta$$

for some constant $C_\tau > 0$ independent of δ and ε. It then follows from (5.1) that

$$\limsup_{\delta\searrow 0}\limsup_{\varepsilon\searrow 0}\Big(\rho\big(d\delta(1+|\bar{x}|_\beta^2)\big) - \delta|\bar{x}|_1^2\Big) \leqslant 0. \tag{5.15}$$

Setting

$$\omega_1(\varepsilon,\delta) := \rho\big(d\delta(1+|\bar{x}|_\beta^2)\big) - \delta|\bar{x}|_1^2,$$

using (5.14) and (F_4) in (5.13), we obtain

$$
\begin{aligned}
&u(\bar{x}) + \frac{1}{\varepsilon}\big\langle A^{\frac{1-\eta}{2}}\bar{x}, A^{\frac{1-\eta}{2}}(\bar{x}-\bar{y})\big\rangle + \frac{1}{\varepsilon}\langle B(\bar{x},\bar{x}), A^{-\eta}(\bar{x}-\bar{y})\rangle \\
&\quad + \delta|\bar{x}|_1^2 + F\big(\bar{x}, \frac{1}{\varepsilon}A^{-\eta}(\bar{x}-\bar{y}), X_N\big) \\
&\quad \leqslant \omega_1(\varepsilon,\delta) + \omega_2(N;\varepsilon,\delta),
\end{aligned} \tag{5.16}
$$

where $\lim_{N\to\infty}\omega_2(N;\varepsilon,\delta) = 0$ if ε, δ are fixed.

Similarly, we have

$$
\begin{aligned}
&v(\bar{y}) + \frac{1}{\varepsilon}\big\langle A^{\frac{1-\eta}{2}}\bar{y}, A^{\frac{1-\eta}{2}}(\bar{x}-\bar{y})\big\rangle + \frac{1}{\varepsilon}\langle B(\bar{y},\bar{y}), A^{-\eta}(\bar{x}-\bar{y})\rangle \\
&\quad - \delta|\bar{y}|_1^2 + F\big(\bar{y}, \frac{1}{\varepsilon}A^{-\eta}(\bar{x}-\bar{y}), Y_N\big) \\
&\quad \geq -\omega_1(\varepsilon,\delta) - \omega_2(N;\varepsilon,\delta).
\end{aligned} \tag{5.17}
$$

Step 4: Finally, we will prove that

$$\limsup_{\delta\searrow 0}\limsup_{\varepsilon\searrow 0}(u(\bar{x}) - v(\bar{y})) \leqslant 0.$$

If this is true, for any $x \in \mathbb{H}$ we have

$$u(x) - v(x) - 2\delta|x|^2 = \Phi(x,x;\varepsilon,\delta) \leqslant \Phi(\bar{x},\bar{y};\varepsilon,\delta) \leqslant u(\bar{x}) - v(\bar{y}).$$

Letting $\varepsilon \searrow 0, \delta \searrow 0$ we will get $u \leqslant v$ on $\mathbb{H}$.

Subtracting (5.17) from (5.16), using $(F3)$, and letting $N \to \infty$, we obtain

$$\begin{aligned} u(\bar{x}) - v(\bar{y}) &+ \delta(|\bar{x}|_1^2 + |\bar{y}|_1^2) \\ &+ \frac{1}{\varepsilon}\Big(b(\bar{x}, \bar{x}, A^{-\eta}(\bar{x} - \bar{y})) - b(\bar{y}, \bar{y}, A^{-\eta}(\bar{x} - \bar{y}))\Big) \\ &\leqslant \omega\Big(|\bar{x} - \bar{y}|_\beta\big(1 + \frac{1}{\varepsilon}|\bar{x} - \bar{y}|_\beta\big)\Big) - \frac{1}{\varepsilon}|\bar{x} - \bar{y}|_{1-\eta}^2 \\ &\quad + 2\omega_1(\varepsilon, \delta). \end{aligned} \tag{5.18}$$

We are now going to estimate the crucial second line in (5.18). The following techniques are originated from [5]. We have

$$\begin{aligned} b(\bar{x}, \bar{x}, A^{-\eta}(\bar{x} - \bar{y})) &- b(\bar{y}, \bar{y}, A^{-\eta}(\bar{x} - \bar{y})) \\ &= b(\bar{x} - \bar{y}, \bar{x}, A^{-\eta}(\bar{x} - \bar{y})) + b(\bar{y}, \bar{x} - \bar{y}, A^{-\eta}(\bar{x} - \bar{y})). \end{aligned}$$

The two terms on the right-hand side of this equality can be estimated similarly, hence we will only show how to estimate the first one. We have, by the Hölder inequality,

$$\begin{aligned} \frac{1}{\varepsilon}|b(\bar{x} - \bar{y}, \bar{x}, &A^{-\eta}(\bar{x} - \bar{y}))| \\ &= \frac{1}{\varepsilon}|b(\bar{x} - \bar{y}, A^{-\eta}(\bar{x} - \bar{y}), \bar{x})| \\ &\leqslant \frac{c}{\varepsilon}|\bar{x}|_{0,q}|A^{-\eta+\frac{1}{2}}(\bar{x} - \bar{y})|_{0,2p}|\bar{x} - \bar{y}|_{0,2p}, \end{aligned} \tag{5.19}$$

where we took p and q such that $\frac{1}{p} + \frac{1}{q} = 1$, and later p will be sufficiently small. To continue we choose τ such that $0 < \tau < \eta - \frac{1}{2}$ (this choice is possible because of $\eta \in (\frac{1}{2}, 1)$), and notice that if p is sufficiently close to 1, the Sobolev imbedding (2.2) guarantees that

$$|A^{-\eta+\frac{1}{2}}(\bar{x} - \bar{y})|_{0,2p} \leqslant C|A^{-\eta+\frac{1}{2}+\frac{\tau}{2}}(\bar{x} - \bar{y})|$$

We now set $S = A^{\frac{1}{2}}$ and $z = A^{-\frac{\eta}{2}}(\bar{x} - \bar{y})$ in (2.3). Then

$$\begin{aligned} |A^{-\eta+\frac{1}{2}+\frac{\tau}{2}}(\bar{x} - \bar{y})| &= |S^{1-\eta+\tau}z| \\ &\leqslant C|Sz|^{1-\eta+\tau}|z|^{\eta-\tau} \\ &= C|A^{\frac{1-\eta}{2}}(\bar{x} - \bar{y})|^{1-\eta+\tau}|\bar{x} - \bar{y}|_{-\eta}^{\eta-\tau}. \end{aligned}$$

To estimate $|\bar{x} - \bar{y}|_{0,2p}$ in (5.19) we again use the Sobolev imbedding

$$|\bar{x} - \bar{y}|_{0,2p} \leqslant C|\bar{x} - \bar{y}|_{\tau(1-\eta)}$$

that holds if p is small enough, and set $S = A^{\frac{1-\eta}{2}}$ and $z = \bar{x} - \bar{y}$ in (2.3). We then obtain

$$\begin{aligned} |\bar{x} - \bar{y}|_{\tau(1-\eta)} = |S^\tau z| &\leqslant C|Sz|^\tau|z|^{1-\tau} \\ &= C|A^{\frac{1-\eta}{2}}(\bar{x} - \bar{y})|^\tau|\bar{x} - \bar{y}|^{1-\tau}. \end{aligned}$$

Therefore, plugging above results in (5.19), and estimating further

$$|\bar{x}|_{0,q} \leqslant C|\bar{x}|_1$$

we get

$$\begin{aligned}&\frac{1}{\varepsilon}|b(\bar{x}-\bar{y},\bar{x},A^{-\eta}(\bar{x}-\bar{y}))| \\ &\qquad\leqslant \frac{C}{\varepsilon}|\bar{x}|_1|\bar{x}-\bar{y}|_{1-\eta}^{1-\eta+2\tau}|\bar{x}-\bar{y}|_{-\eta}^{\eta-\tau}|\bar{x}-\bar{y}|^{1-\tau}\end{aligned}$$

which, upon using $|\bar{x}-\bar{y}| \leqslant C|\bar{x}-\bar{y}|_{1-\eta}$, yields

$$\begin{aligned}&\frac{1}{\varepsilon}\big|b(\bar{x}-\bar{y},\bar{x},A^{-\eta}(\bar{x}-\bar{y}))\big| \\ &\qquad\leqslant C|\bar{x}|_1\frac{|\bar{x}-\bar{y}|_{1-\eta}}{\sqrt{\varepsilon}}\frac{|\bar{x}-\bar{y}|_{-\eta}^{\eta-\tau}}{\sqrt{\varepsilon}}|\bar{x}-\bar{y}|^{1-\eta+\tau}.\end{aligned} \tag{5.20}$$

We now notice that

$$\Phi(\bar{x},\bar{y};\varepsilon,\delta) \geq \max\{\Phi(\bar{x},\bar{x};\varepsilon,\delta),\Phi(\bar{y},\bar{y};\varepsilon,\delta)\}. \tag{5.21}$$

We use the fact that u and v are locally Lipschitz continuous in $|.|_{-\eta}$ norm and $|\bar{x}|,|\bar{y}| \leqslant R_\delta$ independently of ε, for a fixed δ to deduce from (5.21) that

$$\frac{|\bar{x}-\bar{y}|_{-\eta}^2}{\varepsilon} \leqslant K_\delta|\bar{x}-\bar{y}|_{-\eta}$$

for some $K_\delta > 0$. This implies that

$$\frac{|\bar{x}-\bar{y}|_{-\eta}^{\frac{1}{2}}}{\sqrt{\varepsilon}} \leqslant \sqrt{K_\delta}.$$

Therefore

$$\frac{|\bar{x}-\bar{y}|_{-\eta}^{\eta-\tau}}{\sqrt{\varepsilon}} = \frac{|\bar{x}-\bar{y}|_{-\eta}^{\frac{1}{2}}}{\sqrt{\varepsilon}}|\bar{x}-\bar{y}|_{-\eta}^{\eta-\frac{1}{2}-\tau} \leqslant \sqrt{K_\delta}|\bar{x}-\bar{y}|_{-\eta}^{\eta-\frac{1}{2}-\tau} \to 0$$

as $\varepsilon \searrow 0$ (by (5.2) and $\eta-\frac{1}{2}+\tau>0$). Using this in (5.20) we thus obtain

$$\begin{aligned}\frac{1}{\varepsilon}|b(\bar{x}-\bar{y},\bar{x},A^{-\eta}(\bar{x}-\bar{y}))| &\leqslant |\bar{x}|_1\frac{|\bar{x}-\bar{y}|_{1-\eta}}{\sqrt{\varepsilon}}\sigma_1(\varepsilon,\delta) \\ &\leqslant \delta|\bar{x}|_1^2+\frac{|\bar{x}-\bar{y}|_{1-\eta}^2}{2\varepsilon}\sigma_2(\varepsilon,\delta)\end{aligned} \tag{5.22}$$

for some local moduli σ_1 and σ_2. Similarly we obtain

$$\frac{1}{\varepsilon}\big|b(\bar{y},\bar{x}-\bar{y},A^{-\eta}(\bar{x}-\bar{y}))\big| \leqslant \delta|\bar{y}|_1^2+\frac{|\bar{x}-\bar{y}|_{1-\eta}^2}{2\varepsilon}\sigma_2(\varepsilon,\delta). \tag{5.23}$$

Having the results in (5.22), (5.23) and returning to (5.18) we get

$$u(\bar{x})-v(\bar{y}) \leqslant \omega\Big(|\bar{x}-\bar{y}|_\beta\big(1+\frac{|\bar{x}-\bar{y}|_\beta}{\varepsilon}\big)\Big)+\big(\sigma_2(\varepsilon,\delta)-1\big)\frac{|\bar{x}-\bar{y}|_{1-\eta}^2}{\varepsilon}+2\omega_1(\varepsilon,\delta).$$

Set $r = |\bar{x} - \bar{y}|_{1-\eta}$. By the interpolation inequality (2.4) and the properties of the modulus, for all $\mu, \sigma > 0$ there exists $C_\sigma, K_\mu > 0$ such that

$$u(\bar{x}) - v(\bar{y}) \leqslant \mu + \big(\sigma_2(\varepsilon,\delta) - 1\big)\frac{r^2}{\varepsilon} + K_\mu\Big(\sigma\frac{r^2}{\varepsilon} + C_\sigma\frac{|\bar{x}-\bar{y}|_{-\eta}}{\varepsilon}r + r\Big) + 2\omega_1(\varepsilon,\delta).$$

If ε, δ are small enough we will have $\sigma_2(\varepsilon,\delta) - 1 < 0$. Thus, for μ fixed, we can choose σ such that $K_\mu\sigma + \sigma_2(\varepsilon,\delta) - 1 < 0$. Then, in the right-hand side of the previous inequality, we have a polynomial of order 2 in $\frac{r}{\sqrt{\varepsilon}}$ which is bounded from above and we get

$$u(\bar{x}) - v(\bar{y}) \leqslant \mu + \frac{K_\mu^2\big(\sqrt{\varepsilon} + \frac{C_\sigma|\bar{x}-\bar{y}|_{-\eta}}{\sqrt{\varepsilon}}\big)^2}{4(1 - K_\mu\sigma - \sigma_2(\varepsilon,\delta))} + 2\omega_1(\varepsilon,\delta). \tag{5.24}$$

Applying (5.2), (5.15) and (5.24) yields

$$\limsup_{\delta\searrow 0}\limsup_{\varepsilon\searrow 0}\,(u(\bar{x}) - v(\bar{y})) \leqslant \mu$$

for all $\mu > 0$. Thus, we have already proved that $u \leqslant v$ in $\mathbb{H}$. This proves the uniqueness of continuous viscosity solutions in the class $\mathcal{K}$. □

Remark 5.3. In case $\Omega \subset \mathbb{R}^k$ for $k \geq 3$, the verification shows that the proof above is not applicable. For example, in the case $k = 3$ the imbedding (2.2) becomes: for $\gamma \in (0, \frac{3}{2}], q \in [2, \frac{6}{3-2\gamma}]$,

$$\mathbb{V}_\gamma \hookrightarrow L^q(\Omega) \tag{5.25}$$

The imbedding will be used for estimating the absolute value of

$$b(\bar{x} - \bar{y}, \bar{x}, A^{-\eta}(\bar{x} - \bar{y})).$$

In (5.19) we need $\mathbb{V} \hookrightarrow L^q(\Omega)$, which implies from (5.25) that $2 \leqslant q \leqslant 6$. Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p \geq \frac{6}{5}$.

On the other hand, the inequality after (5.19) requires $\mathbb{V}_\tau \hookrightarrow L^{2p}(\Omega)$ for $0 < \tau < \eta - \frac{1}{2}$. Then it follows from (5.25) that

$$2p \leqslant \frac{6}{3 - 2\tau}. \tag{5.26}$$

In general we want to have small values of p and the smallest one is $\frac{6}{5}$. So (5.26) implies $\tau \geq \frac{3}{2}$, this is contrast to the requirement $0 < \tau < \eta - \frac{1}{2}$.

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